1. Prove: if $x \geq 0$ and $y \geq 0$ are normalized floating-point binary numbers such that $x > y$ and

$$2^{-q} \leq 1 - \frac{y}{x} \leq 2^{-p},$$

then at most $q$ and at least $p$ significant binary digits are lost in the subtraction $x - y$.

2. Prove: let $x_0, x_1, \ldots, x_n$ be positive floating-point numbers, then the relative error in computing $\sum_{i=0}^{n} x_i$ in the usual way is at most $(1 + \varepsilon_M)^n - 1 \approx n \varepsilon_M$, where $\varepsilon_M$ is the machine unit roundoff.

3. If $x, y$ and $z$ are not machine numbers, then $fl(x(y + z)) = x(y + z)(1 + \delta)$, $|\delta| \leq \alpha \varepsilon_M$. To find $\alpha =$?

4. Let $f(x) = (((x - 0.5) + x) - 0.5) + x$. Show that if $f$ is valued as shown in single or double precision binary IEEE arithmetic then $f(x) \neq 0$ for all floating point numbers $x$.

5. Show how to rewrite the following expressions to avoid cancellation for the indicated arguments.
   
   (a) $\sqrt{x+1} - 1$, $x \approx 0$
   (b) $\sin x - \sin y$, $x \approx y$.
   (c) $x^2 - y^2$, $x \approx y$.
   (d) $(1 - \cos x)/\sin x$, $x \approx 0$.
   (e) $c = (a^2 + b^2 - 2ab \cos \theta)^{\frac{1}{2}}$, $a \approx b$, $|\theta| \ll 1$.

6. Let $x$ and $y$ be a floating point number in IEEE double precision arithmetic.

   (a) Find some $x$ in the range $1 < x < 2$ such that $x \times (1/x) \neq 1$, that is, $fl(x \times fl(1/x))$ is not exactly 1.
   (b) Find the smallest such number in the range $1 < x < 2$.
   (c) Are there any floating point values of $x$ and $y$ (excepting values both 0, or so huge or tiny to cause overflow or underflow) for which the computed value of $x / \sqrt{x^2 + y^2}$ exceeds 1?
7. Compute the dot product of the following two vectors:

\[ x = [2.718281828, -3.141592654, 1.414213562, 0.5772156649, 0.3010299957], \]
\[ y = [1486.2497, 878366.9879, -22.37492, 4773714.647, 0.000185049]. \]

Evaluate the summation in four ways:

(a) Forward order.

(b) Reverse order.

(c) Largest-to-smallest order (add positive numbers in order from largest to smallest, then add negative numbers in order from smallest to largest, and then add the two partial sums).

(d) Smallest-to-largest order (reverse the order of adding in the previous method).

8. Consider the recurrence

\[ x_{k+1} = 111 - (1130 - 3000/x_{k-1})/x_k, \quad x_0 = 11/2, \quad x_1 = 61/11. \]

In exact arithmetic the \( x_k \) form a monotonically increasing sequence that converges to 6. Implement the recurrence on your computer or pocket calculator and compare the computed \( x_{34} \) with the true value 5.998 (to four correct significant figures). Explain what you see.