Spectra of graphs
Connectedness

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Proposition 1,4,1
Let $G$ be a graph with $n$ connected components $G_1, \ldots, G_n$. Then the spectrum of $G$ will be the union of spectral of $G_i$, $i = 1 \ldots n$. The same statement will hold for Laplace spectrum and signless Laplace spectrum.

Sketch of Proof
Consider the form of the matrices and the corresponding eigenvectors.
Proposition 1.4.2
Let $G$ be an undirected graph and $L$ be the Laplacian of $G$. The multiplicity of zero of $L$ will equal to the number of components of $G$.

Proof
We show that if $G$ is connected, the multiplicity of zero will be 1 and then the theorem follows from Proposition 1.4.1. Let $N$ be the incident matrix of random orientation of $G$, then $L = NN^T$.

Let $u$ be the corresponding eigenvector of 0, then
\[ 0 = u^T Lu = u^T NN^T u = \|N^T u\|^2 \]
Hence $N^T u = 0$ and it implies that if $v_i \sim v_j$ in $G$, then $u_i = u_j$ (Notice that $G$ is oriented.). Since $G$ is connected, $u = (1, \ldots, 1)$. 
Proposition 1.4.3
Let $G$ be an undirected $k$-regular graph. Then $k$ will be the largest eigenvalue of $G$ and its multiplicity will equal to the number of connected components.

Proof
Since $G$ is $k$-regular, $L = kI - A$ and then we can compute the eigenvalues by direct subtraction. Since $A$ is semi-positive definite, by Proposition 1.4.2, done.
Remark

Notice that Proposition 1.4.2 and Proposition 1.4.3 only hold for Laplace eigenvalues. We cannot tell whether a graph is connected by the spectrum of its adjacency matrix. For example, the spectra of both $K_{1,4}$ and $K_1 + C_4$ are $2^1, 0^3, (-2)^1$. 
Proposition 1.4.4

Let $G$ be an undirected graph and $|L|$ be the signless Laplacian of $G$. Then the multiplicity of zero of $|L|$ will equal to the number of bipartite connected components of $G$.

Proof

Let $M$ be the undirected incident matrix, then $|L| = MM^T$. Let $u$ be the eigenvector corresponding to 0, then $0 = |L|u = MM^Tu$.

Hence $M^Tu = 0$

It means that if $v_i \sim v_j$ in $G$, then $u_i = -u_j$. Therefore, the support of $u$ is the union of bipartite component of $G$. 
Proposition 1.4.5

G is a bipartite graph if and only if its Laplace spectrum and signless spectrum are the same.

Proof

- Necessity
  Since G is bipartite, its Laplacian and signless Laplacian are similar by a diagonal matrix D with diagonal entries being $\pm 1$. i.e $|L| = DLD^{-1}$. Therefore, they have the same spectrum.

- Sufficiency
  By Proposition 1.4.2 and Proposition 1.4.4, the number of components equals to the number of bipartite components.