Total Relative Displacements in Graphs

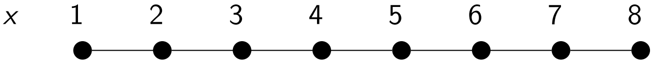
Student: Chia-Fen Chang
Advisor: Hung-Lin Fu
Co-Advisor: Bor-Liang Chen

2008.08.12

- f : a permutation of $V(G)$
- $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$

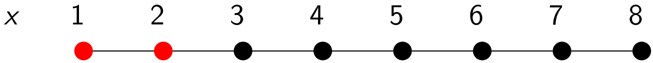
- f : a permutation of $V(G)$
- $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$

x	1	2	3	4	5	6	7	8
$f(x)$	7	5	3	1	8	6	4	2



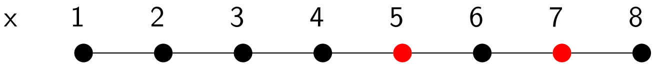
- f : a permutation of $V(G)$
- $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$

x	1	2	3	4	5	6	7	8
$f(x)$	7	5	3	1	8	6	4	2



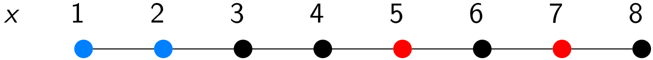
- f : a permutation of $V(G)$
- $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$

x	1	2	3	4	5	6	7	8
$f(x)$	7	5	3	1	8	6	4	2



- f : a permutation of $V(G)$
- $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$
- $\delta_f(1, 2) = |d_G(1, 2) - d_G(f(1), f(2))|$
$= |d_G(1, 2) - d_G(5, 7)| = 1$

x	1	2	3	4	5	6	7	8
$f(x)$	7	5	3	1	8	6	4	2



- $\delta_f(x) = \sum_y \delta_f(x, y)$
- $\delta_f(G) = \sum_{x \neq y} \delta_f(x, y) = \frac{1}{2} \sum_x \delta_f(x)$
- $\pi(G) = \min \delta_f(G) \neq 0$
  $\pi^*(G) = \max \delta_f(G)$
- f with $\delta_f(G) = \pi(G)$: near automorphism
  g with $\delta_g(G) = \pi^*(G)$: chaotic mapping

Origin

[Chartrand, Gavlas, VanderJagt, 1999]

- near-automorphism of graphs
- the 1996 eighth quadrennial international conference on graph theory, combinatorics, algorithms, and applications
- conjectured that $\pi(G) = 2n - 4$ when G is a path with n vertices

Known Results

- [Aitken,1999] $\pi(P_n) = 2n - 4$.
- [Reid,2002] $\pi(K_{m,n}) = \begin{cases} 2m & \text{if } n = m+1, \\ 2(m+n-2) & \text{otherwise.} \end{cases}$
- [Reid,2002] $\pi(K_{n_1,n_2,\ldots,n_t}) =$

$$\begin{cases} 2n_{h+1} - 2 & \text{if } 1 = n_1 = \cdots = n_h < n_{h+1} \leq \cdots \leq n_t, \\ & \text{and } t \geq (h+1), \text{ for some } h \geq 2; \\ 2n_{k_0} & \text{if } 1 = n_1 < n_2 \text{ or } n_h \geq 2 \text{ and} \\ & n_{k+1} = n_k + 1 \text{ for some } k, 1 \leq k \leq t-1, \\ & \text{and } 2 + n_{k_0} \leq n_1 + n_2, \text{ where } k_0 \\ & \text{is the smallest index for which} \\ & n_{k_0+1} = n_{k_0} + 1; \\ 2(n_1 + n_2 - 2) & \text{otherwise.} \end{cases}$$

- $\delta_f(G)$: *even.* $\pi(G)$, $\pi^*(G)$: *even.*
- $2 \leq \pi(G) \leq 2|V(G)| - 4$.

- $\delta_f(G)$: *even.* $\pi(G)$, $\pi^*(G)$: *even.*
- $2 \leq \pi(G) \leq 2|V(G)| - 4.$

let $|V(G)| = n$ and f be a near automorphism

- $\delta_f(G)$: *even.* $\pi(G)$, $\pi^*(G)$: *even.*
- $2 \leq \pi(G) \leq 2|V(G)| - 4.$

let $|V(G)| = n$ and f be a near automorphism

- $\sum \delta_f(G) \leq 4n - 8$
- $\exists v \in V(G)$, $\delta_f(v) < 4$

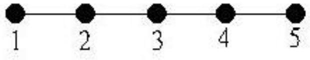
- If f is a permutation and h is an automorphism of a graph G, then $\delta_{h\circ f}(G) = \delta_f(G) = \delta_{f\circ h}(G)$.
- Let f be a permutation of $V(G)$ and $f(v) = v$. Then $\delta_f(v)$ is even.
- If G is a vertex-transitive graph and f is a permutation of $V(G)$, then $\delta_f(v)$ is even for each $v \in V(G)$.
- Let $v \in V(G)$, $f(v) = v$ and $\delta_f(v) \neq 0$. Then there exist at least two distinct vertices u and w such that $\delta_f(u, v)$ and $\delta_f(w, v)$ are positive.

Displacement Graph

Suppose G *is a graph and* f *is a permutation of* $V(G)$*. The displacement graph of* G *with respect to* f *is the directed multigraph* $G[f]$ *whose vertex set* $V(G[f])=\{a_1,a_2,\cdots,a_t\}$*, where* $t=\mathrm{diam}(G)$*, and arc set* $A(G[f])=\{\langle a_i,a_j\rangle : i\neq j$*, there is a pair of vertices* u *and* v *such that* $d(u,v)=i$ *and* $d(f(u),f(v))=j\}$*.*

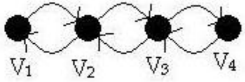
Displacement Graph

Suppose G *is a graph and* f *is a permutation of* $V(G)$. *The displacement graph of* G *with respect to* f *is the directed multigraph* $G[f]$ *whose vertex set* $V(G[f]) = \{a_1, a_2, \cdots, a_t\}$, *where* $t = \text{diam}(G)$, *and arc set* $A(G[f]) = \{\langle a_i, a_j\rangle : i \neq j$, *there is a pair of vertices* u *and* v *such that* $d(u, v) = i$ *and* $d(f(u), f(v)) = j\}$.



Displacement Graph

Suppose G *is a graph and* f *is a permutation of* $V(G)$*. The displacement graph of* G *with respect to* f *is the directed multigraph* $G[f]$ *whose vertex set* $V(G[f]) = \{a_1, a_2, \cdots, a_t\}$, *where* $t = \mathrm{diam}(G)$, *and arc set* $A(G[f]) = \{\langle a_i, a_j\rangle : i \neq j$, *there is a pair of vertices* u *and* v *such that* $d(u, v) = i$ *and* $d(f(u), f(v)) = j\}$.



Main Results

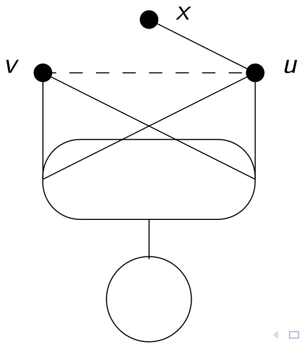
- G with $\pi(G)=2$ and trees T with $\pi(T)=4$.
- $\pi(C_n)=4\lfloor\frac{n}{2}\rfloor-4$ for $n\geq 4$.
- Trees T with $\pi(T)=2n-4$, $n\geq 3$.
- The better lower bound of $\pi^*(G)$ for P_n.

Theorem

Let G be a connected graph which is not complete. Then $\pi(G) = 2$ provided that there exist two vertices u and v satisfying (1) $N[u] \setminus N[v] = x$ for some $x \in V(G)$ and (2) $|d(u,x) - d(v,x)| = 1$.

Theorem

Let G be a connected graph which is not complete. Then $\pi(G) = 2$ provided that there exist two vertices u and v satisfying (1) $N[u] \setminus N[v] = x$ for some $x \in V(G)$ and (2) $|d(u,x) - d(v,x)| = 1$.



Theorem

Suppose G is a connected graph without 3-cycles and 5-cycles, and $|V(G)| \geq 3$. Then, $\pi(G) = 2$ if and only if $G \simeq P_3$.

Trees T *with* $\pi(T) = 4$

If T *is a tree of order at least* 4*. Then* $\pi(T) = 4$ *if and only if there exists a vertex* x *such that* $T - x$ *contains an isolated vertex and a component* K_2*, with a only exception that* $\pi(S_3) = 4$*.*

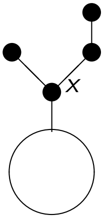
Trees T *with* $\pi(T) = 4$

If T *is a tree of order at least* 4*. Then* $\pi(T) = 4$ *if and only if there exists a vertex* x *such that* $T - x$ *contains an isolated vertex and a component* K_2*, with a only exception that* $\pi(S_3) = 4$*.*

1. $A(T[f]) = \{\langle a_1, a_2\rangle, \langle a_2, a_1\rangle, \langle a_i, a_{i+1}\rangle, \langle a_{i+1}, a_i\rangle : i \geq 1\}$.
2. $A(T[f]) = \{\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \langle a_3, a_1\rangle\}$.
3. $A(T[f]) = \{\langle a_1, a_3\rangle, \langle a_2, a_1\rangle, \langle a_3, a_2\rangle\}$.
4. $A(T[f]) = \{\langle a_1, a_3\rangle, \langle a_3, a_1\rangle\}$.

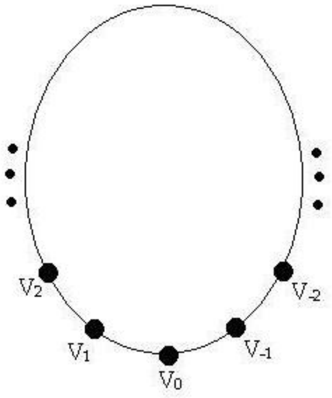
Trees T *with* $\pi(T) = 4$

If T *is a tree of order at least* 4. *Then* $\pi(T) = 4$ *if and only if there exists a vertex* x *such that* $T - x$ *contains an isolated vertex and a component* K_2, *with a only exception that* $\pi(S_3) = 4$.



$$\pi(C_n) = 4\lfloor \frac{n}{2} \rfloor - 4$$

$\pi(C_n) = 4\lfloor \frac{n}{2} \rfloor - 4$



$\pi(C_n) = 4\lfloor \frac{n}{2} \rfloor - 4$

1. C_n : a vertex-transitive graph.
2. f : a non-automorphism and $f(v_0) = v_0$ and $\delta_f(v_0) = \min\{\delta_f(v) : v \in V(C_n)\}$ =0 or 2.
3. $A = \{k : f(v_k) = v_k, k = 1, 2, \cdots, \lceil \frac{n}{2} \rceil - 1\}$
   $B = \{h : f(v_h) = v_{-h}, h = 1, 2, \cdots, \lceil \frac{n}{2} \rceil - 1\}$.

$\pi(P_n) \geq \pi(C_n)$, the equality holds only when n is even.

1. $|d_{P_n}(x, y) - d_{P_n}(f(x), f(y))| \geq |d_{C_n}(x, y) - d_{C_n}(f(x), f(y))|$.
2. f is an automorphism of C_n but not an automorphism of P_n, $\delta_f(P_n) \geq 2n - 4$.
   Clearly, $g(v_i) = v_{n-i+1}$ and $h(v_i) = v_{i+j} \pmod{n}$ are mirror reflection and rotation of C_n here, they can create all the automorphisms of C_n. Obviously, if $\{f(v_1), f(v_n)\} = \{v_1, v_n\}$ for some automorphism f of C_n, then f is also an automorphism of P_n. Otherwise, if $\{f(v_1), f(v_n)\} = \{v_j, v_{j+1}\}$ for $1 \leq j < n$, then $\{f(v_j), f(v_{j+1})\} = \{v_1, v_n\}$ or $\{f(v_{n-j}), f(v_{n-j+1})\} = \{v_1, v_n\}$, and $\delta_f(P_n) \geq \delta_f(v_1, v_n) + \delta_f(v_j, v_{j+1}) + \delta_f(v_{n-j}, v_{n-j+1}) = (n-2) + (n-2) = 2n - 4$.

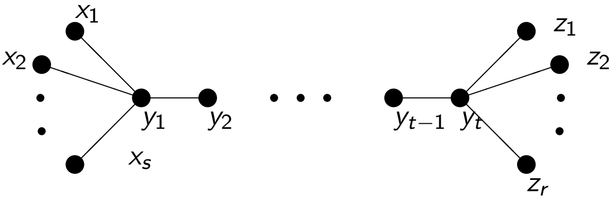
Necessary Conditions

Let T *be a tree of order* n *such that* $\pi(T) = 2n - 4$*. Then, the following conditions hold:*

1. If there exists a vertex x with $deg_T(x) \geq 3$ and $T - x$ has an isolated vertex, then $T - x$ has at most one non-trivial component.
2. For each $y \in V(T)$, if $T - y$ contains only non-trivial components, then $deg_T(y) \leq 3$.

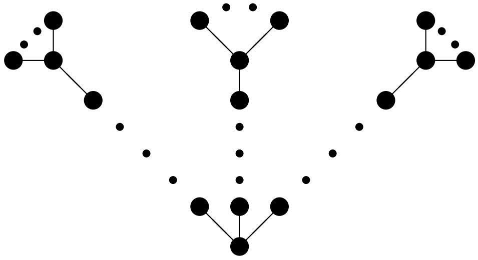
Double Broom $(s \geq r)$.

$T^{(2)}$:



Triple Broom

$T^{(3)}$:



Corollary

Let T be a caterpillar of order n. Then $\pi(T) = 2n - 4$ if and only if $T \in T^{(2)}$, $s \geq r$ and $r + t \geq \frac{n}{2} + 1$.

Known Results

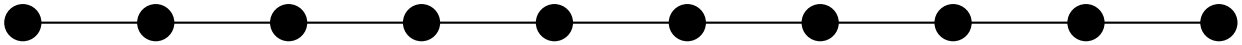
- The problem of finding $\pi^*(K_{n_1,n_2,\cdots,n_t})$ was transformed into a quadratic integer programming problem (QIP), and a characterization of the optimal solution was given.
- Let $A = (a_{ij})$ be a $t \times t$ non-negative matrix. $A = (a_{ij})$ is an optimal solution of Problem (QIP) ,i.e. $\sum a_{ij}^2$ is minimum, if and only if no overweight cycle exists in A.
- Let $X = (x_1, x_2, \cdots, x_k)$ be a non-decreasing sequence. If $Y = (x_k, x_{k-1}, \cdots, x_1)$, then $\eta^*(X) = \zeta_Y(X)$.

- Let $t = \lfloor \frac{\sqrt{2n(n-1)+1}-1}{2} \rfloor$. Then $\pi^*(P_n) \leq \frac{2}{3}t(t+1)(t+2) = \frac{1}{3}n(n-1)(2n-4-3t)$.
- $\pi^*(P_n) \leq \frac{4}{3}(n+1)^3$.
- Suppose that n is even. Then $\pi^*(P_n) \geq \frac{n^3}{12} - \frac{n}{3}$.
- Let C_n be an odd cycle. Then $\pi^*(C_n) \geq \frac{n^3}{12} - \frac{n}{3}$ for $n \equiv 1$ or $5 \pmod 6$; and $\pi^*(C_n) \geq \frac{n^3}{12} - \frac{3n}{4}$ for $n \equiv 3 \pmod 6$.
- $\pi^*(C_n) \leq \begin{cases} \frac{1}{8}n^3 - \frac{1}{4}n^2 + \frac{1}{8}n & \text{, if } n \equiv 1 \pmod 4; \\ \frac{1}{8}n^3 - \frac{1}{4}n^2 - \frac{3}{8}n & \text{, if } n \equiv 3 \pmod 4; \\ \frac{1}{8}n^3 - \frac{1}{4}n^2 & \text{, if } n \text{ is even.} \end{cases}$

$n = 9$

v	1	2	3	4	5	6	7	8	9
$f(v)$	9	7	5	3	1	8	6	4	2

$n = 10$	v	1	2	3	4	5	6	7	8	9	10



$f(v)$	9	7	5	3	1	10	8	6	4	2

$\delta_f(1,2) = 1$	$\delta_f(2,3) = 1$	$\delta_f(3,4) = 1$	$\delta_f(4,5) = 1$	$\delta_f(5,6) = 6$
$\delta_f(1,3) = 2$	$\delta_f(2,4) = 2$	$\delta_f(3,5) = 2$	$\delta_f(4,6) = 3$	$\delta_f(5,7) = 3$
$\delta_f(1,4) = 3$	$\delta_f(2,5) = 3$	$\delta_f(3,6) = 0$	$\delta_f(4,7) = 0$	$\delta_f(5,8) = 0$
$\delta_f(1,5) = 4$	$\delta_f(2,6) = 3$	$\delta_f(3,7) = 3$	$\delta_f(4,8) = 3$	$\delta_f(5,9) = 3$

$\delta_f(1,6) = 4$	$\delta_f(2,7) = 4$	$\delta_f(3,8) = 4$	$\delta_f(4,9) = 4$
$\delta_f(1,7) = 3$	$\delta_f(2,8) = 3$	$\delta_f(3,9) = 3$	
$\delta_f(1,8) = 2$	$\delta_f(2,9) = 2$		
$\delta_f(1,9) = 1$			

$\delta_f(6,7) = 1$	$\delta_f(7,8) = 1$	$\delta_f(8,9) = 1$
$\delta_f(6,8) = 2$	$\delta_f(7,9) = 2$	
$\delta_f(6,9) = 3$		

$\delta_f(1,2) = 1$	$\delta_f(2,3) = 1$	$\delta_f(3,4) = 1$	$\delta_f(4,5) = 1$	$\delta_f(5,6) = 8$
$\delta_f(1,3) = 2$	$\delta_f(2,4) = 2$	$\delta_f(3,5) = 2$	$\delta_f(4,6) = 5$	$\delta_f(5,7) = 5$
$\delta_f(1,4) = 3$	$\delta_f(2,5) = 3$	$\delta_f(3,6) = 2$	$\delta_f(4,7) = 2$	$\delta_f(5,8) = 2$
$\delta_f(1,5) = 4$	$\delta_f(2,6) = 1$	$\delta_f(3,7) = 1$	$\delta_f(4,8) = 1$	$\delta_f(5,9) = 3$
$\delta_f(1,6) = 4$	$\delta_f(2,7) = 4$	$\delta_f(3,8) = 4$	$\delta_f(4,9) = 4$	$\delta_f(5,10) = 4$
$\delta_f(1,7) = 5$	$\delta_f(2,8) = 5$	$\delta_f(3,9) = 5$	$\delta_f(4,10) = 5$	
$\delta_f(1,8) = 4$	$\delta_f(2,9) = 4$	$\delta_f(3,10) = 4$		
$\delta_f(1,9) = 3$	$\delta_f(2,10) = 3$			
$\delta_f(1,10) = 2$				
$\delta_f(6,7) = 1$	$\delta_f(7,8) = 1$	$\delta_f(8,9) = 1$	$\delta_f(9,10) = 1$	
$\delta_f(6,8) = 2$	$\delta_f(7,9) = 2$	$\delta_f(8,10) = 2$		
$\delta_f(6,9) = 3$	$\delta_f(7,10) = 3$			
$\delta_f(6,10) = 4$				

- For n odd, let $f(i)=\begin{cases} n+2-2i & \text{, if } 1\le i\le \frac{n+1}{2};\\ 2(n+1-i) & \text{, if } \frac{n+3}{2}\le i\le n.\end{cases}$
- For n even,let $f(i)=\begin{cases} n+1-2i & \text{, if } 1\le i\le \frac{n}{2};\\ 2(n+1-i) & \text{, if } \frac{n}{2}+1\le i\le n,\end{cases}$

$$
\begin{cases}
\frac{1}{27}n^3 + \frac{1}{9}n^2, & n \equiv 0 \pmod 6; \\
\frac{1}{27}n^3 - \frac{11}{18}n - \frac{23}{54}, & n \equiv 1 \pmod{12}; \\
\frac{1}{27}n^3 - \frac{1}{9}n^2 - \frac{2}{9}n - \frac{8}{27}, & n \equiv 2 \pmod 6; \\
\frac{1}{27}n^3 + \frac{1}{6}n + \frac{1}{2}, & n \equiv 3 \pmod{12}; \\
\frac{1}{27}n^3 + \frac{1}{9}n^2 - \frac{4}{27}, & n \equiv 4 \pmod 6; \\
\frac{1}{27}n^3 - \frac{11}{18}n - \frac{31}{54}, & n \equiv 5 \pmod{12}; \\
\frac{1}{27}n^3 + \frac{7}{18}n + \frac{31}{54}, & n \equiv 7 \pmod{12}; \\
\frac{1}{27}n^3 - \frac{5}{6}n - \frac{1}{2}, & n \equiv 9 \pmod{12}; \\
\frac{1}{27}n^3 + \frac{7}{18}n - \frac{23}{54}, & n \equiv 11 \pmod{12};
\end{cases}
$$

[Hwang,1996]

- $Z = < z_1, z_2, \cdots, z_n >$
  the oscillation Z of P_n :

$$O_{sc}(Z) = \sum_{i=1}^{n-1} (|z_i - z_{i+1}| - 1).$$

Then, finding $\max_{Z \in S_n} O_{sc}(Z)$ reveals the maximum disorderness of the input data arranged on a path.

- $\pi(G) = 2$.
- $T \in T^{(3)}$, $\pi(T) = 2n - 4$.
- $\pi(Q_k) = 2 \cdot 2^k - 8 = 2^{k+1} - 8$.

The End

The End

Thank You