Approximate counting via the Poisson-Laplace-Mellin Method
(joint work with M. Fuchs and H. Prodinger)

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Number of bits needed for counting $n$ objects: $\Theta(\log n)$

**Problem**: What if the space is limited or we want to reduce the number of times of writing the hard disk?
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**R. Morris:** Approximate counting algorithm.

Counter $C_n$ with $C_0 = 0$ and $0 < q < 1$:

$$C_{n+1} = \begin{cases} C_n + 1, & \text{with probability } q^{C_n}; \\ C_n, & \text{with probability } 1 - q^{C_n}. \end{cases}$$  \hspace{1cm} (1)
Origin of Approximate Counting

Number of bits needed for counting $n$ objects: $\Theta(\log n)$

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(1)

It’s rather easy to show that

$$\mathbb{E}q^{-C_n} = n(q^{-1} - 1) + 1.$$

The number of bits needed: $\Theta(\log \log n)$
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Some of the related streaming algorithms:
- Frequency Moment
- Heavy Hitters
- Counting distinct Elements
- Entropy
Applications of approximate counting in different areas:

- Analyzing the Webgraph
- Monitoring network traffic
- Finding patterns in protein
- Data compression
- Artificial intelligent
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- Other variations
• Flajolet (1985), Mellin Transform. The first full analysis.
Analysis of the algorithm

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- Louchard and Prodinger (2006), Analysis of Extreme Value Distribution
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\mathbb{E}(C_n) \sim \log_{1/q} n + F_C(\log_{1/q} n),
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The same analysis of Flajolet can be extended to general \(q\):

\[
\mathbb{E}(C_n) \sim \log_{1/q} n + F_C(\log_{1/q} n),
\]

where \(F_C(z) = \sum_k f_k e^{2k\pi iz}\) is a 1-periodic function with Fourier coefficients

\[
f_0 = \frac{\gamma}{L} + \frac{1}{2} - \alpha, \quad f_k = -\frac{\Gamma(-\chi_k)}{L} \quad (k \neq 0).
\]

Here, \(\gamma\) is Euler’s constant, \(\alpha = \sum_{l \geq 1} q^l/(1 - q^l)\), \(L = \log(1/q)\) and \(\chi_k = 2k\pi i/L\).
As for the variance,

$$\text{Var}(C_n) \sim G_C(\log_{1/q} n),$$
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where $G_C(z) = \sum_k g_k e^{2k\pi i z}$ is again a 1-periodic function with computable Fourier coefficients.
As for the variance,

\[ \text{Var}(C_n) \sim G_C(\log_{1/q} n), \]

where \( G_C(z) = \sum_k g_k e^{2k\pi iz} \) is again a 1-periodic function with computable Fourier coefficients. We also have

\[ g_0 = \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{1}{L} \sum_{l \geq 1} \frac{1}{l \sinh(2l\pi^2/L)}, \]

where \( \beta = \sum_{l \geq 1} q^{2l} / (1 - q^l)^2. \)
First proposed by Coffman and Eve in 1970.

**Example:** A DST built by 9 keys

![Digital Search Tree](image)
Random Model: Bernoulli model.
Bits of keys: i.i.d. Bernoulli random variables
Probability: \( P(I_n = 0) = q^n \)
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Shape Parameters:
- Depth
- Total Path Length
- Peripheral Path Length
- Colless Index
- Number of Occurrence of Patterns
Random Model and Shape Parameters

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Shape Parameters:
- Depth
- Total Path Length
- Peripheral Path Length
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- Number of Occurrence of Patterns
- **Leftmost Path Length**
- Others (Wiener Index, etc.)
$X_n$: Length of the leftmost path in DST built by $n$ strings.
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1. $X_n \overset{d}{=} C_n$
2. $X_{n+1} \overset{d}{=} X_{B_n} + 1$
   - $B_n \overset{d}{=} \text{Binomial}(n, q)$
   - $X_n, B_n$ are i.i.d.
   - Initial condition: $X_0 = 0$
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**Recurrence of the Exponential Generating Functions:**

**First moment:**

$$f'_1(z) = e^{pz} f_1(qz) + e^z$$

**Second Moment:**

$$f'_2(z) = e^{pz} f_2(qz) + 2e^{pz} f_1(qz) + e^z$$
Idea: Combination of **Poissonization**, **Laplace Transform** and **Mellin Transform**.
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Poissonization:
Due to Jacquet-Szpankowski’s theory of depoissonization only the Poisson model has to be analyzed.

\[
\tilde{f}(z) = e^{-z} \sum_{k \geq 0} f_k z^k k!
\]

If \( \tilde{f} \) is an entire function, then
\[
f_n = \sum_{j \geq 0} \tilde{f}(j)(n)_j \tau_j(n) \sim n^n \tilde{f}(n) - n^2 \tilde{f}''(N) + \cdots
\]

where \( \tau_j(n) := n! [z^n] (z - n) e^z \)
Poisson-Laplace-Mellin Method

Idea: Combination of Poissonization, Laplace Transform and Mellin Transform.

Poissonization:
Due to Jacquet-Szpankowski’s theory of depoissonization only the Poisson model has to be analyzed.

Lemma
Let $\tilde{f}(z) = e^{-z} \sum_{k \geq 0} f_k \frac{z^k}{k!}$. If $\tilde{f}$ is an entire function, then

$$f_n = \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n) \sim \tilde{f}(n) - \frac{n}{2} \tilde{f}''(N) + \cdots$$

where

$$\tau_j(n) := n! [z^n] (z - n)^j e^z$$
Jacquet-Szpankowski-asmissibility

\( \tilde{f}(z) \) is JS-Admissible if

1. Uniformly for \( |\arg(z)| \leq \epsilon \),
   \[
   \tilde{f}(z) = O \left( |z|^\alpha \log^\beta |z| \right),
   \]

2. Uniformly for \( \epsilon < |\arg(z)| \leq \pi \),
   \[
   f(z) = e^z \tilde{f}(z) = O \left( e^{(1-\epsilon)|z|} \right).
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Jacquet-Szpankowski-asmissibility

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f(z) = e^z \tilde{f}(z) = O \left( e^{(1-\epsilon)|z|} \right).
\]

Theorem (Jacquet and Szpankowski)

If \( \tilde{f}(z) \) is JS-Admissible, then

\[
f_n \sim \tilde{f}(n) - \frac{n}{2} \tilde{f}''(N) + \cdots
\]
Let $m$ be a nonnegative integer and $\alpha \in (0, 1)$, then

(i) If $\tilde{f}$ is JS-admissible and $\tilde{P}$ is a polynomial, then $\tilde{P}\tilde{f}$ is JS-admissible.

(ii) If $\tilde{f}$ and $\tilde{g}$ are JS-admissible, then $\tilde{f} + \tilde{g}$ and $\tilde{f}(\alpha z)\tilde{g}((1 - \alpha)z)$ are also JS-admissible.

(iii) If $\tilde{f}$ is JS-admissible, then $\tilde{f}'$ is JS-admissible, and thus so are $\tilde{f}^{(m)}$. 
Let \( m \) be a nonnegative integer and \( \alpha \in (0, 1) \), then

(i) If \( \tilde{f} \) is JS-admissible and \( \tilde{P} \) is a polynomial, then \( \tilde{P}\tilde{f} \) is JS-admissible.

(ii) If \( \tilde{f} \) and \( \tilde{g} \) are JS-admissible, then \( \tilde{f} + \tilde{g} \) and \( \tilde{f}(\alpha z)\tilde{g}((1 - \alpha)z) \) are also JS-admissible.

(iii) If \( \tilde{f} \) is JS-admissible, then \( \tilde{f}' \) is JS-admissible, and thus so are \( \tilde{f}^{(m)} \).

**Proposition**

If \( \tilde{f}(z) \) and \( \tilde{g}(z) \) are entire functions satisfying

\[
\tilde{f}(z) + \tilde{f}'(z) = \tilde{f}(qz) + \tilde{g}(z),
\]

then

\( \tilde{g}(z) \) is JS-admissible \( \iff \tilde{f}(z) \) is JS-admissible
Apply poissonization to the exponential generating functions, we get

\[
\begin{align*}
\tilde{f}_1(z) + \tilde{f}_1'(z) &= \tilde{f}_1(qz) + 1 \\
\tilde{f}_2(z) + \tilde{f}_2'(z) &= \tilde{f}_2(qz) + 2\tilde{f}_1(qz) + 1
\end{align*}
\]

⇒ \tilde{f}_1 and \tilde{f}_2 are JS-admissible.

Question: How to compute the variance? In other words, how to handle

\[
V(X_n) = E(X_n^2) - (E(X_n))^2 = \sum_{j \geq 0} \tilde{f}(j^n)j!\tau_j^n - \left(\sum_{j \geq 0} \tilde{f}(j^n)1^n j!\tau_j^n\right)^2
\]
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\[\Rightarrow \tilde{f}_1 \text{ and } \tilde{f}_2 \text{ are JS-admissible.}\]
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\[\implies \text{Apply de-poissionization.}\]
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\[\Rightarrow \text{Apply de-poissonization.} \]

**Question:** How to compute the variance? In other words, how to handle

\[
\mathbb{V}(X_n) = \mathbb{E}(X_n^2) - (\mathbb{E}(X_n))^2
\]

\[
= \sum_{j \geq 0} \frac{\tilde{f}_2^{(j)}(n)}{j!} \tau_j(n) - \left( \sum_{j \geq 0} \frac{\tilde{f}_1^{(j)}(n)}{j!} \tau_j(n) \right)^2
\]
Let $\tilde{D}(z) = \tilde{f}_2(z) - \tilde{f}_1(z)^2$, then we get

$$\nabla(X_n) = \mathbb{E}(X_n^2) - (\mathbb{E}(X_n))^2$$

$$= \sum_{j \geq 0} \frac{\tilde{f}_2^{(j)}(n)}{j!} \tau_j(n) - \left( \sum_{j \geq 0} \frac{\tilde{f}_1^{(j)}(n)}{j!} \tau_j(n) \right)^2$$

$$= \tilde{D}(n) - n\tilde{f}_1(n)^2 - \frac{n}{2} \tilde{D}''(n) + \text{higher derivative terms}$$
Let $\tilde{D}(z) = \tilde{f}_2(z) - \tilde{f}_1(z)^2$, then we get

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$$= \sum_{j \geq 0} \frac{\tilde{f}_2^j(n)}{j!} \tau_j(n) - \left( \sum_{j \geq 0} \frac{\tilde{f}_1^j(n)}{j!} \tau_j(n) \right)^2$$

$$= \tilde{D}(n) - n\tilde{f}_1'(n)^2 - \frac{n}{2} \tilde{D}''(n) + \text{higher derivative terms}$$

### Poissonized variance with correction

After some computation, we will reach the point

$$\tilde{V}(z) = \tilde{f}_2(z) - \sum_{n \geq 0} \tilde{f}_1^n(z) \frac{z^n}{n!}.$$
Our Approach vs. Flajolet-Richmond

EGF $f(z)$ → Laplace transform of $e^{-z}f(z)$ = Euler transform of $A(z)$ → OGF $A(z)$

Asymptotics of $\tilde{f}(z)$ as $|z| \to \infty$

Asymptotics of $\frac{\text{Laplace}}{Q(-s)}$

Laplace transform

Mellin transform

Euler transform

Asymptotics of $\frac{\text{Euler}}{Q(-s)}$

Asymptotics of $A(z)$ as $z \sim 1$

de-Poi by saddle-point

Singularity analysis
After inverse Mellin transform and inverse Laplace transform:

\[
\tilde{f}_1(z) \sim \log_{1/q} z + \frac{\gamma}{L} + \frac{1}{2} - \alpha - \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) z^{\chi_k}
\]
Mean of $X_n$

After inverse Mellin transform and inverse Laplace transform:

$$\tilde{f}_1(z) \sim \log_{1/q} z + \frac{\gamma}{L} + \frac{1}{2} - \alpha - \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) z^{\chi_k}$$

Exactly the same with Flajolet’s result.
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\tilde{f}_1(z) \sim \log_{1/q} z + \frac{\gamma}{L} + \frac{1}{2} - \alpha - \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) z^{\chi_k}
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Exactly the same with Flajolet’s result.

Apply the idea of poissonized variance:

\[
\tilde{V}(z) + \tilde{V}'(z) = \tilde{V}(qz) + \tilde{f}_1'(z)^2.
\]
After inverse Mellin transform and inverse Laplace transform:

\[ \tilde{f}_1(z) \sim \log_1 q z + \frac{\gamma}{L} + \frac{1}{2} - \alpha - \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) z^{\chi_k} \]

Exactly the same with Flajolet’s result.

Apply the idea of poissonized variance:

\[ \tilde{V}(z) + \tilde{V}'(z) = \tilde{V}(qz) + \tilde{f}_1'(z)^2. \]

⇒ We can apply the same method.
Main Result(I)

Theorem (Fuchs, Lee, Prodinger)

For the variance of approximate counting, we have, as $n \to \infty$,

$$\text{Var}(C_n) \sim G_C(\log_{1/q} n),$$

where

$$G_C(z) = \sum_k g_k e^{2k\pi iz}$$

is a 1-periodic function with

$$g_k = \frac{Q_j}{\mathcal{L}(1 + \chi_k)} \sum_{h,l,j \geq 0} \frac{(-1)^j q^{h+l+(j/2)+(1-\chi_k)j}}{Q_h Q_l Q_j} \varphi(\chi_k, q^{h+j} + q^{l+j}).$$

Here, $Q_j = \prod_{l=1}^j (1 - q^l)$, $Q_\infty = \lim_{j \to \infty} Q_j$ and

$$\varphi(\chi; x) := \begin{cases} 
\pi(x^{\chi} - 1)/(\sin(\pi\chi)(x - 1)), & \text{if } x \neq 1, \\
\pi\chi/\sin(\pi\chi), & \text{if } x = 1.
\end{cases}$$
Corollary (Fuchs, Lee, Prodinger)

We have,

\[
\frac{Q_\infty}{L} \sum_{h,l,j \geq 0} \frac{(-1)^j q^{h+l+(j+1)/2}}{Q_h Q_l Q_j} \psi(q^{h+j} + q^{l+j})
\]

\[= \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{1}{L} \sum_{l \geq 1} \frac{1}{l \sinh(2l\pi^2/L)},\]

where

\[
\psi(x) := \begin{cases} 
\log x/(x - 1), & \text{if } x \neq 1; \\
1, & \text{if } x = 1.
\end{cases}
\]

We have a direct proof using q-analysis and help from computer.
Total Path Length ($T_n$): total path length in symmetric digital search tree.

**Theorem (F., Hwang, Zacharovas)**

We have

$$\text{Var}(T_n) \sim n(C_{\text{var}} + G(\log_2 n)),$$

where $G(z)$ is a 1-periodic function with zero average value and

$$C_{\text{var}} = \frac{Q_\infty}{L} \sum_{j,h,l \geq 0} (-1)^j 2^{-\left(\frac{j+1}{2}\right)} \frac{Q_j Q_h Q_l 2^{h+l}}{Q_j Q_h Q_l 2^{h+l}} \delta(2^{-j-h} + 2^{-j-l}),$$

where

$$\delta(x) := \begin{cases} 
(x - \log x - 1)/(x - 1)^2, & \text{if } x \neq 1; \\
1/2, & \text{if } x = 1.
\end{cases}$$
Variance of total path length was also derived by Kirschenhofer,Prodinger and Szpankowski with different expression for $C_{\text{var}}$. 

Let $[FG]_0$ denote the 0-th Fourier coefficient of the product of the two Fourier series $F(z)$ and $G(z)$. 

Put $F(z) = \frac{1}{L} \sum_{l \neq 0} \frac{\Gamma(-1-\chi l)}{\Gamma(-\chi l)} e^{2l\pi iz}$ and $H(z) = -\frac{1}{L} \sum_{l \neq 0} \left(1 - \frac{\chi l^2}{2}\right) \frac{\Gamma(-\chi l)}{\Gamma(-\chi l)} e^{2l\pi iz}$. 

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Variance of total path length was also derived by Kirschenhofer, Prodinger and Szpankowski with different expression for $C_{\text{var}}$.

To describe their expression we need:

- Let $[FG]_0$ denote the 0-th Fourier coefficient of the product of the two Fourier series $F(z)$ and $G(z)$.

- Put

\[ F(z) = \frac{1}{L} \sum_{l \neq 0} \Gamma(-1 - \chi l) e^{2l\pi iz} \]

and

\[ H(z) = -\frac{1}{L} \sum_{l \neq 0} \left(1 - \frac{\chi l}{2}\right) \Gamma(-\chi l) e^{2l\pi iz}. \]
\[
C_{\text{var}} = -\frac{28}{3L} - \frac{39}{4} - 2 \sum_{l \geq 1} \frac{l^2}{(2^l - 1)^2} + \frac{2}{L} \sum_{l \geq 1} \frac{1}{2^l - 1} + \frac{\pi^2}{2L^2} + \frac{2}{L^2} \\
- \frac{2}{L} \sum_{l \geq 3} \frac{(-1)^{l+1}(l - 5)}{(l + 1)l(l - 1)(2^l - 1)} \\
+ \frac{2}{L} \sum_{l \geq 1} (-1)^l 2^{-(l+1)\choose 2} \left( \frac{L(1 - 2^{-l+1})/2 - 1}{1 - 2^{-l}} - \sum_{r \geq 2} \frac{(-1)^{r+1}}{r(r - 1)(2^{r+l} - 1)} \right) \\
- \frac{2Q(1)}{L} + \sum_{l \geq 2} \frac{1}{2^l Q_l} \sum_{r \geq 0} \frac{(-1)^r 2^{-(r+1)\choose 2}}{Q_r} Q_{r+l-2} \cdot \\
\left( - \sum_{j \geq 1} \frac{1}{2^{j+r+l+2} - 1} \left( 2^{l+1} - 2l - 4 + 2 \sum_{i=2}^{l-1} \binom{l+1}{i} \frac{1}{2^{r+i-1} - 1} \right) \\
+ \frac{2}{(1 - 2^{-l-r})^2} + \frac{2l + 2}{(1 - 2^{1-l-r})^2} - \frac{2}{L} \frac{1}{1 - 2^{1-l-r}} + \frac{2}{L} \sum_{j=1}^{l+1} \binom{l+1}{j} \frac{1}{2^{r+j} - 1} \\
- 2 \sum_{j=2}^{l+1} \binom{l+1}{j} \frac{1}{2^{r+j-1} - 1} + 2 \sum_{j=0}^{l+1} \binom{l+1}{j} \sum_{i \geq 1} \frac{(-1)^i}{(i + 1)(2^{r+j+i} - 1)} \right) \\
+ \sum_{l \geq 3} \sum_{r=2}^{l-1} \binom{l+1}{r} \frac{Q_{r-2}Q_{l-r-1}}{2^l Q_l} \sum_{j \geq l+1} \frac{1}{2^j - 1} - 2[FH]_0 - [F^2]_0.
\]
Cichoń and Macyna:
Consider $m$ counters. When counting $n$ objects, for each object we choose one of the counter at random with probability $\frac{1}{m}$ and the counter work as before.

$$D_n = C(1)I_1 + \cdots + C(m)I_m,$$

where $C(n), \ldots, C(m)$ are independent copies of $C_n$ and $P(I_1 = n_1, \ldots, I_m = n_m) = \left(\frac{n}{n_1}, \ldots, \frac{n}{n_m}\right)^m$ with $n_1 + \cdots + n_m = n$. 
Cichoń and Macyna:
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$D_n$: The sum of the counters after counting n objects.
Cichoń and Macyna:
Consider $m$ counters. When counting $n$ objects, for each object we choose one of the counter at random with probability $\frac{1}{m}$ and the counter work as before.

$D_n$: The sum of the counters after counting $n$ objects.

Then,

$$D_n \overset{d}{=} C_{I_1}^{(1)} + \cdots + C_{I_m}^{(m)},$$

where $C_{n}^{(1)}, \ldots, C_{n}^{(m)}$ are independent copies of $C_n$. 
Cichoń and Macyna:
Consider $m$ counters. When counting $n$ objects, for each object we choose one of the counter at random with probability $\frac{1}{m}$ and the counter work as before.

$D_n$: The sum of the counters after counting $n$ objects.

Then,

$$D_n \overset{d}{=} C_{I_1}^{(1)} + \cdots + C_{I_m}^{(m)},$$

where $C_{n}^{(1)}, \ldots, C_{n}^{(m)}$ are independent copies of $C_{n}$ and

$$P(I_1 = n_1, \ldots, I_m = n_m) = \binom{n}{n_1, \ldots, n_m} \frac{1}{m^n}$$

with $n_1 + \cdots + n_m = n$. 
Approximate Counting with \( m \) Counters(II)

\( \tilde{f}_D, \tilde{V}_D \): Poissonized mean and variance of \( D_n \).
\( \tilde{f}_C, \tilde{V}_C \): Poissonized mean and variance of \( C_n \).
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\(\tilde{f}_C, \tilde{V}_C\): Poissonized mean and variance of \(C_n\).

Under Bernoulli model:

\[
\tilde{f}_D(z) = m\tilde{f}_C\left(\frac{z}{m}\right)
\]

\[
\tilde{V}_D(z) = m\tilde{V}_C\left(\frac{z}{m}\right)
\]

Theorem
We have

\[
E(D_n) \sim m \log \frac{1}{q} (\frac{n}{m}) + m\gamma \log 2 - \lambda + \frac{1}{2} + mFc\left(\log \frac{1}{q} (\frac{n}{m})\right)
\]

\[
\text{Var}(D_n) \sim m \text{Var}_\infty + mGc\left(\log \frac{1}{q} (\frac{n}{m})\right)
\]

Where the constants and periodic functions are as before.
\( \tilde{f}_D, \tilde{V}_D \): Poissonized mean and variance of \( D_n \).

\( \tilde{f}_C, \tilde{V}_C \): Poissonized mean and variance of \( C_n \).

Under Bernoulli model:

\[
\tilde{f}_D(z) = m\tilde{f}_C\left(\frac{z}{m}\right)
\]

\[
\tilde{V}_D(z) = m\tilde{V}_C\left(\frac{z}{m}\right)
\]

Theorem

We have

\[
\mathbb{E}(D_n) \sim m \log_{1/q} \left( \frac{n}{m} \right) + m \left( \frac{\gamma}{\log 2} - \lambda + \frac{1}{2} \right) + mF_c(\log_{1/q} \frac{n}{m})
\]

\[
\text{Var}(D_n) \sim m\text{Var}_\infty + mG_c(\log_{1/q} \frac{n}{m}).
\]

Where the constants and periodic functions are as before.
Another variant of $m$ counters:
Consider $m$ counter, label them from 1 to $m$. Use the first counter until it will be increased, then we use the second one and repeat this procedure.
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Then,

\[ D_n \overset{d}{=} X_n, \]

where \(X_n\) satisfies

\[ X_{n+m} \overset{d}{=} X_{B_n} + m, \quad (n \geq 0). \]
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In fact, $D_n$ corresponding to the length of the leftmost path length in random bucket digital search tree with bucket size $m$. 
Variance of Cyclic $m$ Counters

**Theorem (Fuchs, Lee, Prodinger)**

For approximate counting with $m$-counters, where counters are chosen cyclically, we have, as $n \to \infty$,

$$\text{Var}(D_n) \sim G_D(\log_{1/q} n),$$

where $G_D(z) = \sum_k g_k e^{2k\pi i z}$ is a 1-periodic function with Fourier coefficients

$$g_k = \frac{1}{L \Gamma(1 + \chi_k)} \int_0^\infty \frac{e^{s \chi_k}}{Q(-s/q)^m} \left( p(s) + \int_0^\infty e^{-zs} \tilde{g}(z) dz \right) ds$$

and

$$p(s) = \frac{(s + 1)^m - 1 - ms}{s^2}.$$
Possible extensions of this problem and open problems:

- Leader election
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Perspective

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