1. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be strictly diagonally dominant if for $i = 1, 2, \ldots, n$

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|.$$ 

Prove that if $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant, then $A$ is nonsingular.

2. Derive the method for approximating $f''(x)$ that if $f^{(4)}(x)$ is continuous, then

$$f''(x) = \frac{1}{h^2} [f(x + h) - 2f(x) + f(x - h)] - \frac{h^2}{12} f^{(4)}(\xi),$$

where $\xi$ is between $x - h$ and $x + h$.

3. Consider the function

$$e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6} M,$$

where $M$ is a bound for the third derivative of a function. Show that $e(h)$ has a minimum at $\sqrt[3]{3\varepsilon/M}$.

4. Richardson’s extrapolation with centered difference formula is used repeatedly to estimate $f'(x)$. Test your program on the following:

   (a) $\ln(x)$ at $x = 3$;
   (b) $\tan(x)$ at $x = \sin(0.8)$;
   (c) $\sin(x^2 + \frac{1}{3}x)$ at $x = 0$.

5. Carry out the Romberg Integration for a function $f$ defined on an arbitrary interval $[a, b]$ and test your Romberg program on these three examples:

   (a) $\int_{0}^{1} \frac{\sin x}{x} dx$;
   (b) $\int_{-1}^{1} \frac{\cos x - e^x}{\sin x} dx$;
   (c) $\int_{1}^{\infty} (xe^x)^{-1} dx$.

The routines for these integrals should be written to avoid serious loss of significance due to subtraction. Also, it is customary to defined a function $f$ at any questionable point $x_0$ by the equation

$$f(x_0) = \lim_{x \to x_0} f(x).$$

If the limit exists, this method guarantees continuity of $f$ at $x_0$. For the third example, make a suitable change variable, such as $x = 1/t$. Compute seven rows in the Romberg array. Print the array in each case with a format that enables the convergence to be observed.