Operations of graphs and unimodality of independence polynomials

Yi Wang
School of Mathematical Sciences, Dalian University of Technology
wangyi@dlut.edu.cn

Abstract
In this talk, we first give a brief survey on unimodality problems for the independence polynomial of a graph and then present our recent results for certain classes of graphs.

(Joint work with Bao-Xuan Zhu)
Independence polynomials

Let $G$ be a finite and simple graph. An independent set in $G$ is a set of pairwise non-adjacent vertices. A maximum independent set in $G$ is a largest independent set and its size is denoted $\alpha(G)$. Let $i_k(G)$ denote the number of independent sets of cardinality $k$ in $G$. Then its generating function

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k, \quad i_0(G) = 1$$

is called the independence polynomial of $G$ (Gutman and Harary).

- It is an NP-complete problem to determine the independence polynomial, since evaluating $\alpha(G)$ is an NP-complete problem.
A sequence \( \{a_0, a_1, \ldots, a_n\} \) of nonnegative numbers is

- **unimodal** (UM) if \( a_0 \leq \cdots \leq a_m \leq a_m + 1 \geq \cdots \geq a_n \);
  
  \( m \) is called a **mode** of the sequence.

- **log-concave** (LC) if \( a_{i-1}a_{i+1} \leq a_i^2 \) for \( 0 < i < n \);

- **symmetric** (Sym) if \( a_{n-i} = a_i \) for all \( 0 \leq i \leq n \).

**Prop.** \( \text{LC} \iff a_i/a_{i-1} \) is decreasing \( \implies \) UM.

**Ex.** \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \) has a mode \( n/2 \) or two modes \( (n \pm 1)/2 \).
Newton inequality

Newton inequality: Suppose that \( f(x) = \sum_{i=0}^{n} a_i x^i \). Then

\[
a_i^2 \geq a_{i-1}a_{i+1} \frac{(i + 1)(n - i + 1)}{i(n - i)},
\]

and \( a_0, \ldots, a_n \) is therefore LC and UM with at most two modes.

Darroch: \( \lfloor M \rfloor \leq \text{mode} (a_i) \leq \lceil M \rceil \), where

\[
M := \frac{f'(1)}{f(1)} = \frac{\sum_{i=1}^{n} ia_i}{\sum_{i=0}^{n} a_i} = \sum_{j=1}^{n} \frac{1}{1-r_j},
\]

and \( r_j \) are all zeros of \( f(x) \).

Ex. \( \sum_{i=0}^{n} \binom{n}{i} x^i = (x + 1)^n \). \( M = \frac{n}{2} \).
Unimodality of polynomials

Def. We say that a polynomial $\sum_{i=0}^{n} a_i x^i$ is UM (LC, Sym) if the sequence $a_0, a_1, \ldots, a_n$ has the corresponding property.
Denote by $\text{RZ}$ the set of real polynomials with only Real Zeros.

Product Theorem: Let $f(x), g(x) \in \mathbb{R}^+[x]$. Then

1. $f(x), g(x) \in \text{RZ} \implies f(x)g(x) \in \text{RZ}$.
2. $f(x), g(x) \in \text{LC} \implies f(x)g(x) \in \text{LC}$.
3. $f(x) \in \text{LC}, g(x) \in \text{UM} \implies f(x)g(x) \in \text{UM}$.
4. $f(x), g(x) \in \text{Sym} \cap \text{UM} \implies f(x)g(x) \in \text{Sym} \cap \text{UM}$.

Remark: $I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x)$. 
Unimodality of matching polynomials

Let $G$ be a graph with $n$ vertices and let $m_k(G)$ be the number of $k$-edge matchings. The matching polynomial of $G$ is defined as

$$m(G; x) := \sum_{k \geq 0} m_k(G) x^k,$$

or

$$M(G; x) := \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k} = x^n m(G; -x^{-2}).$$

**Heilmann-Lieb:** $M(G; x) \in RZ$, and so $m(G; x) \in RZ$.

- Let $L(G)$ be the line graph of $G$. Then $m(G; x) = I(L(G); x)$. In other words, IP can be regarded as a generalization of MP. Actually, IP was introduced as an analog of MP.
Unimodality of independence polynomials

- **Wilf**: IP share the same unimodality property as MP?
- **Erdős et al**: No! $I(G; x) \in \text{UM}$ if $G$ is a tree or a forest?
- **Chudnovsky-Seymour**: $I(G; x) \in \text{RZ}$ if $G$ is claw-free!

In particular, $I(L(G); x) \in \text{RZ}$ since $L(G)$ is claw-free for any $G$.

- **Brown-Nowakowski**: (1) almost every IP has a nonreal zero; (2) the average IP has only real zeros.
- **Brown**: When does IP of a graph have only real zeros?
Calculation of independence polynomials

Let $G = (V, E)$ be a simple graph and $v \in V$.
Denote $N(v) = \{u : u \in V \text{ and } uv \in E\}$ and $N[v] = N(v) \cup \{v\}$.

**Lemma**  For arbitrary $v \in V$ and $e = uv \in E$, we have

(1) $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$; and

(2) $I(G; x) = I(G - e; x) - x^2I(G - N(u) \cup N(v); x)$.

**Ex.** Let $P_n$ be a path with $n$ vertices. Then

$I(P_n; x) = I(P_{n-1}; x) + xI(P_{n-2}; x), \quad n = 2, 3, 4, \ldots,$

with $I(P_0; x) = 1$ and $I(P_1; x) = 1 + x$.

**Remark** IP for many classes of graphs share recurrence relations.
Polynomials with only real zeros

Let \( f, g \in \text{RZ}. \) Write \( g \preceq f \) if zeros of \( f \) are separated by those of \( g \).

**Liu-Wang (2007a):** Let \( F(x) = a(x)f(x) + \sum_{j=1}^{k} b_j(x)g_j(x) \) and

1. \( f, g_j \in \text{RZ} \) and \( g_j \preceq f \) for each \( j \);
2. \( g_1, \ldots, g_k \) and \( F \) have positive leading coefficients;
3. \( \deg F = \deg f \) or \( \deg f + 1 \).

If \( b_j(r) \leq 0 \) for each \( j \) whenever \( f(r) = 0 \), then \( F \in \text{RZ} \) and \( f \preceq F \).

**Ex.** \( M(G, x) = xM(G - \{v\}, x) - \sum_{u \sim v} M(G - \{v, u\}, x) \).

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Polynomial sequences with only real zeros

Liu-Wang (2007a): Let $f_n(x) \in \mathbb{R}^+[x]$. Suppose that

1) $f_n(x) = a_n(x)f_{n-1}(x) + b_n(x)f_{n-2}(x)$;

2) $f_0(x)$ is a constant and $\deg f_{n-1} \leq \deg f_n \leq \deg f_{n-1} + 1$.

If $b_n(x) \leq 0$ whenever $x \leq 0$, then $f_n(x) \in \mathbb{RZ}$ and $f_{n-1}(x) \preceq f_n(x)$.

Ex. $I(P_0; x) = 1, I(P_1; x) = 1 + x$ and

$$I(P_n; x) = I(P_{n-1}; x) + xI(P_{n-2}; x).$$

Further problem: Factorizations and real zeros of IP!
Linear recurrence relations

**Lemma** Let \( \{z_n\}_{n \geq 0} \) be a sequence satisfying

\[
z_n = az_{n-1} + bz_{n-2}, \quad n = 2, 3, \ldots.
\]

If \( a^2 + 4b > 0 \), then the closed form for the sequence is

\[
z_n = \frac{(z_1 - z_0 \lambda_2) \lambda_1^n + (z_0 \lambda_1 - z_1) \lambda_2^n}{\lambda_1 - \lambda_2}, \quad n = 0, 1, 2, \ldots,
\]

where

\[
\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2}, \quad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2}
\]

are the roots of quadratic equation \( \lambda^2 - a\lambda - b = 0 \).

**Remark** \( \lambda_1 + \lambda_2 = a \) and \( \lambda_1\lambda_2 = -b \).
Factorizations

**Lemma**  Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $n \in \mathbb{N}$.

(1) $\lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2) \prod_{s=1}^{n-1} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{s\pi}{n} \right]$ for odd $n$.

(2) $\lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \prod_{s=1}^{n-2} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{s\pi}{n} \right]$ for even $n$.

(3) $\lambda_1^n + \lambda_2^n = (\lambda_1 + \lambda_2) \prod_{s=1}^{n-1} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{(2s-1)\pi}{2n} \right]$ for odd $n$.

(4) $\lambda_1^n + \lambda_2^n = \prod_{s=1}^{n-2} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{(2s-1)\pi}{2n} \right]$ for even $n$. 

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Independence polynomials of paths

Let $P_n$ be a path with $n$ vertices. Then

$$I(P_n; x) = I(P_{n-1}; x) + xI(P_{n-2}; x), \quad n = 2, 3, 4, \ldots,$$

with $I(P_0; x) = 1$ and $I(P_1; x) = 1 + x$. Thus

$$I(P_n; x) = \frac{(1 + x - \lambda_2)\lambda_1^n + (\lambda_1 - 1 - x)\lambda_2^n}{\lambda_1 - \lambda_2} = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1 - \lambda_2}$$

$$= \prod_{s=1}^{\lfloor(n+1)/2\rfloor} \left(1 + 4x \cos^2 \frac{s\pi}{n + 2}\right),$$

where $\lambda_1$ and $\lambda_2$ are the roots of the quadratic equation $\lambda^2 - \lambda - x = 0$. 

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Concatenation of graphs

**Thm.** Let $G$ be a simple graph and $v$ a vertex of $G$. Let $G_n^-(v)$ denote the graph obtained by gluing each vertex of the path $P_n$ to the vertex $v$ of $n$ copies $G$ respectively. Then

$$I(G_n^-(v); x) = I^\lfloor \frac{n}{2} \rfloor (G - v; x) \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left[ I(G - v; x) + 4xI(G - N[v]; x) \cos^2 \frac{s\pi}{n+2} \right].$$

**Proof.** Let $f_n = I(G_n^-(v); x)$. Denote $a = I(G - v; x)$ and $b = xI(G - N[v]; x)$. Then $f_0 = 1$, $f_1 = a + b$ and $f_n = af_{n-1} + abf_{n-2}$.

Thus

$$f_n = \frac{(a + b - \lambda_2)\lambda_1^n + (\lambda_1 - a - b)\lambda_2^n}{\lambda_1 - \lambda_2} = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{a(\lambda_1 - \lambda_2)},$$

where $\lambda_1$ and $\lambda_2$ are the roots of the equation $\lambda^2 - a\lambda - ab = 0$. 
Claw-free graphs

$f(x)$ and $g(x)$ are compatible if $af(x) + bg(x) \in \mathbb{R}Z$ for all $a, b \geq 0$.

Chudnovsky-Seymour (2007): Let $G$ be a claw-free graph. Then

1. $I(G - v; x)$ and $xI(G - N[v]; x)$ are compatible for any $v \in V(G)$;
2. $I(G; x) \in \mathbb{R}Z$.

Prop. If $G$ is a claw-free graph, then for any vertex $v$ of $G$, the independence polynomial $I(G_n^-(v); x)$ has only real zeros.
Vertebrated graph $V_{n}^{(m)}$

$V_{n}^{(m)}$ is the $n$-concatenation of the star graph $K_{1,m}$ on the center $v$:

Levit-Mandrescu (2002): $I\left(V_{n}^{(1)}; x\right)$ is unimodal.

Conjecture $I(V_{n}^{(1)}; x)$ has only real zeros?

Zhu (2007): (1) $I(V_{n}^{(1)}; x)$ has only real zeros.

(2) $I(V_{n}^{(2)}; x)$ is symmetric and unimodal.

Conjecture $I(V_{n}^{(m)}; x)$ is unimodal?
Our results

Prop. Let \( n, m \geq 1 \). Then

\[
I(V_n^{(m)}; x) = (1 + x)^m \left[ \frac{n+1}{2} \right] \prod_{s=1}^{n/2} \left[ (1 + x)^m + 4x \cos^2 \frac{s\pi}{n+2} \right].
\]

(1) \( I(V_n^{(m)}; x) \) is log-concave and therefore unimodal.

(2) \( I(V_n^{(m)}; x) \) has only real zeros for \( m = 0, 1, 2 \).
Modes of the centipede $V_n^{(1)}$

Levit-Mandrescu conjectured mode $\left(I(V_n^{(1)}; x)\right) = n - f(n)$, where

$$f(n) = \begin{cases} 1 + \lceil n/5 \rceil, & \text{if } 2 \leq n \leq 6; \\ f(2 + (n - 2) \mod 5) + 2 \lfloor (n - 2)/5 \rfloor, & \text{if } n \geq 7. \end{cases}$$

**Counterexample:** For $n = 142$, $n - f(n) = 142 - 57 = 85$, and

$$I(V_{142}^{(1)}; x) = (1 + x)^{71} \prod_{s=1}^{71} \left[ 1 + x \left( 1 + 4 \cos^2 \frac{s\pi}{144} \right) \right].$$

We have $M = \sum_{k=1}^{142} \frac{1}{1-r_k} = \frac{213}{2} - \sum_{s=1}^{71} \frac{1}{2+4 \cos^2 \frac{s\pi}{144}} \approx 86.0487$.

By Darroch’s result, 85 is not one mode of $I(V_{142}^{(1)}; x)$.

Actually, 86 is the unique mode of $I(V_{142}^{(1)}; x) \approx$ 

$$\cdots + 7.18929 \cdot 10^{60} x^{85} + 7.33386 \cdot 10^{60} x^{86} + 7.24852 \cdot 10^{60} x^{87} + \cdots.$$
The $(n, m)$-firecracker graph $F^{(m)}_n$

$F^{(m)}_n$ is the $n$-concatenation of the star graph $K_{1,m}$ on a leaf $v$:

Prop. Let $n, m \geq 1$. Then

$$I(F^{(m)}_n; x) = \left[ (x + 1)^m + x \right]^{\left\lfloor \frac{n}{2} \right\rfloor} \prod_{s=1}^{\frac{n+1}{2}} \left( (x + 1)^m + x + 4x(x + 1)^m \cos^2 \frac{s\pi}{n+2} \right).$$

$I(F^{(m)}_n; x)$ is log-concave and unimodal.
Concatenation of graphs in a circle

**Thm.** Let $C_n$ denote the cycle with $n$ vertices. Let $G$ be a graph and $v$ a vertex of $G$. Let $G_n^o(v)$ denote the graph obtained by gluing each vertex of $C_n$ to the vertex $v$ of $n$ copies $G$ respectively. Then

$$I(G_n^o(v); x) = I\left\lfloor \frac{n+1}{2} \right\rfloor (G - v; x) \times$$

$$\times \prod_{s=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[ I(G - v; x) + 4x I(G - N[v]; x) \cos^2 \frac{(2s - 1)\pi}{2n} \right].$$

In particular, $I(C_n; x) = \prod_{s=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[ 1 + 4x \cos^2 \frac{(2s-1)\pi}{2n} \right].$

**Proof.**

$$g_n = a f_{n-1} + a^2 b f_{n-3} = \lambda_1^n + \lambda_2^n.$$
Graphs of Levit and Mandrescu

Levit and Mandrescu constructed a family of graphs $H_n$ from the path $P_n$ by the so-called “clique cover of a graph” rule:

Clearly, $I(H_0; x) = 1$, $I(H_1; x) = 1 + 3x + x^2$ and

$$
\begin{cases}
I(H_{2m}; x) = I(H_{2m-1}; x) + xI(H_{2m-2}; x), \\
I(H_{2m+1}; x) = (1 + x)^2 I(H_{2m}; x) + xI(H_{2m-1}; x),
\end{cases}
$$

It is showed that $I(H_n; x)$ are symmetric and unimodal.
A conjecture of Levit and Mandrescu

Levit and Mandrescu further proposed the following.

**Conjecture** (1) \( I(H_n; x) \) has only real zeros; and
(2) all zeros of \( I(H_n; x) \) are located in the interval \((-6, 0)\).

**Proof.** Let \( h_n = I(H_n; x) \). Then \( h_{n+2} = (1 + 4x + x^2)h_n - x^2h_{n-2} \), with \( h_{-1} = h_0 = 1, h_1 = 1 + 3x + x^2, h_2 = 1 + 4x + x^2 \). Hence

\[
 h_n = \frac{(\sqrt{\lambda_1})^{n+2} + (\sqrt{\lambda_2})^{n+2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} = \prod_{s=1}^{[(n+1)/2]} \left(1 + 4x + x^2 + 2x \cos \frac{2s\pi}{n+2}\right)
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the roots of \( \lambda^2 - (1 + 4x + x^2)\lambda + x^2 = 0 \).
Remarks

Suppose that \( z_n = az_{n-1} + bz_{n-2} \). Then

\[
z_n = \frac{(z_1 - z_0 \lambda_2) \lambda_1^n + (z_0 \lambda_1 - z_1) \lambda_2^n}{\lambda_1 - \lambda_2},
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the roots of \( \lambda^2 - a\lambda - b = 0 \). If \( z_1 = az_0 \), then

\[
z_n = z_0 \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} = z_0 a \delta_n \prod_{s=1}^{\lfloor n/2 \rfloor} \left[ a^2 + 4b \cos^2 \frac{s\pi}{n+1} \right],
\]

where \( \delta_n = 0 \) for even \( n \) and 1 otherwise. If \( z_1 = a\frac{z_0}{2} \), then

\[
z_n = \frac{1}{2} z_0 (\lambda_1^n + \lambda_2^n) = \frac{1}{2} z_0 a \delta_n \prod_{s=1}^{\lfloor n/2 \rfloor} \left[ a^2 + 4b \cos^2 \frac{(2s - 1)\pi}{2n} \right].
\]
Operations on graphs

Frucht-Harary: The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking $|V(G)|$ copies of $H$, and then joining the $i$th vertex of $G$ to all vertices in the $i$th copy of $H$.

Godsil-Mckay: The rooted product $G\tilde{\circ}H$ of a graph $G$ and a rooted graph $H$ is the graph obtained by gluing the $i$th vertex of $G$ to the root of the $i$th copy of $H$.

Def. Let $\mathcal{C} = \{C_1, \ldots, C_q\}$ be a clique cover of $G$ and $U \subseteq V(H)$. The clique cover product $G^\mathcal{C} \star H^U$ is the graph obtained by joining every vertex of $C_j$ to every vertex of $U$ in the $j$th copy of $H$.

Ex. $G \circ H = G^{V(G)} \star H^{V(H)}$ and $G\tilde{\circ}H = G^{V(G)} \star (H - v)^{N(v)}$. 

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Thm. \[ I(G^\mathcal{C} \star H^U; x) = I^q(H; x)I \left( G; \frac{xI(H-U;x)}{I(H;x)} \right), \quad q = |\mathcal{C}|. \]

Gutman: \[ I(G \circ H; x) = I^n(H; x)I \left( G; \frac{x}{I(H;x)} \right), \quad n = |V(G)|. \]

Gutman, Rosenfeld: If \( H \) is a graph with the root \( v \), then

\[ I(G\!\!\circ H; x) = I^n(H - v; x)I \left( G; \frac{xI(H - N[v]; x)}{I(H - v; x)} \right). \]

Rosenfeld: If \( v \) is a pedant vertex of \( H \) and \( N(v) = u \), then

\[ I(G\!\!\circ H; x) = I^n(H - v; x)I \left( G; \frac{xI(H - u - v; x)}{I(H - v; x)} \right). \]

Ex. \[ G^-(n)(v) = P_n\!\!\circ G \text{ and } G^\circ(n)(v) = C_n\!\!\circ G. \]
References


Thank you for your attention!