Some Results on Sign-balance of Restricted Permutations

Tung-Shan Fu

Pingtung Institute of Commerce

based on joint work with S.-P. Eu, Y.-J. Pan, C.-T. Ting
outline
outline
outline

- backgrounds on sign-balances of restricted permutations
outline

- backgrounds on sign-balance of restricted permutations
- 321-avoiding alternating permutations and plane trees
outline

• backgrounds on sign-balances of restricted permutations

• 321-avoiding alternating permutations and plane trees

• simsun permutations and increasing 1-2 trees
Given a permutation \( \sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n \), define the \textit{inversion number} of \( \sigma \) as

\[
\text{inv}(\sigma) = \left| \left\{ (\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j \right\} \right|. 
\]
Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, define the \textit{inversion number} of $\sigma$ as

$$\text{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j\}|.$$ 

Let $\text{sign}(\sigma)$ denote the \textit{sign} of $\sigma$, defined as

$$\text{sign}(\sigma) = (-1)^{\text{inv}(\sigma)}.$$
Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, define the \textit{inversion number} of $\sigma$ as

$$\text{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j\}|.$$  

Let $\text{sign}(\sigma)$ denote the \textit{sign} of $\sigma$, defined as

$$\text{sign}(\sigma) = (-1)^{\text{inv}(\sigma)}.$$  

The \textit{sign-balance} of $S_n$ is 0, i.e.,

$$\sum_{\sigma \in S_n} \text{sign}(\sigma) = 0.$$  

321-avoiding permutations

Let $\mathcal{S}_n(321) \subseteq \mathcal{S}_n$ be the subset of permutations without decreasing subsequences of length three.
321-avoiding permutations

Let $\mathcal{S}_n(321) \subseteq \mathcal{S}_n$ be the subset of permutations without decreasing subsequences of length three. It is known that

$$|\mathcal{S}_n(321)| = C_n,$$

where $C_n = \frac{1}{2n+1} \binom{2n}{n}$ is the $n$th Catalan number.
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Theorem (Simion-Schmidt)

$$\sum_{\sigma \in \mathcal{S}_n(321)} \text{sign}(\sigma) = \begin{cases} C_{\frac{1}{2}(n-1)} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$
a refined-sign-balance result

Let \( \text{Ides}(\sigma) = \max\{i : \sigma_i > \sigma_{i+1}, 1 \leq i \leq n - 1\} \).
a refined-sign-balance result

Let \( \text{ldes}(\sigma) = \max\{i : \sigma_i > \sigma_{i+1}, 1 \leq i \leq n - 1\} \).

Theorem (Adin-Roichman 2004)

\[
\sum_{\sigma \in S_{2n+1}(321)} \text{sign}(\sigma) q^{\text{ldes}(\sigma)} = \sum_{\sigma \in S_n(321)} q^{2 \cdot \text{ldes}(\sigma)}.
\]

\[
\sum_{\sigma \in S_{2n}(321)} \text{sign}(\sigma) q^{\text{ldes}(\sigma)} = (1 - q) \sum_{\sigma \in S_n(321)} q^{2 \cdot \text{ldes}(\sigma)}.
\]
another refined-sign-balance result

Let $\text{lis}(\sigma)$ be the length of longest increasing subsequence in $\sigma$. 
another refined-sign-balance result

Let \( \text{lis}(\sigma) \) be the length of longest increasing subsequence in \( \sigma \).

**Theorem (Reifegerste 2005)**

\[
\sum_{\sigma \in \mathcal{S}_{2n+1}(321)} \text{sign}(\sigma) q^{\text{lis}(\sigma)} = q \sum_{\sigma \in \mathcal{S}_n(321)} q^{2 \cdot \text{lis}(\sigma)}.
\]

\[
\sum_{\sigma \in \mathcal{S}_{2n+2}(321)} \text{sign}(\sigma) q^{\text{lis}(\sigma)} = q(q - 1) \sum_{\sigma \in \mathcal{S}_n(321)} q^{2 \cdot \text{lis}(\sigma)}.
\]
their distributions
their distributions

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Ides            lis
321-avoiding alternating permutations
alternating permutations
A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is \textit{alternating} if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$. 
A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is *alternating* if
$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.

Let $\text{Alt}_n(321)$ denote the subset of 321-avoiding alternating permutations in $S_n$. 
A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is *alternating* if
$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.

Let $\text{Alt}_n(321)$ denote the subset of 321-avoiding
alternating permutations in $S_n$.

It is known that

$$|\text{Alt}_{2n}(321)| = |\text{Alt}_{2n-1}(321)| = C_n.$$
sign-balance of $\text{Alt}_n(321)$
sign-balance of $\text{Alt}_n(321)$

An observation:

$$\sum_{\sigma \in \text{Alt}_{2n}(321)} \text{sign}(\sigma) = - \sum_{\sigma \in \text{Alt}_{2n-1}(321)} \text{sign}(\sigma)$$

$$= \begin{cases} (-1)^{\frac{n+1}{2}} C_{\frac{1}{2}}(n-1) & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$
a refined-sign-balance of $Alt_n(321)$
a refined-sign-balance of $\text{Alt}_n(321)$

Let $\text{lead}(\sigma)$ denote the first entry of $\sigma$. 
Let $\text{lead}(\sigma)$ denote the first entry of $\sigma$.

**Theorem (Eu, Fu, Pan, Ting 2012)**

\[
\sum_{\sigma \in \text{Alt}_{4n+2}(321)} \text{sign}(\sigma) \cdot q^{\text{lead}(\sigma)} = (-1)^{n+1} \sum_{\sigma \in \text{Alt}_{2n}(321)} q^{2 \cdot \text{lead}(\sigma)}
\]

\[
\sum_{\sigma \in \text{Alt}_{4n}(321)} \text{sign}(\sigma) \cdot q^{\text{lead}(\sigma)} = (-1)^{n+1} (1 - q) \sum_{\sigma \in \text{Alt}_{2n}(321)} q^{2(\text{lead}(\sigma) - 1)}
\]
plane trees and Dyck paths
plane trees and Dyck paths

Let $\mathcal{T}_n$ be the set of plane trees with $n$ edges.
plane trees and Dyck paths

Let $\mathcal{T}_n$ be the set of plane trees with $n$ edges.

Figure: The trees in $\mathcal{T}_3$. 
plane trees and Dyck paths

Let $\mathcal{T}_n$ be the set of plane trees with $n$ edges.

Figure: The trees in $\mathcal{T}_3$.

Figure: Dyck paths of length 3.
a bijection between $\text{Alt}_{2n}(321)$ and $\mathcal{T}_n$
a bijection between $\text{Alt}_{2n}(321)$ and $\mathcal{T}_n$

Let $\sigma = 3 \ 1 \ 6 \ 2 \ 7 \ 4 \ 8 \ 5 \in \text{Alt}_8(321)$. 
a bijection between $\text{Alt}_{2n}(321)$ and $\mathcal{T}_n$

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a bijection between \( \text{Alt}_{2n}(321) \) and \( \mathcal{T}_n \)

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a bijection between $\operatorname{Alt}_{2n}(321)$ and $\mathcal{T}_n$

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a bijection between $\text{Alt}_{2n}(321)$ and $\mathcal{T}_n$

Let $\sigma = 3 \ 1 \ 6 \ 2 \ 7 \ 4 \ 8 \ 5 \in \text{Alt}_8(321)$. 

\[ \begin{array}{ccccccc}
  & & & & 6 & & 7 & 8 \\
 & & & 5 & & & 8 & 5 \\
 & 3 & & 4 & & 7 & 4 \\
2 & & 6 & & 2 & & 3 \\
1 & 1 & & & & & \\
3 & & & & & & \\
\end{array} \]
a bijection between $\text{Alt}_{2n}(321)$ and $\mathcal{T}_n$

Let $\sigma = 3 \ 1 \ 6 \ 2 \ 7 \ 4 \ 8 \ 5 \in \text{Alt}_8(321)$.

\[
\text{inv}(\sigma) = \text{hsum}(T), \text{ sum of vertex-heights of } T
\]
\[
\text{lead}(\sigma) = \text{Imp}(T), \# \text{ vertices in the leftmost path}
\]
legal trees
legal trees

A **legal tree** satisfies the condition:

- every leaf is the first child of its parent;
- every internal node is not the first child.
legal trees

A legal tree satisfies the condition:
  • every leaf is the first child of its parent;
  • every internal node is not the first child.

Let \( G_n \) be the set of legal trees with \( n \) edges.
legal trees

A legal tree satisfies the condition:

- every leaf is the first child of its parent;
- every internal node is not the first child.

Let $G_n$ be the set of legal trees with $n$ edges.

Construct $G_{2n+1}$ from $T_n$. 
a sign-reversing involution on $T_{2n+1} - G_{2n+1}$ by Chen-Shapiro-Yang
a sign-reversing involution on $\mathcal{T}_{2n+1} - \mathcal{G}_{2n+1}$ by Chen-Shapiro-Yang
a sign-reversing involution on $\mathcal{T}_{2n+1} - \mathcal{G}_{2n+1}$ by Chen-Shapiro-Yang

$v$ is the last illegal vertex in right-to-left preorder. $u$ is the parent of $v$. 
sign-balance of $\text{Alt}_{4n+2}$
We have proved

$$\sum_{\sigma \in \text{Alt}_{4n+2}} \text{sign}(\sigma) = \sum_{T \in \mathcal{T}_{2n+1}} \text{hsum}(T) = (-1)^{n+1} C_n.$$
sign-balance of $\text{Alt}_{4n+2}$

We have proved

$$
\sum_{\sigma \in \text{Alt}_{4n+2}} \text{sign}(\sigma) = \sum_{T \in \mathcal{T}_{2n+1}} \text{hsum}(T) = (-1)^{n+1} C_n.
$$

$$
\sum_{\sigma \in \text{Alt}_{4n}} \text{sign}(\sigma) = \sum_{T \in \mathcal{T}_{2n}} \text{hsum}(T) = 0.
$$
refined sign-balance of $\text{Alt}_{4n+2}$
refined sign-balance of $\text{Alt}_{4n+2}$

We want to prove

$$\sum_{T \in \mathcal{T}_{2n+1}} \text{hsum}(T) q^{\text{imp}(T)} = (-1)^{n+1} \sum_{T \in \mathcal{T}_n} q^{2 \cdot \text{imp}(T)}.$$
refined sign-balance of $\text{Alt}_{4n+2}$

We want to prove

$$\sum_{T \in \mathcal{T}_{2n+1}} \text{hsum}(T)q^{\text{imp}(T)} = (-1)^{n+1} \sum_{T \in \mathcal{T}_n} q^{2 \cdot \text{imp}(T)}.$$

$$\sum_{T \in \mathcal{T}_{2n}} \text{hsum}(T)q^{\text{imp}(T)} = (-1)^{n+1}(1 - q) \sum_{T \in \mathcal{T}_n} q^{2(\text{imp}(T)-1)}.$$
generalized legal trees
generalized legal trees
picture of the map $\Phi$ on $T_5$
picture of the map $\Phi$ on $\mathcal{T}_5$

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$\Phi$
simsun permutations
descents of a permutation
descents of a permutation

\( \mathfrak{S}_n \): the set of permutations of \([n] = \{1, \ldots, n\}\).
descents of a permutation

\( \mathfrak{S}_n \): the set of permutations of \( [n] = \{1, \ldots, n\} \).

Let \( \sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n \). A \textit{descent} of \( \sigma \) is a pair \((\sigma_i, \sigma_{i+1})\) with \( \sigma_i > \sigma_{i+1} \) for some \( i \in [n-1] \).
descents of a permutation

\( S_n \): the set of permutations of \([n] = \{1, \ldots, n\} \).

Let \( \sigma = \sigma_1 \cdots \sigma_n \in S_n \). A descent of \( \sigma \) is a pair \((\sigma_i, \sigma_{i+1})\) with \( \sigma_i > \sigma_{i+1} \) for some \( i \in [n-1] \).

A double descent of \( \sigma \) a triple \((\sigma_i, \sigma_{i+1}, \sigma_{i+2})\) with \( \sigma_i > \sigma_{i+1} > \sigma_{i+2} \) for some \( i \in [n-2] \).
simsun permutations
simsun permutations

The permutation $\sigma$ is called \textit{simsun} if for all $k$, the subword of $\sigma$ restricted to $\{1, \ldots, k\}$ (in the order they appear in $\sigma$) contains no double descents.
simsun permutations

The permutation $\sigma$ is called *simsun* if for all $k$, the subword of $\sigma$ restricted to \{1, $\ldots$, $k$\} (in the order they appear in $\sigma$) contains no double descents.

For example, 42351 is simsun,
simsun permutations

The permutation $\sigma$ is called \textit{simsun} if for all $k$, the subword of $\sigma$ restricted to \{1, \ldots, $k$\} (in the order they appear in $\sigma$) contains no double descents.

For example, 42351 is simsun, but 24351 is not.
Euler number
Euler number

Let $\mathcal{R}_n$ be the set of simsun permutations in $\mathcal{S}_n$. 
Euler number

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Simion and Sundaram proved that

$$|\mathcal{R}_n| = E_{n+1},$$
Euler number

Let $\mathcal{R}_n$ be the set of simsun permutations in $\mathfrak{S}_n$.

Simion and Sundaram proved that

$$|\mathcal{R}_n| = E_{n+1},$$

where $E_n$ is the $n$th Euler number and satisfies

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$
sign-balance of $\mathcal{R}_n$
An observation

1. \[ \sum_{\sigma \in \mathcal{R}_{2n-1}} \text{sign}(\sigma) = (2n - 3)!!. \]
sign-balance of $\mathcal{R}_n$

An observation

1. $\sum_{\sigma \in \mathcal{R}_{2n-1}} \text{sign}(\sigma) = (2n - 3)!!.$

2. $\sum_{\sigma \in \mathcal{R}_{2n}} \text{sign}(\sigma) = 0.$
a refined sign-balance for $\mathcal{R}_{2n-1}$
Consider the sign-balance of simsun permutations, refined by descent numbers. Define

\[ A_n(y) = \sum_{\sigma \in \mathcal{R}_{2n-1}} \text{sign}(\sigma) y^{\text{des}(\sigma)}. \]
a refined sign-balance for $\mathcal{R}_{2n-1}$

Consider the sign-balance of simsun permutations, refined by descent numbers. Define

$$A_n(y) = \sum_{\sigma \in \mathcal{R}_{2n-1}} \text{sign}(\sigma) y^{\text{des} (\sigma)}.$$

Some initial polynomials

$$A_1(y) = 1$$
$$A_2(y) = 1$$
$$A_3(y) = 1 + 2y$$
$$A_4(y) = 1 + 8y + 6y^2$$
$$A_5(y) = 1 + 22y + 58y^2 + 24y^3.$$
second-order Eulerian triangle
second-order Eulerian triangle

Let $\{\begin{array}{c} n \\ k \end{array}\}$ denote the second-order Eulerian numbers.
second-order Eulerian triangle

Let \( \{\binom{n}{k}\} \) denote the second-order Eulerian numbers.

They satisfy the recurrence relation

\[
\{\binom{n}{k}\} = (k + 1)\left\{\binom{n-1}{k}\right\} + (2n - k - 1)\left\{\binom{n-1}{k-1}\right\},
\]

with initial conditions \( \{\binom{0}{0}\} = 1 \) and \( \{\binom{0}{k}\} = 0, \ k \neq 0. \)
increasing 1-2 trees
increasing 1-2 trees

Let $\mathcal{T}_n$ be the set of increasing 1-2 trees on $[0, n]$. 

```
0
 |   |
1   1
 |   |
2   2
 |   |
3   3

0
 |   |
0   1
 |   |
3   2
 |   |
0   1

0
 |   |
0   1
 |   |
3   2
 |   |
1   1

0
 |   |
0   1
 |   |
3   2
 |   |
1   3
```
canonical form of increasing 1-2 trees
canonical form of increasing 1-2 trees

- If a vertex $x$ has two children $u, v$ with $u > v$ then $u$ is the left child, and $v$ is the right child.

- If $x$ has only one child then it is the right child of $x$. 
a bijection $\phi : \mathcal{T}_n \to \mathcal{R}_n$
a bijection $\phi : \mathcal{T}_n \rightarrow \mathcal{R}_n$

Theorem

There is a bijection $\phi : \mathcal{T}_n \rightarrow \mathcal{R}_n$ such that a tree $T \in \mathcal{T}_n$ with $k + 1$ leaves is carried to a permutation $\phi(T) \in \mathcal{R}_n$ with $k$ descents.
the construction of $\phi : \mathcal{T}_n \rightarrow \mathcal{R}_n$
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the construction of $\phi : \mathcal{T}_n \rightarrow \mathcal{R}_n$

\begin{array}{cccc}
\text{stages} & \text{subwords} \\
(a) & 3846 \\
\end{array}
the construction of \( \phi : \mathcal{T}_n \rightarrow \mathcal{R}_n \)

\[
\begin{array}{c}
\text{stages} & \text{subwords} \\
(a) & 3846 \\
(b) & 38461
\end{array}
\]
the construction of $\phi : \mathcal{T}_n \to \mathcal{R}_n$

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(a)  
(b)  
(c)  
(d)
the construction of $\phi : \mathcal{T}_n \rightarrow \mathcal{R}_n$

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vertex-exchangeable trees
**vertex-exchangeable trees**

Given a tree $T \in \mathcal{T}_n$ with $\sigma = \phi(T) \in \mathcal{R}_n$, let $s_i(T)$ be the tree obtained from $T$ by interchanging the vertex-labels $i$ and $i + 1$. 
vertex-exchangeable trees

Given a tree $T \in \mathcal{T}_n$ with $\sigma = \phi(T) \in \mathcal{R}_n$, let $s_i(T)$ be the tree obtained from $T$ by interchanging the vertex-labels $i$ and $i + 1$.

(a)

\begin{align*}
&\begin{array}{c}
0 \\
5
\end{array} \\
&\begin{array}{ccc}
1 & 2 & 3 \\
6 & 4 & 7
\end{array}
\end{align*}

(b)

\begin{align*}
&\begin{array}{c}
0 \\
4
\end{array} \\
&\begin{array}{ccc}
1 & 2 & 3 \\
5 & 6 & 7
\end{array}
\end{align*}
Given a tree $T \in \mathcal{T}_n$ with $\sigma = \phi(T) \in \mathcal{R}_n$, let $s_i(T)$ be the tree obtained from $T$ by interchanging the vertex-labels $i$ and $i + 1$. $T$ is called $(i, i + 1)$-exchangeable if $s_i(T) \in \mathcal{T}_n$ is in the canonical form.
partitioning the set $\mathcal{T}_{2n-1}$
partitioning the set $\mathcal{T}_{2n-1}$

A tree $T \in \mathcal{T}_{2n-1}$ is in $\mathcal{T}_{2n-1}^{(1)}$ if there exists a number $i$ ($1 \leq i \leq n - 1$) such that $T$ is $(2i, 2i + 1)$-exchangeable; otherwise, it is in $\mathcal{T}_{2n-1}^{(2)}$. 
partitioning the set $\mathcal{T}_{2n-1}$

A tree $T \in \mathcal{T}_{2n-1}$ is in $\mathcal{T}_{2n-1}^{(1)}$ if there exists a number $i \ (1 \leq i \leq n - 1)$ such that $T$ is $(2i, 2i + 1)$-exchangeable; otherwise, it is in $\mathcal{T}_{2n-1}^{(2)}$.

Therefore, we partition the set $\mathcal{R}_{2n-1}$ into two subsets

$$\mathcal{R}_{2n-1}^{(1)} = \{ \phi(T) : T \in \mathcal{T}_{2n-1}^{(1)} \}$$

$$\mathcal{R}_{2n-1}^{(2)} = \{ \phi(T) : T \in \mathcal{T}_{2n-1}^{(2)} \}. $$
an immediate observation
an immediate observation

Proposition

The following identity holds.

\[
\sum_{\sigma \in \mathcal{R}_{2n-1}^{(1)}} \text{sign}(\sigma) q^{\text{des}(\sigma)} = 0.
\]
an immediate observation

Proposition

The following identity holds.

\[ \sum_{\sigma \in \mathcal{R}_{2n-1}^{(1)}} \text{sign}(\sigma) q^{\text{des}(\sigma)} = 0. \]

\( \mathcal{R}_{2n-1} \) has the same refined sign-balance as \( \mathcal{R}_{2n-1}^{(2)} \).
characterization of the trees in $\mathcal{T}^{(2)}_{2n-1}$
characterization of the trees in $\mathcal{T}_{2n-1}^{(2)}$

Lemma

$T \in \mathcal{T}_{2n-1}^{(2)}$ if and only if for every $i$ $(1 \leq i \leq n - 1)$ the vertex $2i + 1$ is either the right child or the sibling of the vertex $2i$ in $T$. 
Lemma

$T \in \mathcal{T}_{2n-1}^{(2)}$ if and only if for every $i$ ($1 \leq i \leq n-1$) the vertex $2i + 1$ is either the right child or the sibling of the vertex $2i$ in $T$. 

(a) $T_0$

(b) $T_1$ $T_2$ $T_3$
inductive construction of $\mathcal{T}^{(2)}_{2n-1}$
inductive construction of $T_{2n-1}^{(2)}$

<table>
<thead>
<tr>
<th>leaf($T$)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign($\sigma$)</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_3^{(2)}$</th>
<th>0 1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_5^{(2)}$</td>
<td>0 1 2 3 4 5</td>
</tr>
</tbody>
</table>

Figure: The trees of $T_3^{(2)}$ and $T_5^{(2)}$. 
recurrence relation for the size of $\tau^{(2)}_{2n-1}$
recurrence relation for the size of $\mathcal{T}_{2n-1}^{(2)}$

$$|\mathcal{T}_{2n+1}^{(2)}| = (2n + 1)|\mathcal{T}_{2n-1}^{(2)}|.$$
recurrence relation for the size of $\mathcal{T}^{(2)}_{2n-1}$

$$|\mathcal{T}^{(2)}_{2n+1}| = (2n + 1)|\mathcal{T}^{(2)}_{2n-1}|.$$ 

Along with the initial value $|\mathcal{T}^{(2)}_1| = 1$, we have

$$|\mathcal{T}^{(2)}_{2n+1}| = (2n + 1)!!.$$
sign-balance of $\mathcal{T}_{2n-1}^{(2)}$
sign-balance of $\mathcal{T}_{2n-1}^{(2)}$

For $k \geq 0$, let $\mathcal{R}_{2n-1,k}^{(2)} = \{ \sigma \in \mathcal{R}_{2n-1}^{(2)} : \text{des}(\sigma) = k \}$. 
sign-balance of $\mathcal{T}_{2n-1}^{(2)}$

For $k \geq 0$, let $\mathcal{R}_{2n-1,k}^{(2)} = \{ \sigma \in \mathcal{R}_{2n-1}^{(2)} : \text{des}(\sigma) = k \}$.

We partition the even and odd permutations in $\mathcal{R}_{2n-1,k}^{(2)}$ into two subsets $\mathcal{E}_{2n-1,k}^{(2)}$ and $\mathcal{O}_{2n-1,k}^{(2)}$, respectively.
sign-balance of $\mathcal{T}^{(2)}_{2n-1}$

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**Proposition**

The following recurrence relations hold.

\[
|\mathcal{E}^{(2)}_{2n+1,k}| = (k + 1)|\mathcal{E}^{(2)}_{2n-1,k}| + (2n - k)|\mathcal{E}^{(2)}_{2n-1,k-1}| + |\mathcal{O}^{(2)}_{2n-1,k-1}|, \\
|\mathcal{O}^{(2)}_{2n+1,k}| = (k + 1)|\mathcal{O}^{(2)}_{2n-1,k}| + (2n - k)|\mathcal{O}^{(2)}_{2n-1,k-1}| + |\mathcal{E}^{(2)}_{2n-1,k-1}|.
\]
shifted second-order Eulerian numbers
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\[ A_n(y) = \sum_{\sigma \in \mathcal{R}_{2n-1}^{(1)}} \text{sign}(\sigma)y^\text{des}(\sigma) + \sum_{\sigma \in \mathcal{R}_{2n-1}^{(2)}} \text{sign}(\sigma)y^\text{des}(\sigma) \]
shifted second-order Eulerian numbers

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\[ = \sum_{k \geq 0} (|E_{2n-1,k}^{(2)}| - |O_{2n-1,k}^{(2)}|) y^k. \]
shifted second-order Eulerian numbers

\[ A_n(y) = \sum_{\sigma \in \mathcal{R}^{(1)}_{2n-1}} \text{sign}(\sigma) y^{\text{des}(\sigma)} + \sum_{\sigma \in \mathcal{R}^{(2)}_{2n-1}} \text{sign}(\sigma) y^{\text{des}(\sigma)} \]

\[ = \sum_{\sigma \in \mathcal{R}^{(2)}_{2n-1}} \text{sign}(\sigma) y^{\text{des}(\sigma)} \]

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Let \( a_{n,k} = |\mathcal{E}_{2n-1,k}^{(2)}| - |\mathcal{O}_{2n-1,k}^{(2)}|. \)
shifted second-order Eulerian numbers

\[ A_n(y) = \sum_{\sigma \in R_{2n-1}^{(1)}} \text{sign} (\sigma) y^{\text{des} (\sigma)} + \sum_{\sigma \in R_{2n-1}^{(2)}} \text{sign} (\sigma) y^{\text{des} (\sigma)} \]

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Let \( a_{n,k} = |E_{2n-1,k}^{(2)}| - |O_{2n-1,k}^{(2)}| \). Then \( a_{n,k} \) satisfies the recurrence relation

\[ a_{n,k} = (k + 1) a_{n-1,k} + (2n - 3 - k) a_{n-1,k-1}. \]
results for odd-length sets
results for odd-length sets

Multiplying the recurrence relation for $a_{n,k}$ by $y^k$ and summing over $k \geq 1$, we derive the recurrence relation for $A_n(y)$

$$A_n(y) = (1 + (2n - 4)y)A_{n-1}(y) - (y^2 - y)A'_{n-1}(y).$$
exponential generating function for $A_n(y)$
exponential generating function for $A_n(y)$

Let $A_0(y) = 1$. Define

$$A = A(x, y) = \sum_{n \geq 0} A_n(y) \frac{x^n}{n!}.$$
exponential generating function for $A_n(y)$

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Multiplying the recurrence relation for $A_n(y)$ by $\frac{x^{n-1}}{(n-1)!}$ and summing over $n \geq 2$, we obtain the partial differential equation for $A(x, y)$

$$\left(\frac{1}{y} - 2x\right) \frac{\partial A}{\partial x} + (y - 1) \frac{\partial A}{\partial y} = \left(\frac{1}{y} - 2\right) A + 2.$$
solve the PDE by the method of characteristic
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Solving the differential equation

\[
\frac{dx}{dy} = \frac{y^{-1} - 2x}{y - 1}
\]

yields the characteristic curves \( ye^{x(y-1)^2-y} = c_1 \).
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Solving the differential equation

\[
\frac{dA}{dy} = \frac{(y^{-1} - 2)A + 2}{y - 1}
\]

yields \( (y^2 - y)A - y^2 = c_2 \).
the solution of the PDE
the solution of the PDE

The general solution of this PDE is $c_2 = f(c_1)$, where $f$ is an arbitrary function, i.e.,

$$(y^2 - y)A(x, y) - y^2 = f(ye^{x(y-1)^2-y}).$$
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By the initial value \( A(0, y) = 1 \), \(-y = f(y e^{-y})\).
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By the initial value \( A(0, y) = 1, \quad -y = f(ye^{-y}) \).

We derive the e.g.f for \( A_n(y) \)

\[
A(x, y) = \frac{y^2 - T(ye^{x(y-1)^2-y})}{y^2 - y},
\]

where \( T(z) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!} \).
the tree function
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The tree function $T(z) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}$ satisfies the equation

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Note that \( n^{n-1} \) is the number of labeled rooted trees with \( n \) vertices.
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Note that \( n^{n-1} \) is the number of labeled rooted trees with \( n \) vertices.

It is the inverse of the function \( g(x) = xe^{-x} \) (since \( g(T(z)) = T(z)e^{-T(z)} = z \)).
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It is the inverse of the function \( g(x) = x e^{-x} \) (since \( g(T(z)) = T(z) e^{-T(z)} = z \)).

It follows from \( T(g(x)) = x \) that \( T(x e^{-x}) = x \).
expansion in Maple
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> alias(W=LambertW);
expansion in Maple

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W
expansion in Maple

> alias(W=LambertW);

> T:=solve(x*exp(-x)=z,x);
expansion in Maple

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\[ W \]

> T := solve(x*exp(-x)=z, x);

\[ T := -W(-z) \]
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\[ z + z^2 + \frac{3}{2} z^3 + \frac{8}{3} z^4 + \frac{125}{24} z^5 + \frac{54}{5} z^6 + \frac{16807}{720} z^7 + O(z^8) \]
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> A:=(y^2-subs(z=y*exp(x*(y-1)^2-y),T))/(y^2-y):
expansion in Maple

> alias(W=LambertW);
  
  \text{W}

> T:=solve(x*exp(-x)=z,x);
  
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> A:=(y^2-subs(z=y*exp(x*(y-1)^2-y),T))/(y^2-y):

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\[ W \]

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\[
\begin{align*}
z + z^2 + \frac{3}{2}z^3 + \frac{8}{3}z^4 + \frac{125}{24}z^5 + \frac{54}{5}z^6 + \frac{16807}{720}z^7 + O(z^8)
\end{align*}
\]

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\[
\begin{align*}
A := (y^2 - \text{subs}(z=y*\exp(x*(y-1)^2-y),T))/(y^2-y):
\end{align*}
\]

> simplify(subs(W(-y*exp(-y))=-y,series(A,x,5)));
refined sign-balance for even-length sets
refined sign-balance for even-length sets

Define

\[ B_n(y) = \sum_{\sigma \in \mathcal{R}_{2n}} \text{sign}(\sigma) y^{\text{des}(\sigma)}. \]
refined sign-balance for even-length sets

Define

\[ B_n(y) = \sum_{\sigma \in \mathcal{R}_{2n}} \text{sign}(\sigma)y^{\text{des}(\sigma)}. \]

Some initial polynomials

\[
\begin{align*}
B_1(y) &= 1 - y \\
B_2(y) &= 1 - y \\
B_3(y) &= 1 + y - 2y^2 \\
B_4(y) &= 1 + 7y - 2y^2 - 6y^3 \\
B_5(y) &= 1 + 21y + 36y^2 - 34y^3 - 24y^4. \\
\end{align*}
\]
results for even-length sets
results for even-length sets

We prove that $B_n(y)$ satisfies the recurrence relation

$$B_n(y) = (1 + (2n - 3)y)B_{n-1}(y) - (y^2 - y)B'_{n-1}(y).$$
exponential generating function for $B_n(y)$
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1. $B(x, y)$ satisfies the partial differential equation

$$\left(\frac{1}{y} - 2x\right) \frac{\partial B}{\partial x} + (y - 1) \frac{\partial B}{\partial y} = \left(\frac{1}{y} - 1\right) B.$$
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1. $B(x, y)$ satisfies the partial differential equation

$$(\frac{1}{y} - 2x) \frac{\partial B}{\partial x} + (y - 1) \frac{\partial B}{\partial y} = (\frac{1}{y} - 1) B.$$ 

2. Let $T(z) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}$. Then

$$B(x, y) = \frac{T(\mu)}{y}, \text{ where } \mu = ye^{x(y-1)^2-y}.$$