Vertex-graceful labelings for some double cycles

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Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges.

$G$ is said to be vertex-graceful if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the induced labeling $f^+ : E(G) \rightarrow \mathbb{Z}_q$ defined by $f^+(uv) \equiv f(u) + f(v) \pmod{q}$, for each edge $uv$, is a bijection.
Vertex-graceful

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$f$ is called a vertex-graceful labeling of $G$.

This concept were first introduced by Lee, Pan and Tsai in 2005.
Another induced labeling $f^* : E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = f(u) + f(v)$. If $f^*(E(G))$ consists of consecutive integers, then $f$ is called a strong vertex-graceful labeling. And $G$ is called strong vertex-graceful.

Acharya and Hegde, in 1991, called a strong vertex-graceful graph as strongly $s$-indexable graph, where $s$ is the minimum value of the mapping $f^*$. It will be introduced later.
Total edge-magic

$G$ is said to be **total edge magic** if there a bijection
$f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ such that
$f(u) + f(uv) + f(v)$ is a constant, for each edge $uv$.
The study of total edge-magic graphs is initially introduced by Kotzig and Rosa in 1970. They called
the total edge magic graph as magic graph.
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In 1998, Enomoto et al. called a total edge-magic graph as super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. 
A relation

Chen, in 2001, claimed (also proved by Figuero-Centeno et al. in 2001) that a graph is super edge-magic if and only if there exists a vertex labeling such that two sets $f(V(G))$ and 
$$\{f(u) + f(v) \mid uv \in E(G)\}$$ are both consecutive.

So strong vertex-graceful and super edge-magic are equivalent.
There is an equivalent concept similar to strongly vertex-graceful.

For integers $s, d \geq 1$, a $(p, q)$-graph $G$ is called strongly $(s, d)$-indexable if there is a bijection $f : V(G) \rightarrow \{0, 1, \ldots, p - 1\}$ such that the induced map $f^* : E(G) \rightarrow \{s, s + d, s + 2d, \ldots, s + (q - 1)d\}$ is bijective, where $f^*(uv) = f(u) + f(v)$ for $uv \in E(G)$. 
There is an equivalent concept similar to strongly vertex-graceful. For integers $s, d \geq 1$, a $(p, q)$-graph $G$ is called **strongly $(s, d)$-indexable** if there is a bijection $f : V(G) \rightarrow \{0, 1, \ldots, p - 1\}$ such that the induced map $f^* : E(G) \rightarrow \{s, s + d, s + 2d, \ldots, s + (q - 1)d\}$ is bijective, where $f^*(uv) = f(u) + f(v)$ for $uv \in E(G)$.

Note that if we only request that the values $f^*$ are all distinct, then $G$ is called **indexable**.
A \((s, 1)\)-strongly indexable graph is simply called \textbf{strongly }\(s\)-\textbf{indexable} graph, and a \((1, 1)\)-strongly indexable graph is simply called \textbf{strongly indexable} graph.
A \((s, 1)\)-strongly indexable graph is simply called **strongly \(s\)-indexable** graph, and a \((1, 1)\)-strongly indexable graph is simply called **strongly indexable** graph.

**Remark:** If \(f : V(G) \rightarrow \{0, 1, \ldots, p - 1\}\) is a strongly \((s, 1)\)-indexable labeling of \(G\), then \(f + 1\) is strong vertex-graceful labeling of \(G\) with minimum value \(s + 2\). Hence, **strongly \(s\)-indexable**, **strong vertex-graceful** and **super edge-magic** are equivalent.
General Properties

The following lemma was found by Enomoto et al.

**Lemma 1**  If a \((p, q)\)-graph is super edge-magic, then \(q \leq 2p - 3\).
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**Lemma 1** *If a* $(p, q)$*-graph is super edge-magic, then* $q \leq 2p - 3$.

**Corollary 2** *Every super edge-magic* $(p, q)$*-graph contains at least two vertices of degree less than 4.*
General Properties

Let $f$ be any vertex labeling of a graph $G$ which contains $q$ edges. Then

$$\sum_{e \in E(G)} f^*(e) = \sum_{uv \in E(G)} (f(u) + f(v)) = \sum_{x \in V(G)} \deg(x)f(x).$$
General Properties

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$$

If $f$ is a vertex-graceful, then by the above equation

$$
\sum_{x \in V(G)} \deg(x) f(x) \equiv \sum_{e \in E(G)} f^+(e) \equiv \sum_{i=1}^{q} i \equiv \frac{q(q + 1)}{2} \equiv \begin{cases} 
0 & \text{if } q \text{ is odd} \\
\frac{q}{2} & \text{if } q \text{ is even} 
\end{cases} \pmod{q}. \quad (1)
$$
Some Known Results

\( P_n \) is super edge-magic for \( n \geq 1 \) which was proved by Ringel and Lladó in 1996.
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Results of Enomoto et al. in 1998:

★ $C_n$ is super edge-magic if and only if $n$ is odd.
Some Known Results

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Results of Enomoto et al. in 1998:

- $C_n$ is super edge-magic if and only if $n$ is odd.
- $K_{m,n}$ with $m \leq n$ is super edge-magic if and only if $m = 1$. 
Some Known Results

R.M. Figueroa-Centeno et al. in 2001 proved that

★ The fan $F_n = P_n \lor K_1$ is super edge-magic if and only if $1 \leq n \leq 6$. 
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- The fan $F_n$ is edge-magic for every positive integer $n$. 
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★ The ladder $L_n = P_n \times P_2$ is super edge-magic for odd $n$. (Also proved by Tsuchiya and Yokomura independently.)
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★ The ladder $L_n = P_n \times P_2$ is super edge-magic for odd $n$. (Also proved by Tsuchiya and Yokomura independently.)

★ The generalized prism $C_m \times P_n$ is super edge-magic if $m$ is odd and $n \geq 2$. (Also proved by Tsuchiya and Yokomura independently.)
Some Known Results


★ $K_1 \lor K_{1,n}$ is super edge-magic for $n \geq 1$. 
Some Known Results


★ $K_1 \vee K_{1,n}$ is super edge-magic for $n \geq 1$.
★ There is a connected cubic super edge-magic graph of order $p$ if and only if $p \equiv 2 \pmod{4}$. 
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There are also some super edge-magic disconnected graphs.
Double cycle

Double cycle $C(m, n)$
In 2005, Lee et al. claimed without proof that
\( C(3, n) \) for \( n = 4, 6, 7, 8; \)
\( C(4, m) \) for \( m = 5, 6, 7, 9; \)
\( C(5, k) \) for \( k = 5, 6, 8, 9 \)
are not vertex-graceful.

They also showed that \( C(3, 5), C(3, 9), C(4, 4), \)
\( C(4, 8), C(5, 7) \) are strong vertex-graceful.
Theorem 3  A double cycle $C(m, n)$ is vertex-graceful only if $m + n \equiv 0 \pmod{4}$. 
Necessary condition for vertex-graceful double cycle

**Theorem 3** A double cycle $C(m, n)$ is vertex-graceful only if $m + n \equiv 0 \pmod{4}$.

**Proof:**

$$
\sum_{x \in V(G)} \deg(x)f(x) = 2f(c) + 2 \sum_{x \in V(G)} f(x) = 2f(c) + (m + n - 1)(m + n),
$$

where $c$ is the coalesced vertex.
Theorem 3  A double cycle \( C(m, n) \) is vertex-graceful only if \( m + n \equiv 0 \pmod{4} \).

Proof:

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\sum_{x \in V(G)} \deg(x)f(x) = 2f(c) + 2 \sum_{x \in V(G)} f(x) \\
= 2f(c) + (m + n - 1)(m + n),
\]

where \( c \) is the coalesced vertex.

By (1) we have

\[
2f(c) \equiv \begin{cases} 
0 & \text{if } m + n \text{ is odd} \\
\frac{m+n}{2} & \text{if } m + n \text{ is even}
\end{cases} \pmod{m + n}
\]
Proof

\[ 2f(c) \equiv \begin{cases} 
0 & \text{if } m + n \text{ is odd} \\
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\end{cases} \pmod{m+n} \]

Suppose \( m + n \) is odd. Then \( f(c) \equiv 0 \pmod{m+n} \).

But it is impossible since

\[ f(c) \in \{1, 2, \ldots, m + n - 1\}. \]
Proof

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Suppose \( m + n \) is odd. Then \( f(c) \equiv 0 \pmod{m+n} \).

But it is impossible since \( f(c) \in \{1, 2, \ldots, m+n-1\} \).

Suppose \( m + n \) is even. Then \( 4f(c) \equiv m + n \pmod{2(m+n)} \).

This implies \( m + n \equiv 0 \pmod{4} \). \( \square \)
From the proof above we can see that $f(c) = \frac{m+n}{4}$ or $\frac{3(m+n)}{4}$ for a vertex-graceful labeling of $C(m, n)$. 
A remark

From the proof above we can see that \( f(c) = \frac{m+n}{4} \) or \( \frac{3(m+n)}{4} \) for a vertex-graceful labeling of \( C(m, n) \).

But these cases are equivalent. It is because that if \( f \) is a vertex-graceful labeling, then \( (m + n) - f \) is also a vertex-graceful labeling. This result also holds for any strong vertex-graceful labeling of \( C(m, n) \).
Theorem 4  For $k \geq 2$, $C(3, 4k - 3)$ is strong vertex-graceful.
Proof: Let the two cycles in $C(3, 4k - 3) = (V, E)$ be $u_0 u_1 u_2 u_0$ and $v_0 v_1 \cdots v_{4k-4} v_0$, where $u_0 = v_0$. 
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Define $f : V \rightarrow \{1, 2, \ldots, 4k - 1\}$ by

- $f(v_{2i}) = k + i$ for $0 \leq i \leq 2k - 2$;
- $f(v_{2j+1}) = 3k + 1 + j$ for $0 \leq j \leq k - 2$;
- $f(v_{2j+1}) = j + 2 - k$ for $k - 1 \leq j \leq 2k - 3$;
- $f(u_1) = 3k - 1$ and $f(u_2) = 3k$. 
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Define $f : V \rightarrow \{1, 2, \ldots, 4k - 1\}$ by

$f(v_{2i}) = k + i$ for $0 \leq i \leq 2k - 2$;

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$f(v_{2j+1}) = j + 2 - k$ for $k - 1 \leq j \leq 2k - 3$;

$f(u_1) = 3k - 1$ and $f(u_2) = 3k$.

It is easy to see that

$\{f(v_{2i}) | 0 \leq i \leq 2k - 2\} = [k, 3k - 2]$;

$\{f(v_{2j+1}) | 0 \leq j \leq k - 2\} = [3k + 1, 4k - 1]$;

$\{f(v_{2j+1}) | k - 1 \leq j \leq 2k - 3\} = [1, k - 1]$. Thus, $f$ is a bijection.
Proof

Also we can check that $f^*(E) = [2k, 6k - 1]$. Hence $f$ is a strong vertex-graceful labeling of $C(3, 4k - 3)$. □
Example 1 This is the strong vertex-graceful labeling for $C(3, 13)$ constructed in the proof of Theorem 4 ($k = 4$).

- $f(v_{2i}) = k + i$ for $0 \leq i \leq 2k - 2$;
- $f(v_{2j+1}) = 3k + 1 + j$ for $0 \leq j \leq k - 2$;
- $f(v_{2j+1}) = j + 2 - k$ for $k - 1 \leq j \leq 2k - 3$;
- $f(u_1) = 3k - 1$ and $f(u_2) = 3k$. 
Theorem 5  The graph $C(2n + 3, 2n + 1)$ is strong vertex-graceful for $n \geq 1$. 
Proof: First we consider $C_{4n+4}$.

Now we want to define a labeling $f : V(C_{4n+4}) \to [1, 4n + 3]$ such that $f(x_1) = f(y_1) = n + 1$. 
Proof

\[ f(x_{2i+1}) = n + 1 - i, \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(x_{2i}) = 3n + 3 - i, \quad 1 \leq i \leq \left\lfloor \frac{n + 1}{2} \right\rfloor \]

\[ f(y_{2i+1}) = n + 1 + i, \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(y_{2i}) = 3n + 2 + i, \quad 1 \leq i \leq \left\lfloor \frac{n + 1}{2} \right\rfloor \]

\[ f(z_{2i+1}) = 2n + 2 + i, \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(z_{2i}) = i, \quad 1 \leq i \leq \left\lfloor \frac{n + 1}{2} \right\rfloor \]

\[ f(w_{2i+1}) = 4n + 3 - i, \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(w_{2i}) = 2n + 2 - i, \quad 1 \leq i \leq \left\lfloor \frac{n + 1}{2} \right\rfloor \]
Proof

Clearly $f$ is onto with $f(x_1) = f(y_1) = n + 1$. 
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Clearly $f$ is onto with $f(x_1) = f(y_1) = n + 1$.

Now we merge $x_1$ with $y_1$ to get the graph $C(2n + 3, 2n + 1)$ and keep the labeling $f$. Then $f$ is a bijection between $V(C(2n + 3, 2n + 1))$ and $[1, 4n + 3]$. 
Proof

Clearly $f$ is onto with $f(x_1) = f(y_1) = n + 1$.
Now we merge $x_1$ with $y_1$ to get the graph $C'(2n + 3, 2n + 1)$ and keep the labeling $f$. Then $f$ is a bijection between $V(C(2n + 3, 2n + 1))$ and $[1, 4n + 3]$.

It can be checked that $f$ is a strong vertex-graceful labeling of $C(2n + 3, 2n + 1)$. 
Examples

Example 2 This is the strong vertex-graceful labeling for $C(7, 5)$ constructed in the proof of Theorem 5.
**Example 3** This is the strong vertex-graceful labeling for $C(9, 7)$ constructed in the proof of Theorem 5.
Strong vertex-graceful labeling for $C(4, 12)$.
END