Sequence

Monotonic Sequence Theorem
Every bounded monotonic sequence is convergent.

Series

Test of Divergence
If \( \lim_{n \to \infty} a_n \neq 0 \) or the limit does not exist, then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

Integral Test
If \( f \) is continuous, positive, decreasing on \([1, \infty)\) and \( a_n = f(n) \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.

Comparison Test
Assume that \( \sum a_n \) and \( \sum b_n \) are series with positive terms, then
1. If \( \sum b_n \) converges and \( a_n \leq b_n \) for all \( n \), then \( \sum a_n \) converges.
2. If \( \sum b_n \) diverges and \( a_n \geq b_n \) for all \( n \), then \( \sum a_n \) diverges.

Limit Comparison Test
Assume that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) and \( c > 0 \), then either both series converges or diverges.

Alternating Series Test
If the alternating series \( \sum_{n=1}^{\infty} (-1)^{n+1} b_n \) satisfies
1. \( b_n \geq b_{n+1} \) for all \( n \).
2. \( \lim_{n \to \infty} b_n = 0 \)
then the series is convergent.

Ratio Test
For a given series \( \sum a_n \),
1. If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = L < 1 \), then \( \sum a_n \) is absolutely convergent.
2. If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = L > 1 \) or the limit does not exist, then \( \sum a_n \) is absolutely convergent.
3. If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = 1 \), then we cannot get any conclusion directly.

Root Test
For a given series \( \sum a_n \),
1. If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1 \), then \( \sum a_n \) is absolutely convergent.
2. If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 0 \) or the limit does not exist, then \( \sum a_n \) is absolutely convergent.

3. If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \), then we cannot get any conclusion directly.

Approximation of the Series

Since the partial sum \( S_n = \sum_{k=1}^{n} a_k \) and \( \lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n \), it is natural to use the partial sum as an approximation of the series. Let \( R_n = \sum_{n=1}^{\infty} a_n - S_n \), then \( R_n \) is the error when we use \( S_n \) to approximate the series. There are two tools introduced so far to estimate the error:

1. If \( a_n = f(n) \) with \( f \) being continuous, positive and decreasing for \( x \geq n \) and \( \sum a_n \) is convergent, then
   \[
   \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx.
   \]

2. If \( S = \sum_{n=1}^{\infty} (-1)^{n+1} b_n \) and the series satisfies
   (a) \( 0 \leq b_{n+1} \leq b_n \)
   (b) \( \lim_{n \to \infty} b_n = 0 \)
   then \( |R_n| = |S - S_n| \leq b_{n+1} \)

Common Series

Geometric Series

The geometric series
\[
\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots
\]
is convergent if and only if \( |r| < 1 \). The sum is
\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, |r| < 1
\]

\( p \)-Series

The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).