Multiple Almost Periodic Solutions in Nonautonomous Delayed Neural Networks

Kuang-Hui Lin
hs3893@mail.nc.hcc.edu.tw
Chih-Wen Shih
cwshih@math.nctu.edu.tw
Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, R.O.C.

A general methodology that involves geometric configuration of the network structure for studying multistability and multiperiodicity is developed. We consider a general class of nonautonomous neural networks with delays and various activation functions. A geometrical formulation that leads to a decomposition of the phase space into invariant regions is employed. We further derive criteria under which the \( n \)-neuron network admits \( 2^n \) exponentially stable sets. In addition, we establish the existence of \( 2^n \) exponentially stable almost periodic solutions for the system, when the connection strengths, time lags, and external bias are almost periodic functions of time, through applying the contraction mapping principle. Finally, three numerical simulations are presented to illustrate our theory.

1 Introduction

Rhythms are ubiquitous in nature. Experimental and theoretical studies suggest that a mammalian brain may be exploiting dynamic attractors for its storage of associative memories (Kopell, 2000; Skarda & Freeman, 1987; Izhikevich, 1999; Yau, Freeman, Burke, & Yang, 1991). Therefore, investigation of dynamic attractors as limit cycles and strange attractors is essential in neural networks. Many important studies of neural networks, however, focus mainly on static attractors, that is, equilibrium type. On the other hand, cyclic behaviors or rhythmic activities may not be represented exactly by periodic functions from a phenomenological point of view. It is therefore interesting to investigate almost periodic functions, the naturally generalized notion of periodic functions.

In this letter, we consider the following neural network model,

\[
\frac{dx_i(t)}{dt} = -\mu_i(t)x_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) + I_i(t),
\]

(1.1)
where \( i = 1, 2, \ldots, n \); the integer \( n \) corresponds to the number of units in the network; \( x_i(t) \) corresponds to the state of the \( i \)th unit at time \( t \); \( \mu_i(t) \) represents the rate with which the \( i \)th unit resets its potential to the resting state in isolation when disconnected from the networks and external input; \( \alpha_{ij}(t) \) and \( \beta_{ij}(t) \) denote the connection strength of the \( j \)th unit on the \( i \)th unit at time \( t \) and time \( t - \tau_{ij}(t) \), respectively; \( g_j(x_j(t)) \) denotes the output of the \( j \)th unit at time \( t \); \( \tau_{ij}(t) \) denotes the transmission delay of the \( i \)th unit along the axon of the \( j \)th unit at time \( t \) and is a nonnegative function; and \( I_i(t) \) corresponds to the external bias on the \( i \)th unit at time \( t \).

Many theoretical studies of neural networks are predominantly concerned with autonomous systems; namely, all the parameters in system 1.1 are constants. Such an autonomous network system is often referred to in the literature as a delayed cellular neural network (Chua & Yang, 1988; Roska & Chua, 1992). The autonomous cellular neural networks with delay and without delay have been successfully applied to many fields, such as optimization, associative memory, signal and image processing, and pattern recognition. The theory on the existence and stability of equilibrium and periodic solution for such autonomous systems has been extensively investigated (Cao, 2003; Chen, Lu, & Chen, 2005; Juang & Lin, 2000; Mohamad & Gopalsamy, 2003; Peng, Qiao, & Xu, 2002; Shih, 2000; Tu & Liao, 2005; van den Driessche & Zou, 1998; Zhou, Liu, & Chen, 2004). The autonomous neural network systems are modeled on the assumption that the environment is stationary, that is, its characteristics do not change with time. However, frequently the environment of interest is not stationary, and the parameters of the information-bearing signal generated by the environment vary with time. In general, nonautonomous phenomena often occur in many realistic systems. Particularly when we consider a long-term dynamical behavior of a system, the parameters of the system usually change with time. The investigations of nonautonomous delayed neural networks (see equation 1.1) are important in understanding the dynamical characteristics of neural behavior in time varying environments.

In this letter, we develop a geometrical approach that leads to a decomposition of the phase space into several positively invariant and stable sets. We then study the existence of multiple, almost periodic solutions for the nonautonomous delayed neural network 1.1, under the assumption that all the parameters and delays in system 1.1 are almost periodic. Most of the studies of autonomous systems and nonautonomous systems center around the existence of a unique equilibrium, periodic, multiple almost periodic solutions and their global convergence. A unique periodic solution and its globally exponentially robust stability were investigated by Cao and Chen (2004) for system 1.1 with periodic external bias \( I_i(t) \), using the Lyapunov method. The globally exponential stability of a unique almost periodic solution has been analyzed by Cao, Chen, and Huang (2005), Chen and Cao (2003), Fan and Ye (2005), Huang and Cao (2003), Huang, Cao, and Ho (2006), Liu & Huang (2005), Mohamad and Gopalsamy (2000), Zhang,
Wei, and Xu (2005), Mohamad (2003), and Zhao (2004). The techniques they employed include exponential dichotomy, fixed-point theorem, contraction mapping principle, Lyapunov methods and Halanay inequality. A neural network in a more general form, which includes system 1.1 and distributed delays, has been analyzed in Lu and Chen (2004, 2005), who obtained the globally exponential stability of a single periodic solution and a single almost periodic solution. Multistability in neural networks is essential in numerous applications such as content-addressable memory storage and pattern recognition. In addition, the application of multistability is important in decision making, digital selection, and analogy amplification (Hahnloser, 1998). Recently the theory of the existence of multiple stable equilibria for autonomous networks has been established in Cheng, Lin, and Shih (2006); multiple stable periodic orbits for autonomous networks with periodic inputs have been exploited in Cheng, Lin, and Shih (2007), and Zeng and Wang (2006). This presentation moves this direction of studies to wider and more general considerations. We establish the existence of $2^n$ exponentially stable almost periodic solutions for general $n$-dimensional nonautonomous networks, 1.1, with various activation functions. Our derivation also indicates the basins of attraction for these almost periodic solutions.

The presentation is organized as follows. In section 2, we justify that all the solutions for system 1.1 are bounded, under the assumption that all the parameters in this system are bounded continuous functions of $t$. $2^n$ subsets of the phase space are shown to be positively invariant under the flows generated by system 1.1. Within these $2^n$ subsets, we further construct $2^n$ exponentially stable subsets (to be defined later) for system 1.1. In section 3, we derive the existence of $2^n$ almost periodic solutions for system 1.1 with almost periodic parameters and time lags through the contraction mapping principle. Finally, in section 4, we arrange three numerical simulations on the dynamics of system 1.1 to illustrate the theory.

2 Boundedness, Invariant Sets, and Exponentially Stable Regions

We consider that $\mu_i(t), \alpha_{ij}(t), \beta_{ij}(t), I_i(t), \tau_{ij}(t)$ in system 1.1 are bounded continuous functions defined for $t \in \mathbb{R}$, the set of all real numbers. We denote that

$$0 < \underline{\mu}_i \leq \mu_i(t) \leq \bar{\mu}_i, \quad \underline{\alpha}_{ij} \leq \alpha_{ij}(t) \leq \bar{\alpha}_{ij}, \quad \underline{\beta}_{ij} \leq \beta_{ij}(t) \leq \bar{\beta}_{ij},$$

$$\underline{I}_i \leq I_i(t) \leq \bar{I}_i, \quad 0 \leq \underline{\tau}_{ij} \leq \tau_{ij}(t) \leq \bar{\tau}_{ij},$$

for all $i, j = 1, 2, \ldots, n$ and $t \in \mathbb{R}$ with

$$\inf_{t \in \mathbb{R}} \mu_i(t) = \underline{\mu}, \quad \inf_{t \in \mathbb{R}} \alpha_{ij}(t) = \underline{\alpha}_{ij}, \quad \inf_{t \in \mathbb{R}} \beta_{ij}(t) = \underline{\beta}_{ij}, \quad \inf_{t \in \mathbb{R}} I_i(t) = \underline{I}_i,$$
\[
\begin{align*}
\inf_{t \in \mathbb{R}} \tau_{ij}(t) &= \tau_{ij}, & \sup_{t \in \mathbb{R}} \mu_i(t) &= \bar{\mu}_i, & \sup_{t \in \mathbb{R}} \alpha_{ij}(t) &= \bar{\alpha}_{ij}, & \sup_{t \in \mathbb{R}} \beta_{ij}(t) &= \bar{\beta}_{ij}, \\
\sup_{t \in \mathbb{R}} I_i(t) &= \bar{I}_i, & \sup_{t \in \mathbb{R}} \tau_{ij}(t) &= \bar{\tau}_{ij}.
\end{align*}
\]

Moreover, we set
\[
\alpha^*_{ij} = \max(|\alpha_{ij}|, |\bar{\alpha}_{ij}|), \quad \beta^*_{ij} = \max(|\beta_{ij}|, |\bar{\beta}_{ij}|), \quad I^*_{i} = \max(|I_{i}|, |\bar{I}_{i}|). \tag{2.2}
\]

The activation functions \(g_j\) usually admit sigmoidal configuration or are nondecreasing with saturation. In this presentation, we mainly adopt
\[
g_j(\xi) = g(\xi) := \tanh(\xi), \quad \text{for all} \ j = 1, 2, \ldots, n. \tag{2.3}
\]

Its graph is depicted in Figure 1a. Our theory can be applied to more general activation functions:
\[
g_j \in C^2, \quad \begin{cases} 
  u_i < g_j(\xi) < v_i, & g_j'(\xi) > 0, (\xi - \sigma_j)g_j''(\xi) < 0, \text{for all} \ \xi \in \mathbb{R}, \\
  \lim_{\xi \to +\infty} g_j(\xi) = v_i, & \lim_{\xi \to -\infty} g_j(\xi) = u_i,
\end{cases}
\]

where \(u_i, v_i, \sigma_i\) are constants with \(u_i < v_i\), and the nondecreasing continuous functions with saturation
\[
g_i(\xi) = \begin{cases} 
  u_i & \text{if} \ -\infty < \xi < p_i, \\
  \text{increasing} & \text{if} \ p_i \leq \xi \leq q_i, \\
  v_i & \text{if} \ q_i < \xi < \infty,
\end{cases}
\]

where \(u_i, v_i, q_i, p_i\) are constants with \(u_i < v_i, p_i < q_i, i = 1, 2, \ldots, n\). The latter contains the standard piecewise linear activation function for cellular neural networks: \(\max(|\xi - 1| + |\xi + 1|)/2\). Notably, the analysis in Zeng and Wang (2006) is performed within the framework of this piecewise linear activation function. The configurations for these functions are depicted in Figures 2a to 2c, respectively.

We set \(\tau := \max_{1 \leq i, j \leq n} \{\bar{\tau}_{ij}\}\). System 1.1 is one of nonautonomous delayed differential equations. The evolution of the system is described by the solution \(x(t; t_0, \varphi) = x(t; t_0) = (x_1(t; t_0), x_2(t; t_0), \ldots, x_n(t; t_0))\), \(t \geq t_0 - \tau\), starting from the initial condition: \(x_i(t_0 + \theta; t_0) = \varphi_i(\theta)\) for \(\theta \in [-\tau, 0]\), where \(\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C([-\tau, 0), \mathbb{R}^n)\). We note that the fundamental theory for the delayed equation \(\dot{x}(t) = F(t, x_t)\) employs the conventional expression \(x_i(\theta) = x(t + \theta; t_0), \ \theta \in [-\tau, 0]\) with phase space \(C([-\tau, 0], \mathbb{R}^n)\). One usually regards the image of solution \(x_i(\theta), t \geq t_0\) as a trajectory \(\{x(t; t_0) : t \geq t_0 - \tau\}\) lying in \(\mathbb{R}^n\).

We first illustrate that all solutions of system 1.1 are bounded and then establish criteria under which system 1.1 admits \(2^n\) positively invariant
subsets of phase space $C([-\tau, 0], \mathbb{R}^n)$. Each of these invariant sets induces an exponentially stable region in $\mathbb{R}^n$ in which the trajectories $\{x(t; t_0) : t \geq t_0 - \tau\}$ lie. Note that the norm $\|\phi\| = \max_{1 \leq i \leq n} \{\sup_{s \in [-\tau, 0]} |\phi_i(s)|\}$ on $C([-\tau, 0], \mathbb{R}^n)$ is adopted here.

**Proposition 1.** All solutions of system 1.1 with bounded activation functions are bounded. Moreover, for system 1.1 with activation function 2.3,

$$|x_i(t_0; t_0)| \leq M_i \text{ implies } |x_i(t; t_0)| \leq M_i \text{, for } t > t_0,$$

$$|x_i(t_0; t_0)| > M_i \text{ implies } \lim_{t \to \infty} \sup_{t_0 < t} |x_i(t; t_0)| \leq M_i,$$

for all $i = 1, 2, \ldots, n$, where $M_i = \frac{\sigma_i \beta_i + \beta_i^*}{\mu_i}.$
Proof. We prove only the situation of activation function 2.3. Using equations 2.1 to 2.3 in system 1.1, we derive

\[
\frac{d^+}{dt} |x_i(t; t_0)| \leq -\mu_i |x_i(t; t_0)| + \sum_{j=1}^n (\alpha_{ij}^* + \beta_{ij}^*) + I_i^*, \quad t > t_0, \ i = 1, 2, \ldots, n,
\]

where \(d^+/dt\) denotes the right-hand derivative. It follows that

\[
|x_i(t; t_0)| \leq M_i + e^{-\mu_i (t-t_0)} (|x_i(t_0; t_0)| - M_i), \quad t > t_0, \ i = 1, 2, \ldots, n,
\]

where \(M_i\) is defined above. Hence, \(|x_i(t_0; t_0)| \leq M_i\) implies \(|x_i(t; t_0)| \leq M_i, \ t > t_0, \ i = 1, 2, \ldots, n\). On the other hand, if \(|x_i(t_0; t_0)| - M_i = C_i > 0\), then \(|x_i(t; t_0)| \leq M_i + e^{-\mu_i (t-t_0)} C_i\) for \(t > t_0\) and \(e^{-\mu_i (t-t_0)} C_i \to 0\) as \(t \to \infty\).
Let us denote by \( F = (F_1, F_2, \ldots, F_n) \) the right-hand side of equation 1.1 with activation function 2.3,

\[
F_i(t, x_t) := -\mu_i(t)x_i(t) + \sum_{j=1}^n \alpha_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^n \beta_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) + I_i(t),
\]

with \( g_j(\xi) = \tanh(\xi) \), for every \( j \). Next, we define, for \( i = 1, 2, \ldots, n \),

\[
\hat{f}_i(\xi) = \begin{cases} 
-\mu_i \xi + (\bar{a}_{ii} + \bar{b}_{ii})g_i(\xi) + k^+_i, & \xi \geq 0 \\
-\bar{\mu}_i \xi + (\bar{a}_{ii} + \bar{b}_{ii})g_i(\xi) + k^+_i, & \xi < 0,
\end{cases}
\]

\[
\check{f}_i(\xi) = \begin{cases} 
-\mu_i \xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + k^-_i, & \xi \geq 0 \\
-\bar{\mu}_i \xi + (\bar{a}_{ii} + \bar{b}_{ii})g_i(\xi) + k^-_i, & \xi < 0,
\end{cases}
\]

where

\[
k^+_i := \sum_{j=1, j \neq i}^n (\alpha_{ij}^* + \beta_{ij}^*) + I_i, \quad k^-_i := -\sum_{j=1, j \neq i}^n (\alpha_{ij}^* + \beta_{ij}^*) + I_i,
\]

with \( \alpha_{ij}^* \) and \( \beta_{ij}^* \) defined in equation 2.2. It follows that

\[
\check{f}_i(x_t) \leq F_i(t, x_t) \leq \hat{f}_i(x_t),
\]

for all \( x_t \in C([-\tau, 0], \mathbb{R}^n) \), \( x_t \in \mathbb{R} \), \( t \geq t_0 \) and \( i = 1, 2, \ldots, n \), since \( |g_j| \leq 1 \) for all \( j \).

Our approach is motivated by the geometric configurations of \( \check{f}_i \) and \( \hat{f}_i \) depicted in Figure 1b. To establish such a configuration, several conditions are required. The first parameter condition is

\[
(H_1): 0 = \inf_{\xi \in \mathbb{R}} g'_i(\xi) < \frac{\mu_i}{\alpha_{ii} + \beta_{ii}}, \quad \frac{\bar{\mu}_i}{\bar{\alpha}_{ii} + \bar{b}_{ii}} < \max_{\xi \in \mathbb{R}} g'_i(\xi) = 1, \quad i = 1, 2, \ldots, n.
\]

**Lemma 1.** Assume that condition \( H_1 \) holds. Then (i) there exist two points \( \bar{p}_i \) and \( \bar{q}_i \) with \( \bar{p}_i < 0 < \bar{q}_i \) such that \( \check{f}'_i(\bar{p}_i) = 0 \) and \( \hat{f}'_i(\bar{q}_i) = 0 \), \( i = 1, 2, \ldots, n \); and (ii) there exist two points \( p_i \) and \( q_i \) with \( p_i < 0 < q_i \) such that \( \check{f}'_i(p_i) = 0 \) and \( \hat{f}'_i(q_i) = 0 \), \( i = 1, 2, \ldots, n \).
Proof. We prove only case i. For each $i$, when $\xi \geq 0$, since $\hat{f}_i'(\xi) = -\mu_i + (\bar{\alpha}_{ii} + \bar{\beta}_{ii})g_i'(\xi)$, we have $\hat{f}_i'(\xi) = 0$ if and only if

$$g_i'(\xi) = \frac{\mu_i}{\bar{\alpha}_{ii} + \bar{\beta}_{ii}}.$$  

Note that the graph of function $g_i'(\xi)$ is concave down. In addition, $g_i'$ attains its maximal value at $\xi = 0$ and $\lim_{\xi \to \pm\infty} g_i'(\xi) = 0$. Hence, since $g_i'$ is continuous, if

$$0 = \inf_{\xi \in \mathbb{R}} g_i'(\xi) < g_i'(\xi) = \frac{\mu_i}{\bar{\alpha}_{ii} + \bar{\beta}_{ii}} < \max_{\xi \in \mathbb{R}} g_i'(\xi) = 1, \quad i = 1, 2, \ldots, n,$$

there exists a point $\bar{q}_i$ with $\bar{q}_i > 0$ such that

$$g_i'\left(\bar{q}_i\right) = \frac{\mu_i}{\bar{\alpha}_{ii} + \bar{\beta}_{ii}}.$$  

Similarly, when $\xi < 0$, there exists a point $\bar{p}_i$ with $\bar{p}_i < 0$, such that

$$g_i'\left(\bar{p}_i\right) = \frac{-\mu_i}{\bar{\alpha}_{ii} + \bar{\beta}_{ii}}.$$  

Note that condition $H_1$ implies $\bar{\alpha}_{ii} + \bar{\beta}_{ii} > 0$ for all $i = 1, 2, \ldots, n$, since each $\mu_i$ is already assumed a positive constant. The second parameter condition is

$$(H_2) : \hat{f}_i(\bar{p}_i) < 0, \hat{f}_i(\bar{q}_i) > 0, \quad i = 1, 2, \ldots, n.$$  

The configuration that motivates $H_2$ is depicted in Figure 1. Such a configuration is due to the characteristics of the activation functions $g_i$. Under assumptions $H_1$ and $H_2$, there exist points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$.

We consider the following $2^n$ subsets of $C([-\tau, 0], \mathbb{R}^n)$. Let $\mathbf{e} = (e_1, \ldots, e_n)$ with $e_i = 1$ or $r$, where, $l, r$ mean, respectively, “left,” and “right.” Set

$$\Lambda^{e} = \{\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \mid \varphi_i \in \Lambda^1_i \text{ if } e_i = 1, \ \varphi_i \in \Lambda^2_i \text{ if } e_i = r\}.$$  

where

$$\Lambda^1_i := \{\varphi_i \in C([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) < \hat{b}_i, \ \text{for all } \theta \in [-\tau, 0]\},$$

$$\Lambda^2_i := \{\varphi_i \in C([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) > \hat{b}_i, \ \text{for all } \theta \in [-\tau, 0]\}.$$
Theorem 1. Assume that $H_1$ and $H_2$ hold and $\beta_{ii}(t) > 0$, for all $t \in \mathbb{R}, i = 1, 2, \ldots, n$. Then each $\mathbf{A}^e$ is positively invariant under the solution flow generated by system 1.1.

Proof. Under assumptions $H_1$, $H_2$, and by lemma 1, we obtain the configuration as in Figure 1b and $2^n$ subsets $\mathbf{A}^e$ of $C([-\tau, 0], \mathbb{R}^n)$ defined in equation 2.6. Consider one of the $2^n$ sets $\mathbf{A}^e$, that is, fix an $\mathbf{e} = (e_1, e_2, \ldots, e_n)$, $e_i = 1, r, i = 1, 2, \ldots, n$. Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in \mathbf{A}^e$. Then there exists a sufficiently small constant $\epsilon_0 > 0$ such that $\phi_i(\theta) \geq \hat{b}_i + \epsilon_0$ for all $\theta \in [-\tau, 0]$, if $e_i = r$ and $\phi_i(\theta) \leq \hat{b}_i - \epsilon_0$ for all $\theta \in [-\tau, 0]$, if $e_i = 1$. We assert that the solution $x(t) = x(t; t_0; \phi)$ remains in $\mathbf{A}^e$ for all $t \geq t_0$. If this is not true, there exists a component of $x(t)$ that is the first one or among the first ones escaping from $\Lambda^1_1$ or $\Lambda^1_r$. More precisely, there exist some $i \in \{1, 2, \ldots, n\}$ and $t_0$ such that either $x_i(t_0) = \hat{b}_i + \epsilon_0$, $\frac{dx_i}{dt}(t_0) \leq 0$, and $x_i(t) > \hat{b}_i + \epsilon_0$ for $t_0 - \tau \leq t < t_1$ or $x_i(t_0) = \hat{b}_i - \epsilon_0$, $\frac{dx_i}{dt}(t_0) \geq 0$ and $x_i(t) < \hat{b}_i - \epsilon_0$ for $t_0 - \tau \leq t < t_1$. For the first case with $x_i(t_0) = \hat{b}_i + \epsilon_0$ and $\frac{dx_i}{dt}(t) \leq 0$, it follows from equation 1.1 that

$$
\frac{dx_i}{dt}(t_1) = -\mu_i(t_1)(\hat{b}_i + \epsilon_0) + \alpha_{ii}(t_1)g_i(\hat{b}_i + \epsilon_0) + \beta_{ii}(t_1)g_i(x_i(t_1 - \tau_{ii}(t_1))) \\
+ \sum_{j=1, j\neq i}^n \alpha_{ij}(t_1)g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^n \beta_{ij}(t_1)g_j(x_j(t_1 - \tau_{ij}(t_1))) + I_i(t_1) \leq 0.
$$

(2.7)

On the other hand, $H_2$ and the previous description of $\hat{b}_i$ yield $\tilde{f}_i(\hat{b}_i + \epsilon_0) > 0$;

$$
\begin{cases}
-\mu_i(\hat{b}_i + \epsilon_0) + (\alpha_{ii} + \beta_{ii})g_i(\hat{b}_i + \epsilon_0) - \sum_{j=1, j\neq i}^n (\alpha_{ij}^* + \beta_{ij}^*) + L_i > 0, \\
\text{if } \hat{b}_i + \epsilon_0 \geq 0,
\end{cases}
$$

$$
\begin{cases}
-\mu_i(\hat{b}_i + \epsilon_0) + (\alpha_{ii} + \beta_{ii})g_i(\hat{b}_i + \epsilon_0) - \sum_{j=1, j\neq i}^n (\alpha_{ij}^* + \beta_{ij}^*) + L_i > 0, \\
\text{if } \hat{b}_i + \epsilon_0 < 0.
\end{cases}
$$

(2.8)

Notice that $g_i(x_i(t_1 - \tau_{ii}(t_1))) > g_i(\hat{b}_i + \epsilon_0)$, by the monotonicity of function $g_i$. In addition, $t_1$ is the first time for $x_i$ to decrease across the value $\hat{b}_i + \epsilon_0$. By $\beta_{ii}(t_1) > 0$ and $|g_i(\cdot)| \leq 1$, we obtain from equation 2.8 that

$$
-\mu_i(t_1)(\hat{b}_i + \epsilon_0) + \alpha_{ii}(t_1)g_i(\hat{b}_i + \epsilon_0) + \beta_{ii}(t_1)g_i(x_i(t_1 - \tau_{ii}(t_1))) \\
+ \sum_{j=1, j\neq i}^n \alpha_{ij}(t_1)g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^n \beta_{ij}(t_1)g_j(x_j(t_1 - \tau_{ij}(t_1))) + I_i(t_1) \\
$$
\[ \begin{align*}
&\geq \begin{cases} 
-\mu_i(\hat{b}_i + \varepsilon_0) + (\alpha_{ij} + \beta_{ij})g_i(\hat{b}_i + \varepsilon_0) - \sum_{j=1, j \neq i}^{n} (\alpha_{ij}^+ + \beta_{ij}^+) \\
&+ L_j > 0, \text{ if } \hat{b}_i + \varepsilon_0 \geq 0 \\
-\mu_i(\hat{b}_i + \varepsilon_0) + (\alpha_{ij} + \beta_{ij})g_i(\hat{b}_i + \varepsilon_0) - \sum_{j=1, j \neq i}^{n} (\alpha_{ij}^+ + \beta_{ij}^+) \\
&+ L_j > 0, \text{ if } \hat{b}_i + \varepsilon_0 < 0, 
\end{cases}
\end{align*} \]

which contradicts equation 2.7. Hence, \(x_i(t) \geq \hat{b}_i + \varepsilon_0\) for all \(t > t_0\). Similar arguments justify that \(x_i(t) \leq \hat{b}_i - \varepsilon_0\) for all \(t > t_0\) for the situation that \(x_i(t_1) = \hat{b}_i - \varepsilon_0\) and \(\frac{dx_i}{dt}(t_1) \geq 0\). Therefore, \(\Lambda^e\) is positively invariant under the flow generated by equation 1.1.

**Remark 1.** The assertion of theorem 1 actually holds for the subset \(\Lambda^e\) in equation 2.6 with \(\hat{b}_i\) replaced by any number between \(\bar{b}_i\) and \(\bar{c}_i\), and \(\hat{b}_i\) replaced by any number between \(\bar{b}_i\) and \(\hat{a}_i\), as observed from the proof. Such an observation provides further understanding of the dynamics of the system.

**Definition 1.** A positively invariant set \(\Lambda \subset C([-\tau, 0], \mathbb{R}^n)\) is called an exponentially stable set for system 1.1 if there exist constants \(\gamma > 0\) and \(L > 0\) such that

\[ ||x_i(\cdot; t_0; \varphi) - x_i(\cdot; t_0; \psi)|| \leq Le^{\gamma(t-t_0)}||\varphi - \psi|| \text{ for all } t \geq t_0, \]

for solutions \(x_i(\cdot; t_0; \varphi)\) and \(x_i(\cdot; t_0; \psi)\) of system 1.1, starting from any two initial conditions \(\varphi, \psi \in \Lambda\). In this situation, \(\Omega := \{\psi(\theta), \theta \in [-\tau, 0]\} \text{ for all } \psi \in \Lambda \subset \mathbb{R}^n\), the images of all elements in \(\Lambda\), is called an exponentially stable region for system 1.1.

Let \(\eta_j \in \mathbb{R}\) such that \(\eta_j > \max\{g_j'(\xi) \mid \xi = \hat{c}_j, \hat{a}_j\}, j = 1, 2, \ldots, n\). We consider the following criterion, which yields exponentially stable sets for system 1.1:

\[ (H_3) : \inf_{t \in \mathbb{R}} \left\{ \mu_i(t) - \sum_{j=1}^{n} \eta_j [|\alpha_{ij}(t)| + |\beta_{ij}(t)|] \right\} > 0, i = 1, 2, \ldots, n. \]

We define \(\bar{d}_j\) and \(\tilde{d}_j\) as

\[ \bar{d}_j := \min\{\xi | g_j'(\xi) = \eta_j\}, \tilde{d}_j := \max\{\xi | g_j'(\xi) = \eta_j\}. \]

Then \(\bar{d}_j > \hat{a}_j, \tilde{d}_j < \hat{c}_j\). We consider the following \(2^n\) subsets of \(C([-\tau, 0], \mathbb{R}^n)\).

Let \(e = (e_1, e_2, \ldots, e_n)\) with \(e_i = 1\) or \(r\), and set

\[ \bar{\Lambda}^e = \{\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \mid \varphi_i \in \bar{\Lambda}_i^1 \text{ if } e_i = 1, \varphi_i \in \bar{\Lambda}_i^r \text{ if } e_i = r\}. \]
where

\[ \tilde{\Lambda}^1_l := \{ \varphi_i \in C([\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \leq d_i, \text{ for all } \theta \in [-\tau, 0] \}, \]
\[ \tilde{\Lambda}^1_r := \{ \varphi_i \in C([\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \geq \bar{d}_i, \text{ for all } \theta \in [-\tau, 0] \}. \]

Notably, \( \tilde{\Lambda}^1_l \subseteq \Lambda^1_l \) and \( \tilde{\Lambda}^1_r \subseteq \Lambda^1_r \).

In the following, we shall derive that each of these \( 2^n \) subsets \( \tilde{\Lambda}^e \) of \( C([\tau, 0], \mathbb{R}^n) \) is an exponentially stable set for system 1.1.

**Theorem 2.** Assume that conditions \( H_1, H_2, H_3 \) hold and \( \beta_{ii}(t) > 0, \) for all \( t \in \mathbb{R}, \ i = 1, 2, \ldots, n \). Then each of the \( 2^n \) sets \( \tilde{\Lambda}^e \) is exponentially stable for system 1.1.

**Proof.** We consider any one of these \( 2^n \) subsets \( \tilde{\Lambda}^e \), defined in equation 2.10. The positive invariance property of \( \tilde{\Lambda}^e \) follows from remark 1. Let \( x(t) = x(t; t_0; \phi) \) and \( y(t) = x(t; t_0; \psi) \) be two solutions of system 1.1 with respective initial conditions \( \phi, \psi \in \tilde{\Lambda}^e \). For each \( i = 1, 2, \ldots, n \), we define the single-variable functions \( G_i(\cdot) \) by

\[
G_i(\xi) = \inf_{t \in \mathbb{R}} \left\{ \mu_i(t) - \xi - \sum_{j=1}^{n} \eta_j |\alpha_{ij}(t)| - \sum_{j=1}^{n} \eta_j |\beta_{ij}(t)| e^{\xi \tau_{ij}} \right\}.
\]

Then \( G_i \) is continuous and \( G_i(0) > 0 \) from \( H_3 \). Moreover, there exists a constant \( \lambda > 0 \) such that \( G_i(\lambda) > 0 \), for all \( i = 1, 2, \ldots, n \), due to continuity of \( G_i \). From equations 2.1 to 2.3 and by using \( \eta_j > \max\{g_j'(\xi) \mid \xi = \tilde{c}_j, \tilde{a}_j\} \), we obtain

\[
\frac{d^+}{dt} |x_i(t) - y_i(t)| \leq -\mu_i(t)|x_i(t) - y_i(t)| + \sum_{j=1}^{n} \eta_j |\alpha_{ij}(t)||x_j(t) - y_j(t)|
\]

\[ + \sum_{j=1}^{n} \eta_j |\beta_{ij}(t)||x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t))|. \quad (2.11)\]

We assume that

\[
K := \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_0 - \tau, t_0]} |x_i(s) - y_i(s)| \right\} > 0. \quad (2.12)
\]

Now consider functions \( z_i(\cdot) \) defined by

\[
z_i(t) = e^{\lambda(t-t_0)}|x_i(t) - y_i(t)| \quad \text{for } t \geq t_0 - \tau. \quad (2.13)
\]
Then $z_i(t) \leq K$ for $t \in [t_0 - \tau, t_0]$. Using equations 2.11 and 2.13 we obtain

$$
\frac{d^+}{dt} z_i(t) \leq -[\mu_i(t) - \lambda]z_i(t) + \sum_{j=1}^{n} \eta_j |\alpha_{ij}(t)|z_j(t) + \sum_{j=1}^{n} \eta_j |\beta_{ij}(t)| e^{\lambda t_i} \left( \sup_{s \in [t-t_1]} z_j(s) \right)
$$

(2.14)

for $i = 1, 2, \ldots, n$, $t \geq t_0$. We assert that

$$
z_i(t) \leq K \quad \text{for all } t \geq t_0, \ i = 1, 2, \ldots, n. \quad (2.15)
$$

Suppose on the contrary, there exists an $i \in \{1, 2, \ldots, n\}$ (say, $i = k$) and a $t_1 \geq t_0$ at the earliest time such that

$$z_i(t) \leq K, \ t \in [-\tau, t_1], \ i = 1, 2, \ldots, n, \ i \neq k,$$

$$z_k(t) \leq K, \ t \in [-\tau, t_1), \ z_k(t_1) = K, \ \text{with } \frac{d}{dt} z_k(t_1) \geq 0. \quad (2.16)
$$

From equations 2.14 and 2.16, due to $G_{i}(\lambda) > 0$, we obtain

$$0 \leq \frac{d^+}{dt} z_k(t_1)$$

$$\leq -[\mu_k(t_1) - \lambda]z_k(t_1) + \sum_{j=1}^{n} \eta_j |\alpha_{kj}(t_1)|z_j(t_1)$$

$$+ \sum_{j=1}^{n} \eta_j |\beta_{kj}(t_1)| e^{\lambda t_k} \left( \sup_{s \in [t_1-t, t_1]} z_j(s) \right)$$

$$\leq -\left( \mu_k(t_1) - \lambda - \sum_{j=1}^{n} \eta_j |\alpha_{kj}(t_1)| - \sum_{j=1}^{n} \eta_j |\beta_{kj}(t_1)| e^{\lambda t_k} \right) K < 0,$$

which is a contradiction. Hence assertion 2.15 holds. It then follows from equations 2.12, 2.13, and 2.15 that

$$|x_i(t) - y_i(t)| \leq e^{-\lambda(t-t_0)} \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_0-\tau, t_0]} |x_i(s) - y_i(s)| \right\},$$

for all $i = 1, 2, \ldots, n$, $t \geq t_0$. 


3 Existence and Stability of Almost Periodic Solutions

In this section, we investigate the existence and stability of almost periodic solutions for system 1.1. We consider that \( \mu_i(t), \alpha_{ij}(t), \beta_{ij}(t), \tau_{ij}(t), \text{ and } I_i(t), \) \( i, j = 1, 2, \ldots, n, \) are almost periodic functions defined for all \( t \in \mathbb{R}. \) First, let us recall some basic definitions and properties of almost periodic functions.

**Definition 2.** (Bohr, 1947; Yoshizawa, 1975). Let \( h: \mathbb{R} \to \mathbb{R} \) be continuous. \( h \) is said to be (Bohr) almost periodic on \( \mathbb{R} \) if, for any \( \epsilon > 0, \) the set \( T(h, \epsilon) = \{ p : |h(t + p) - h(t)| < \epsilon, \text{ for all } t \in \mathbb{R} \} \) is relatively dense, that is, for any \( \epsilon > 0, \) it is possible to find a real number \( l = l(\epsilon) > 0, \) and for any interval with length \( l(\epsilon), \) there exists a number \( p = p(\epsilon) \) in the interval such that

\[
|h(t + p) - h(t)| < \epsilon, \text{ for all } t \in \mathbb{R}.
\]

The number \( p \) is called the \( \epsilon \)-translation number or \( \epsilon \)-almost period of \( h. \)

We denote by \( \mathcal{AP} \) the set of all such functions. Notably, \( (\mathcal{AP}, \| \cdot \|) \) with the supremum norm is a Banach space. We collect some properties of almost periodic functions, which will be used in the following discussions.

**Proposition 2.** (Corduneanu, 1968; Fink, 1974).

i. Every periodic function is almost periodic.

ii. Each \( h \) in \( \mathcal{AP} \) is bounded and uniformly continuous.

iii. \( \mathcal{AP} \) is an algebra (closed under addition, product, and scalar multiplication).

iv. If \( h \in \mathcal{AP} \) and \( H \) is uniformly continuous on the range of \( h, \) then \( H \circ h \in \mathcal{AP}. \)

v. If \( h \in \mathcal{AP}, \) and \( \inf_{t \in \mathbb{R}} |h(t)| > 0, \) then \( 1/h \in \mathcal{AP}. \)

vi. \( \mathcal{AP} \) is closed under uniform limits on \( \mathbb{R}. \)

vii. Suppose \( h_1, h_2, \ldots, h_m \in \mathcal{AP}. \) Then for every \( \epsilon > 0, \) there exist a common \( l(\epsilon) \) and a common \( \epsilon \)-translation number \( p(\epsilon) \) for these functions.

Proposition 2(vii) shows that definition 2 can be extended to vector-valued functions, as the situation for our solutions \( x(t; t_0) \) to system 1.1. We denote by \( \mathcal{AP}(\mathbb{R}^n) \) the space of all almost periodic functions from \( \mathbb{R} \) to \( \mathbb{R}^n \) with norm \( \| h \| = \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} |h_i(t)|. \)

**Proposition 3.** Assume that \( H, h \in \mathcal{AP}, \) and define \( \tilde{H}(t) = H(t + h(t)). \) Then \( \tilde{H} \in \mathcal{AP}. \)

**Proof.** \( H \) is uniformly continuous since \( H \in \mathcal{AP}. \) Thus, for any \( \epsilon > 0, \) there exists \( \delta = \delta(\epsilon) > 0 \) (we choose \( \delta < \epsilon/2 \)) such that

\[
|H(\xi) - H(\eta)| < \epsilon/2, \text{ provided } |\xi - \eta| < \delta.
\]
On the other hand, for the $\delta = \delta(\epsilon) > 0$ in equation 3.1, there exists a real number $l = l(\delta) > 0$, and for any interval of length $l(\delta)$, there exists a number $p = p(\delta)$ in the interval such that

$$|H(t + p) - H(t)| < \delta, \text{ and } |h(t + p) - h(t)| < \delta, \text{ for all } t \in \mathbb{R},$$
due to $H, h \in \mathcal{AP}$ and proposition 2(vii). Therefore, for any $\epsilon > 0$, there exists $l = l(\epsilon) = l(\delta(\epsilon))$, and $p(\epsilon) = p(\delta(\epsilon))$ in any interval of length $l$ such that

$$|H(t + h(t + p)) - H(t + h(t))| < \epsilon/2,$$

$$|H(t + h(t + p) + p) - H(t + h(t + p))| < \delta < \epsilon/2.$$

Subsequently,

$$|H(t + p + h(t + p)) - H(t + h(t))|$$

$$\leq |H(t + h(t + p) + p) - H(t + h(t + p))| + |H(t + h(t + p))$$

$$- H(t + h(t))|$$

$$< \epsilon.$$

If we set $\mu_i(t), \alpha_{ij}(t), \beta_{ij}(t), \tau_{ij}(t)$ and $I_i(t), i, j = 1, 2, \ldots, n$, to be almost periodic functions, then theorems 1 and 2 can also be established under the same conditions, since every almost periodic function is bounded. In the following discussion, we investigate the existence of almost periodic solutions of system 1.1 under those conditions in theorem 2 and $\alpha_{ii}(t) > 0$, for all $t$.

**Theorem 3.** Let $\mu_i(t), \alpha_{ij}(t), \beta_{ij}(t), \tau_{ij}(t)$, and $I_i(t), i, j = 1, 2, \ldots, n$, be almost periodic functions. Assume that conditions $H_1, H_2, H_3$, and $\alpha_{ii}(t) > 0, \beta_{ii}(t) > 0, t \in \mathbb{R}, i = 1, 2, \ldots, n$, hold. Then there exist $2^n$ exponentially stable almost periodic solutions for system 1.1.

**Proof.** We consider the following $2^n$ subsets of $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$. For each $e = (e_1, \ldots, e_n), e_i = 1$ or $r$, we set

$$\bar{\Lambda}^e = \{ \phi = (\phi_1, \phi_2, \ldots, \phi_n) \mid \phi_i \in \bar{\Lambda}_i^1 \text{ if } w_i = 1, \phi_i \in \bar{\Lambda}_i^r \text{ if } w_i = r \}, \quad (3.2)$$

where

$$\bar{\Lambda}_i^1 := \{ \phi_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \mid \phi_i(\theta) \leq d_i, \text{ for all } \theta \in \mathbb{R} \},$$

$$\bar{\Lambda}_i^r := \{ \phi_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \mid \phi_i(\theta) \geq d_i, \text{ for all } \theta \in \mathbb{R} \}.$$
and $d_i, \bar{d}_i$ are defined in equation 2.9. Consider a fixed $\Lambda^e$. Let us define a mapping $P, P(\phi) = (P_1(\phi), P_2(\phi), \ldots, P_n(\phi))$, by

$$P_i(\phi)(t) = \int_0^\infty \left[ \sum_{j=1}^n \alpha_{ij}(t-r)g_j(\phi_j(t-r)) ight.$$ 

$$\left. + \sum_{j=1}^n \beta_{ij}(t-r)g_j(\phi_j(t-r-\tau_{ij}(t-r))) + I_i(t-r) \right] e^{-\int_0^t \mu_i(s)ds} dr, \quad t \in \mathbb{R}. \quad (3.3)$$

It will be shown below that $P : \bar{\Lambda}^e \cap \mathcal{AP}(\mathbb{R}^n) \to \bar{\Lambda}^e \cap \mathcal{AP}(\mathbb{R}^n)$. We divide the following justifications into four steps.

**Step 1.** Let $\phi \in \bar{\Lambda}^e$. We derive from equation 3.3 that

$$P_i(\phi)(t) = \int_0^\infty \left[ \alpha_{ii}(t-r)g_i(\phi_i(t-r)) + \beta_{ii}(t-r)g_i(\phi_i(t-r-\tau_{ii}(t-r))) ight.$$ 

$$\left. + \sum_{j=1, j \neq i}^n \alpha_{ij}(t-r)g_j(\phi_j(t-r)) + \sum_{j=1, j \neq i}^n \beta_{ij}(t-r)g_j(\phi_j(t-r-\tau_{ij}(t-r))) + I_i(t-r) \right] e^{-\int_0^t \mu_i(s)ds} dr$$

$$\geq \int_0^\infty \left[ \alpha_{ii}(t-r)\bar{g}_i(\bar{d}_i) + \beta_{ii}(t-r)\bar{g}_i(\bar{d}_i) + \sum_{j=1, j \neq i}^n \alpha_{ij}(t-r)\bar{g}_j(\bar{d}_j) + \sum_{j=1, j \neq i}^n \beta_{ij}(t-r)\bar{g}_j(\bar{d}_j) + I_i(t-r) \right] e^{-\int_0^t \bar{\mu}_i(s)ds} dr$$

$$\geq \int_0^\infty \left[ (\alpha_{ii} + \beta_{ii})\bar{g}_i(\bar{d}_i) - \sum_{j=1, j \neq i}^n (\alpha_{ij} + \beta_{ij}) \right] e^{-\int_0^t \bar{\mu}_i(s)ds} dr,$$ 

$$\quad (3.4)$$
since \( \alpha_i(t), \beta_i(t) \geq 0, \ t \in \mathbb{R}, \) \( |g_i(\cdot)| \leq 1, \) \( \bar{d}_i > 0, \) and \( g_i(\phi_i(t - r)), g_i(\phi_i(t - r - \tau_i(t - r))) \geq g_i(\bar{d}_i) > 0, \) due to the monotonicity of functions \( g_i \) and \( H_1, H_2, \) and \( H_3. \) On the other hand, since \( \dot{f}_i(\bar{d}_i) > 0 \) and \( \bar{d}_i > 0, \) we have

\[
\dot{f}_i(\bar{d}_i) = -\bar{\mu}_i \bar{d}_i + (\alpha_{ii} + \beta_{ii})g_i(\bar{d}_i) - \sum_{j=1, j \neq i}^{n} (\alpha_{ij}^* + \beta_{ij}^*) + L_i > 0,
\]

that is, \( (\alpha_{ii} + \beta_{ii})g_i(\bar{d}_i) - \sum_{j=1, j \neq i}^{n} (\alpha_{ij}^* + \beta_{ij}^*) + L_i > \bar{\mu}_i \bar{d}_i. \) With this, it follows from equation 3.4 that

\[
P_i(\phi)(t) > \bar{\mu}_i \bar{d}_i \int_{0}^{\infty} e^{-\int_{\tau}^{t} \bar{\mu}_i ds} dr = \bar{d}_i, \quad \text{for all } t \in \mathbb{R}.
\]

Similar arguments can be employed to show that \( P_i(\phi)(t) < \bar{d}_i, \) for all \( t \in \mathbb{R} \) and \( i = 1, 2, \ldots, n. \) Therefore, \( P: \Lambda^e \to \Lambda^e. \)

**Step 2.** We justify that \( P : \mathcal{AP}(\mathbb{R}^n) \to \mathcal{AP}(\mathbb{R}^n). \) It suffices to show that \( \phi \in \mathcal{AP}(\mathbb{R}^n) \) implies \( P_i(\phi) \in \mathcal{AP}(\mathbb{R}^1) \) for all \( i = 1, 2, \ldots, n. \) Let \( \epsilon > 0 \) be arbitrary. Choose \( \epsilon' \) such that

\[
\epsilon' = \min \left\{ \frac{\epsilon}{5 \sum_{k=1}^{n} (\alpha_{ik}^* + \beta_{ik}^*) + I_i^*}, \frac{\epsilon}{5 \sum_{k=1}^{n} (\alpha_{ik}^* + \beta_{ik}^*)}, i = 1, 2, \ldots, n \right\}.
\]

By propositions 2 and 3, there exists a common \( \epsilon' \)-almost period \( p = p(\epsilon') \) for \( \mu_i(\cdot), \alpha_{ij}(\cdot), \beta_{ij}(\cdot), I_i(\cdot), \) and \( H_{ij}(\cdot) \) such that for each \( i = 1, 2, \ldots, n, \)

\[
|\mu_i(t + p - s) - \mu_i(t - s)| \leq \epsilon' \leq \frac{\epsilon}{5 \sum_{k=1}^{n} (\alpha_{ik}^* + \beta_{ik}^*) + I_i^*},
\]

\[
\sum_{j=1}^{n} |\alpha_{ij}(t + p - r) - \alpha_{ij}(t - r)| \leq \epsilon' \leq \frac{\epsilon}{5 \mu_i},
\]

\[
\sum_{j=1}^{n} |\beta_{ij}(t + p - r) - \beta_{ij}(t - r)| \leq \epsilon' \leq \frac{\epsilon}{5 \mu_i},
\]

\[
|I_i(t + p - r) - I_i(t - r)| \leq \epsilon' \leq \frac{\epsilon}{5 \mu_i},
\]

\[
|\phi_i(t + p - r) - \phi_i(t - r)| \leq \epsilon' \leq \frac{\epsilon}{5 \sum_{k=1}^{n} (\alpha_{ik}^* + \beta_{ik}^*)},
\]

\[
|\phi_i(t + p - r - \tau_{ii}(t + p - r)) - \phi_i(t - r - \tau_{ii}(t - r))| \leq \epsilon' \leq \frac{\epsilon}{5 \sum_{k=1}^{n} (\alpha_{ik}^* + \beta_{ik}^*)},
\]

\[
(3.5)
\]
for all $\ell = 1, 2, \ldots, n$, and $r, s, t \in \mathbb{R}$. We compute that

\[
|P_\ell(\phi)(t + p) - P_\ell(\phi(t))| \\
\leq \int_0^\infty \sum_{j=1}^{n} |\alpha_{ij}(t + p - r) - \alpha_{ij}(t - r)||g_j(\phi_j(t + p - r))| e^{-\int_0^t \mu_i(t+p-s)ds} dr \\
+ \int_0^\infty \sum_{j=1}^{n} |\beta_{ij}(t + p - r) - \beta_{ij}(t - r)||g_j(\phi_j(t + p - \tau_i(t + p - r)))| e^{-\int_0^t \mu_i(t+p-s)ds} dr \\
+ \int_0^\infty \sum_{j=1}^{n} |\beta_{ij}(t - r)||g_j(\phi_j(t + p - \tau_i(t + p - r)))| e^{-\int_0^t \mu_i(t+p-s)ds} dr \\
- g_j(\phi_j(t - r - \tau_i(t - r)))| e^{-\int_0^t \mu_i(t+p-s)ds} dr \\
+ \int_0^\infty \sum_{j=1}^{n} |\alpha_{ij}(t - r)||g_j(\phi_j(t - r))| e^{-\int_0^t \mu_i(t+p-s)ds} - e^{-\int_0^t \mu_i(t-s)ds} |dr \\
+ \int_0^\infty \sum_{j=1}^{n} |\beta_{ij}(t - r)||g_j(\phi_j(t - r - \tau_i(t - r)))| e^{-\int_0^t \mu_i(t+p-s)ds} - e^{-\int_0^t \mu_i(t-s)ds} |dr \\
\times |e^{-\int_0^t \mu_i(t+p-s)ds} - e^{-\int_0^t \mu_i(t-s)ds} |dr \\
+ \int_0^\infty \sum_{j=1}^{n} |I_j(t + p - r) - I_j(t - r)| e^{-\int_0^t \mu_i(t+p-s)ds} dr \\
+ \int_0^\infty \sum_{j=1}^{n} |I_j(t - r)||e^{-\int_0^t \mu_i(t+p-s)ds} - e^{-\int_0^t \mu_i(t-s)ds} |dr \tag{3.6}
\]

for $\phi \in \mathcal{A}(\mathbb{R}^n)$. By the mean value theorem, we obtain the following:

\[
|g_j(\phi_j(t + p - r)) - g_j(\phi_j(t - r))| = |\tanh(\phi_j(t + p - r)) - \tanh(\phi_j(t - r))| = |\operatorname{sech}^2(\theta_i)[\phi_j(t + p - r) - \phi_j(t - r)]| \\
\leq |\phi_j(t + p - r) - \phi_j(t - r)| \leq \frac{\epsilon}{5} \frac{\mu_i}{\sum_{k=1}^{n} (\alpha_{ik} + \beta_{ik}^+)}, \tag{3.7}
\]
for all $i = 1, 2, \ldots, n$, where $\theta_0$ lies between $\phi_j(t + p - r)$ and $\phi_j(t - r)$, and

$$
|g_j(\phi_j(t + p - r - \tau_j(t + p - r))) - g_j(\phi_j(t - r - \tau_j(t - r)))| = |\text{tanh}(\phi_j(t + p - r - \tau_j(t + p - r))) - \text{tanh}(\phi_j(t - r - \tau_j(t - r)))| = |\text{sech}^2(\theta_\tau)[\phi_j(t + p - r - \tau_j(t + p - r)) - \phi_j(t - r - \tau_j(t - r))]| \\
\leq |\phi_j(t + p - r - \tau_j(t + p - r)) - \phi_j(t - r - \tau_j(t - r))| \\
\leq \epsilon \frac{\mu_i}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) \ .
$$

(3.8)

for all $i = 1, 2, \ldots, n$, where $\theta_\tau$ lies between $\phi_j(t + p - r - \tau_j(t + p - r))$ and $\phi_j(t - r - \tau_j(t - r))$. Similarly,

$$
|e^{-\int_0^r \mu_i(t+s)ds} - e^{-\int_0^r \mu_i(t-s)ds}| = |e^{\theta_\tau} ( - \int_0^r \mu_i(t + s)ds + \int_0^r \mu_i(t - s)ds)| \\
\leq e^{-\int_0^r \mu_i ds} \int_0^r |\mu_i(t + s) - \mu_i(t - s)|ds \\
\leq re^{-\mu_i r} \epsilon \frac{\mu_i^2}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) + I_i^* \ .
$$

(3.9)

for all $i = 1, 2, \ldots, n$, where $\theta_\tau$ lies between $-\int_0^r \mu_i(t+s)ds$ and $-\int_0^r \mu_i(t-s)ds$. By substituting equations 3.5 and 3.7 to 3.9 into 3.6, we have

$$
|P_i(\phi)(t + p) - P_i(\phi)(t)| \\
\leq \epsilon \frac{\mu_i}{5} \int_0^\infty e^{-\int_0^r \mu_i ds} dr + \epsilon \frac{\mu_i}{5} \int_0^\infty e^{-\int_0^r \mu_i ds} dr \\
+ \sum_{j=1}^n \alpha_{ij}^* \frac{\mu_i}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) \int_0^\infty e^{-\int_0^r \mu_i ds} dr \\
+ \sum_{j=1}^n \beta_{ij}^* \frac{\mu_i}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) \int_0^\infty e^{-\int_0^r \mu_i ds} dr \\
+ \sum_{j=1}^n \alpha_{ij}^* \frac{\mu_i^2}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) + I_i^* \int_0^\infty re^{-\mu_i r} dr \\
+ \sum_{j=1}^n \beta_{ij}^* \frac{\mu_i^2}{5} \sum_{k=1}^n (\alpha_{ik}^* + \beta_{ik}^*) + I_i^* \int_0^\infty re^{-\mu_i r} dr
$$
\[ + \varepsilon \frac{\mu_i}{5} \int_0^\infty e^{-\int_0^r \mu_i \, dr} \, dr + I_i^* \frac{\mu_i^2}{5} \sum_{k=1}^n (\alpha_{ik}^n + \beta_{ik}^n) + I_i^* \int_0^\infty e^{-\mu_i \, dr} \]
\[ \leq \varepsilon + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon \]  

(3.10)

for \( \phi \in \mathcal{AP}(\mathbb{R}^n) \). Therefore, we obtain that the common \( \epsilon' \)-almost period \( p \) that satisfies equation 3.5 is actually an \( \epsilon \)-almost period of \( P_i(\phi) \) for all \( i = 1, 2, \ldots, n \).

**Step 3.** We shall prove that \( P \) is a contraction mapping. Notably, condition \( H_3 \) yields

\[
\mu_i(t) > \sum_{j=1}^n \eta_j [\|\alpha_{ij}(t)\| + \|\beta_{ij}(t)\|] \quad \text{for all } t \in \mathbb{R}, \ i = 1, 2, \ldots, n. \quad (3.11)
\]

For arbitrary \( \varphi, \psi \in \mathcal{A} e \cap \mathcal{AP}(\mathbb{R}^n) \), we derive

\[
\|P(\varphi) - P(\psi)\| = \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} |P_i(\varphi)(t) - P_i(\psi)(t)|
\]

\[
= \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left| \int_0^\infty e^{-\int_0^r \mu_i(t-s) \, ds} \left\{ \sum_{j=1}^n \alpha_{ij}(t-r)[g_j(\varphi_j(t-r)) - g_j(\psi_j(t-r))] + \sum_{j=1}^n \beta_{ij}(t-r)[g_j(\varphi_j(t-r - \tau_{ij}(t-r))] - g_j(\psi_j(t-r - \tau_{ij}(t-r))] \right\} \, dr \right|
\]

\[
\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left| e^{-\int_0^r \mu_i(t-s) \, ds} \sum_{j=1}^n \eta_j [\|\alpha_{ij}(t-r)\| + \|\beta_{ij}(t-r)\|] \right| \, dr
\]

\[
\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left( \int_0^\infty e^{-\int_0^r \mu_i(t-s) \, ds} \, dr \right) \|\varphi - \psi\| = \kappa \|\varphi - \psi\|,
\]
where

$$\kappa := \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \int_{0}^{\infty} \left\{ \sum_{j=1}^{n} \eta_j [|\alpha_{ij}(t-r)| + |\beta_{ij}(t-r)|] \right\} e^{-\int_{0}^{t} \mu_i(t-s)ds} dr$$

$$< \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \int_{0}^{\infty} \mu_i(t-r)e^{-\int_{0}^{t} \mu_i(t-s)ds} dr = 1,$$

due to equation 3.11, and that $\mu_i(t)$ are positive almost periodic functions of $t \in \mathbb{R}$. Accordingly, $P$ is a contraction mapping. Thus, $P$ has a unique fixed point $\bar{x} = \bar{x}(\cdot) \in \bar{A}^e \cap AP(\mathbb{R}^n)$, that is, $P(\bar{x}) = \bar{x}$, by the contraction mapping principle.

**Step 4.** Finally, we show that $\bar{x} = \bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))$ defined for $t \in \mathbb{R}$ is a solution of equation 1.1. Indeed,

$$\frac{d}{dt} \bar{x}_i(t) = \frac{d}{dt} P_i(\bar{x})(t)$$

$$= \frac{d}{dt} \int_{0}^{\infty} \left[ \sum_{j=1}^{n} \alpha_{ij}(t-r)g_j(\bar{x}_j(t-r)) + \sum_{j=1}^{n} \beta_{ij}(t-r)g_j(\bar{x}_j(t-r) - \tau_{ij}(t-r)) + I_i(t-r) \right] e^{-\int_{0}^{t} \mu_i(t-s)ds} dr$$

$$= \frac{d}{dt} \int_{-\infty}^{t} \left[ \sum_{j=1}^{n} \alpha_{ij}(r)g_j(\bar{x}_j(r)) + \sum_{j=1}^{n} \beta_{ij}(r)g_j(\bar{x}_j(r) - \tau_{ij}(r)) + I_i(r) \right] e^{-\int_{0}^{r} \mu_i(s)ds} dr$$

$$= -\mu_i(t)\bar{x}_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t)g_j(\bar{x}_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t)g_j(\bar{x}_j(t - \tau_{ij}(t))) + I_i(t).$$

Hence system 1.1 admits $2^n$ almost periodic solutions. In addition, each of the $2^n$ regions $\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq \bar{d}_i, \text{ or } x_i \leq \underline{d}_i \}$ contains one of these almost periodic solutions. Moreover, according to theorem 2, these $2^n$ almost periodic solutions are exponentially stable. This completes the proof.

Theorems 2 and 3 yield corollary 1, and the proof of theorem 3 can be modified to justify corollary 2:

**Corollary 1.** Each $\bar{A}^e$ defined in equation 3.2 contains the basin of attraction for the almost periodic solution lying in $\bar{A}^e$.

**Corollary 2.** Let $\mu_i(t)$, $\alpha_{ij}(t)$, $\beta_{ij}(t)$, $\tau_{ij}(t)$ and $I_i(t)$, $i, j = 1, 2, \ldots, n$, be $T$-periodic functions defined for all $t \in \mathbb{R}$. Assume that conditions $H_1$, $H_2$, $H_3$,...
Using equation 2.1, we obtain, for all $t \in \mathbb{R}$, $i = 1, 2, \ldots, n$, hold. Then there exist exactly $2^n$ exponentially stable $T$-periodic solutions for system 1.1.

4 Numerical Illustrations

In this section, we present three examples to illustrate our theory.

4.1 Example 1. Consider the two-dimensional system 1.1 with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$:

\[
\frac{dx_1(t)}{dt} = -\mu_1(t)x_1(t) + \alpha_{11}(t)g_1(x_1(t)) + \alpha_{12}(t)g_2(x_2(t)) + \beta_{11}(t)g_1(x_1(t - \tau_{11}(t))) + \beta_{12}(t)g_2(x_2(t - \tau_{12}(t))) + I_1(t)
\]
\[
\frac{dx_2(t)}{dt} = -\mu_2(t)x_2(t) + \alpha_{21}(t)g_1(x_1(t)) + \alpha_{22}(t)g_2(x_2(t)) + \beta_{21}(t)g_1(x_1(t - \tau_{21}(t))) + \beta_{22}(t)g_2(x_2(t - \tau_{22}(t))) + I_2(t).
\]

The parameters are set as follows:

\[
\begin{align*}
\alpha_{11}(t) &= 4 + 0.3 \sin(t) & \alpha_{12}(t) &= 0.6 + 0.1 \cos(\pi t) \\
\beta_{11}(t) &= 3 + 0.2 \sin(t) & \beta_{12}(t) &= 0.4 + 0.1 \cos(\pi t) \\
\alpha_{21}(t) &= 0.7 + 0.2 \cos(t) & \alpha_{22}(t) &= 5 + 0.5 \sin(\pi t) \\
\beta_{21}(t) &= 0.8 + 0.2 \cos(t) & \beta_{22}(t) &= 7 + 0.5 \sin(\pi t) \\
\mu_1(t) &= 1 + |0.1 \sin(t)| + |0.2 \sin(\pi t)| & \mu_2(t) &= 3 + |0.2 \cos(t)| + |0.1 \cos(\pi t)| \\
I_1(t) &= \cos(t) + \cos(\pi t) & I_2(t) &= \sin(t) + \sin(\sqrt{2}t) \\
\tau_{11}(t) &= \tau_{21}(t) = 10 + \sin(t) + \sin(\pi t) & \tau_{12}(t) &= \tau_{22}(t) = 5 + \cos(\sqrt{2}t)
\end{align*}
\]

Using equation 2.1, we obtain, for all $t \in \mathbb{R}$,

\[
\begin{align*}
3.7 &= \alpha_{11} \leq \alpha_{11}(t) \leq \bar{\alpha}_{11} = 4.3 & 0.5 &= \alpha_{12} \leq \alpha_{12}(t) \leq \bar{\alpha}_{12} = 0.7 \\
2.8 &= \beta_{11} \leq \beta_{11}(t) \leq \bar{\beta}_{11} = 3.2 & 0.3 &= \beta_{12} \leq \beta_{12}(t) \leq \bar{\beta}_{12} = 0.5 \\
0.5 &= \alpha_{21} \leq \alpha_{21}(t) \leq \bar{\alpha}_{21} = 0.9 & 4.5 &= \alpha_{22} \leq \alpha_{22}(t) \leq \bar{\alpha}_{22} = 5.5 \\
0.6 &= \beta_{21} \leq \beta_{21}(t) \leq \bar{\beta}_{21} = 1 & 6.5 &= \beta_{22} \leq \beta_{22}(t) \leq \bar{\beta}_{22} = 7.5 \\
1 &= \mu_1 \leq \mu_1(t) \leq \bar{\mu}_1 = 1.3 & 3 &= \mu_2 \leq \mu_2(t) \leq \bar{\mu}_2 = 3.3 \\
-2 &= L_1 \leq L_1(t) \leq \bar{L}_1 = 2 & -2 &= L_2 \leq L_2(t) \leq \bar{L}_2 = 2
\end{align*}
\]

By equation 2.4, a direct computation gives

\[
\hat{f}_1(x_1) = \begin{cases} 
-x_1 + 7.5g_1(x_1) + 3.2, & x_1 \geq 0 \\
-1.3x_1 + 6.5g_1(x_1) + 3.2, & x_1 < 0 
\end{cases}
\]
Table 1: Local Extreme Points and Zeros of \( \tilde{f}_1, \tilde{f}_1', \tilde{f}_2, \tilde{f}_2'. \)

<table>
<thead>
<tr>
<th>( \bar{a}_1 )</th>
<th>( \bar{p}_1 )</th>
<th>( \bar{b}_1 )</th>
<th>( \bar{q}_1 )</th>
<th>( \bar{c}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2.466999)</td>
<td>(-1.443655)</td>
<td>(-0.768372)</td>
<td>(1.665432)</td>
<td>(10.7)</td>
</tr>
<tr>
<td>(-10.7)</td>
<td>(-1.665432)</td>
<td>(0.768372)</td>
<td>(1.443655)</td>
<td>(2.466999)</td>
</tr>
<tr>
<td>(-2.331622)</td>
<td>(-1.209935)</td>
<td>(-0.440329)</td>
<td>(1.362874)</td>
<td>(5.364478)</td>
</tr>
<tr>
<td>(-5.364478)</td>
<td>(-1.362874)</td>
<td>(0.440329)</td>
<td>(1.209935)</td>
<td>(2.331622)</td>
</tr>
</tbody>
</table>

and

\[
\tilde{f}_2(x_2) = \begin{cases} 
-3x_2 + 13g_2(x_2) + 3.1, & x_2 \geq 0 \\
-3.3x_2 + 11g_2(x_2) + 3.1, & x_2 < 0
\end{cases}
\]

\[
\tilde{f}_2'(x_2) = \begin{cases} 
-3.3x_2 + 11g_2(x_2) - 3.1, & x_2 \geq 0 \\
-3x_2 + 13g_2(x_2) - 3.1, & x_2 < 0
\end{cases}
\]

Herein, the parameters satisfy our conditions in theorem 3:

**Condition H₁**

\[
0 < \frac{\mu_1}{\bar{a}_{11} + \bar{b}_{11}} = \frac{1}{7.5}, \quad \frac{\bar{\mu}_1}{\bar{a}_{11} + \bar{b}_{11}} = \frac{1.3}{6.5} < 1
\]

\[
0 < \frac{\mu_2}{\bar{a}_{22} + \bar{b}_{22}} = \frac{3}{13}, \quad \frac{\bar{\mu}_2}{\bar{a}_{22} + \bar{b}_{22}} = \frac{3.3}{11} < 1
\]

**Condition H₂**

\[
\tilde{f}_1(\bar{p}_1) = -0.737051 < 0, \quad \tilde{f}_1(\bar{q}_1) = 0.737051 > 0
\]

\[
\tilde{f}_2(\bar{p}_2) = -2.110474 < 0, \quad \tilde{f}_2(\bar{q}_2) = 2.110474 > 0
\]

**Condition H₃**

\[
\inf_{t \in \mathbb{R}} \left\{ \mu_1(t) - \eta_1(|\alpha_{11}(t)| + |\beta_{11}(t)|) - \eta_2(|\alpha_{12}(t)| + |\beta_{12}(t)|) \right\} = 0.106 > 0
\]

\[
\inf_{t \in \mathbb{R}} \left\{ \mu_2(t) - \eta_1(|\alpha_{21}(t)| + |\beta_{21}(t)|) - \eta_2(|\alpha_{22}(t)| + |\beta_{22}(t)|) \right\} = 1.25 > 0
\]

where \( \eta_1 = 0.1 > g'(\bar{a}_1) = g'(\bar{c}_1) = 0.028381 \) and \( \eta_2 = 0.12 > g'(\bar{a}_2) = g'(\bar{c}_2) = 0.037041 \). Local extreme points and zeros of \( \tilde{f}_1, \tilde{f}_1', \tilde{f}_2, \tilde{f}_2' \) are listed in Table 1.

The components \( x_1(t), x_2(t) \) of evolutions from several constant initial values and time-dependent initial values are depicted in Figures 3 and 4, respectively. The two-dimensional phase diagram is arranged in Figure 5.
4.2 Example 2. Consider the two-dimensional system 1.1 with activation functions \( g_1(\xi) = g_2(\xi) = \frac{1}{2}(|\xi - 1| + |\xi + 1|) \) and the following parameters:

\[
\begin{align*}
\alpha_{11} (t) &= 2 + 0.2 \cos \left( \frac{t}{2} \right) \\
\beta_{11} (t) &= 3 + 0.2 \cos \left( \frac{t}{2} \right) + 0.1 \sin \left( \frac{\sqrt{2}t}{2} \right) \\
\alpha_{21} (t) &= -1 + 0.2 \sin \left( \frac{t}{2} \right) \\
\beta_{21} (t) &= 2 + 0.3 \sin \left( \frac{t}{2} \right) + 0.3 \cos \left( \frac{\sqrt{2}t}{2} \right) \\
\mu_1 (t) &= 1 + 0.1 \cos \left( \frac{t}{2} \right) + 0.2 \sin \left( \frac{\sqrt{2}t}{2} \right) + 0.1 \sin \left( \frac{t}{2} \right) + 0.3 \sin \left( \frac{\sqrt{2}t}{2} \right) \\
\alpha_{12} (t) &= 1 + 0.1 \sin \left( \frac{\sqrt{2}t}{2} \right) \\
\beta_{12} (t) &= 1 + 0.2 \sin \left( \frac{\sqrt{2}t}{2} \right) \\
\alpha_{22} (t) &= 4 + 0.5 \cos \left( \frac{\sqrt{2}t}{2} \right) \\
\beta_{22} (t) &= 5 + 0.3 \sin \left( \frac{t}{2} \right) + 0.3 \cos \left( \frac{\sqrt{2}t}{2} \right)
\end{align*}
\]
These parameters satisfy the conditions analogous to the ones of theorem 3, as adapted to the activation functions considered here. The evolutions of components $x_1(t), x_2(t)$ are depicted in Figures 6 and 7, respectively.

4.3 Example 3. Consider the three-dimensional system 1.1 with activation functions $g_j(\xi) = g(\xi) := \frac{1}{1+e^{-\xi}}, \quad \varepsilon = 0.5 > 0, \quad j = 1, 2, 3$. Let $\alpha_{ij} = 0, \quad i, j = 1, 2, 3$, that is, the three-dimensional nonautonomous delayed Hopfield neural networks,

$$
\frac{dx_1(t)}{dt} = -\mu_1(t)x_1(t) + \beta_{11}(t)g_1(x_1(t - \tau_{11}(t))) + \beta_{12}(t)g_2(x_2(t - \tau_{12}(t))) \\
+ \beta_{13}(t)g_3(x_3(t - \tau_{13}(t))) + I_1(t)
$$
Figure 5: Phase portrait for example 1.

\[
\frac{dx_2(t)}{dt} = -\mu_2(t)x_2(t) + \beta_{21}(t)g_1(x_1(t - \tau_{21}(t))) + \beta_{22}(t)g_2(x_2(t - \tau_{22}(t))) \\
+ \beta_{23}(t)g_3(x_3(t - \tau_{23}(t))) + I_2(t)
\]

\[
\frac{dx_3(t)}{dt} = -\mu_3(t)x_3(t) + \beta_{31}(t)g_1(x_1(t - \tau_{31}(t))) + \beta_{32}(t)g_2(x_2(t - \tau_{32}(t))) \\
+ \beta_{33}(t)g_3(x_3(t - \tau_{33}(t))) + I_3(t).
\]

The parameters are set as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_1(t))</td>
<td>(1 + 0.1\sin(t))</td>
</tr>
<tr>
<td>(\mu_2(t))</td>
<td>(1 + 0.1\cos(t))</td>
</tr>
<tr>
<td>(\mu_3(t))</td>
<td>(3 + 0.2\sin(t))</td>
</tr>
<tr>
<td>(\beta_{11}(t))</td>
<td>(19 + \sin(t))</td>
</tr>
<tr>
<td>(\beta_{12}(t))</td>
<td>(1 + \cos(t))</td>
</tr>
<tr>
<td>(\beta_{13}(t))</td>
<td>(\sin(t))</td>
</tr>
<tr>
<td>(\beta_{21}(t))</td>
<td>(1 + \sin(t))</td>
</tr>
<tr>
<td>(\beta_{22}(t))</td>
<td>(22 + \cos(t) + \sin(t))</td>
</tr>
<tr>
<td>(\beta_{23}(t))</td>
<td>(1 + \sin(t))</td>
</tr>
<tr>
<td>(\beta_{31}(t))</td>
<td>(\sin(t))</td>
</tr>
<tr>
<td>(\beta_{32}(t))</td>
<td>(1 + \cos(t))</td>
</tr>
<tr>
<td>(\beta_{33}(t))</td>
<td>(32 + 2\sin(t))</td>
</tr>
<tr>
<td>(I_1(t))</td>
<td>(-9 + \sin(t))</td>
</tr>
<tr>
<td>(I_2(t))</td>
<td>(-10 + \cos(t))</td>
</tr>
<tr>
<td>(I_3(t))</td>
<td>(-15 + \sin(t))</td>
</tr>
<tr>
<td>(\tau_{k1}(t))</td>
<td>(10 + \sin(t))</td>
</tr>
<tr>
<td>(\tau_{k2}(t))</td>
<td>(20 + \cos(t))</td>
</tr>
<tr>
<td>(\tau_{k3}(t))</td>
<td>(30 + \sin(t))</td>
</tr>
</tbody>
</table>
Figure 6: Evolution of state variable $x_1(t)$ in example 2.

Figure 7: Evolution of state variable $x_2(t)$ in example 2.
Figure 8: The dynamics in example 3.

where \( k = 1, 2, 3 \). These parameters satisfy the conditions analogous to the ones of corollary 2, as adapted to the activation functions considered here. Hence, there are eight stable periodic solutions. The dynamics of the system are illustrated in Figure 8.

5 Conclusion

We have established a new methodology for investigating multistability and multiperiodicity for general nonautonomous neural networks with delays. The presentation is mainly formulated for the system with activation function 2.3. All the conditions of the theorems can be modified to adapt to other typical activation functions. Our approach, which makes use of the geometric configuration of network structure, is effective and illuminating. The treatment is expected to contribute toward exploring further fruitful dynamics in neural networks.

Acknowledgments

We are grateful to the reviewers for their comments and suggestions, which led to an improvement of this presentation. This work is partially supported by the National Science Council, the National Center of Theoretical Sciences, and the MOEATU program, of R.O.C. on Taiwan.


Received September 3, 2006; accepted October 29, 2006.