Universality of random matrices, Dyson Brownian Motion and Quantum Unique Ergodicity

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May 27, 2014

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Outlines

1 Historical aspects
2 Eigenvalue universality
3 Quantum unique ergodicity for random matrices
4 Perspectives
Eugene Wigner (1956): Successive energy levels of large nuclei have universal spacing statistics that can be described by random matrices.

— a grand vision

Freeman Dyson (1962): A gas of particles with a logarithmic interaction reaches local equilibrium very fast.

— a seminal idea
De Giorgi, Nash, Moser (1957-60):
Regularity theory for parabolic equations

Quantum Chaos conjecture (1977-84) and the Anderson Model (1958)

Rudnick-Sarnak (1991) eigenfunctions on negative curved manifolds are “flat” (Quantum unique ergodicity).
Experimental data for excitation spectra of heavy nuclei:

Wigner (1955): This pattern can be modeled by the spacing distribution of eigenvalues of random matrices.

Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.

— E. Wigner
Gaussian Orthogonal Ensemble (GOE):

\[ H = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \ldots & h_{NN} \end{pmatrix} \]

\[ h_{jk} = h_{kj} \quad (\text{for } j < k) \] are real independent normal random variables

\[ \mathbb{E} h_{jk} = 0, \quad \mathbb{E} |h_{jk}|^2 = \frac{1}{N} \]

The eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \) are of order one.

Also, Hermitian or quaternion self-dual (GUE, GSE) ensemble. 

Wigner ensembles: \( h_{ij} \) are just independent (not necessarily normal) distributions.
Wigner semicircle law: Density of eigenvalues

\[ \rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \]

\[ \mathbb{P}\left( \lambda_1 \sim E + \frac{x_1}{N}, \lambda_2 \sim E + \frac{x_2}{N} \right) \sim 1 - \left( \frac{\sin \pi (x_1 - x_2)}{\pi (x_1 - x_2)} \right)^2 \]

Gaudin-Mehta and Dyson
Is this exact computation for Gaussian models valid for general systems?

Fundamental belief of universality: Random matrix statistics are a new class of universal laws for highly correlated systems.

A simple manifestation (Wigner-Dyson-Mehta conjecture): If $h_{ij}$ are independent, then the local eigenvalue statistics are the same as those of the Gaussian ensembles.

Only symmetry types of the ensembles matter.

Central limit theorem:

$$\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2),$$

normal (Gaussian) distribution
Quantum Chaos and Anderson Model

Anderson (1958): $V_\omega$ random potential on $\mathbb{R}^d$ or $\mathbb{Z}^d$.
$H = -\Delta + \lambda V_\omega$. Local eigenvalue statistics:
Depending on $\lambda$ and $d$, there are two distinct regimes.

GOE statistics in the delocalization regime (e.g., small $\lambda$ in 3 dimension)

Poisson statistics (Minami) in the localization regime (Frohlich-Spencer, Aizenman-Malchonov).

Similarly for $-\Delta$ on a domain with chaotic classical dynamics.
For integrable dynamics: Berry-Tabor (1977)
Quantum unique ergodicity conjecture (QUE) [Rudnick-Sarnak]:

\[(\psi_k)_{k \geq 1}\] orthogonal eigenfunctions of \(M\) with negative curvature:

\[
\int a(x)|\psi_j(x)|^2d\text{Vol}(x) \to \int a(x)d\text{Vol}(x)
\]

for any large energy limit.

Question: Does QUE hold for Wigner matrices? Yes for GUE/GOE because the eigenvectors are uniformly distributed on $U(N)/O(N)$.

Quantum (unique) ergodicity/delocalization
$\iff$ random matrix statistics?

Two basic mathematical questions:
1. Universality conjecture for Wigner matrices
2. QUE for Wigner matrices.

Key ingredient in solving these problems: Dyson Brownian motion.
Generalized Wigner Ensembles, $H = (h_{ij})_{1 \leq i,j \leq N}$

\[ h_{ji} = h_{ij}, \mathbb{E}h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = s_{ij}, \quad \sum_i s_{ij} = 1, \quad \frac{c}{N} \leq s_{ij} \leq \frac{C}{N} \]

Solution to the Universality conjecture

Theorem [Erdos-Schlein-Y-Yin, 2009-2010] Suppose that $4 + \varepsilon$ moments of matrix elements are uniformly bounded. Then local eigenvalue statistics for generalized Wigner ensembles are universal in the bulk (and at the edges).

Matrices with Bernoulli entries with varying variances are included.

[Tao-Vu, 2009-10] Wigner matrices with four moments matching those of $\mathcal{N}(0,1)$. Hermitian case via combination with Johansson's result.

Extensions to sparse matrices, $\beta$-ensembles, single gap universality, the edge universality [jointly with Bourgade, Erdoes, Knowles, Yin]
Theorem [Bourgade-Y, 2013] [Probabilistic local QUE]

\( u_i, \ i = 1, \ldots, N \): normalized e-vectors of generalized Wigner matrices.

\( \mathcal{N}_1, \mathcal{N}_2 \): independent normal distribution \( \mathcal{N}(0,1) \). Then for any \( j, k \) in the bulk and \( \mathbf{q} \) a unit vector in \( \mathbb{R}^N \), we have

\[
\sqrt{N} \left( |\langle \mathbf{q}, u_j \rangle|, |\langle \mathbf{q}, u_k \rangle| \right) \rightarrow (|\mathcal{N}_1|, |\mathcal{N}_2|)
\]

Probabilistic local QUE holds: for any \( k \in \{1, \ldots, N\} \) and for any set \( A \subset \{1, \ldots, N\} \) with \( |A| \geq N^{C\varepsilon} \), we have with high probability that

\[
\left| \frac{1}{|A|} \sum_{a \in A} \left( N|u_k(a)|^2 - 1 \right) \right| \ll 1
\]

Our results hold for Bernoulli random matrices and the extension to the Erdos-Renyi graphs also hold. This gives an example of prob-QUE for the Laplacian on a random graph.
Previous results: Delocalization [Erdos-Schlein-Y],

Knowles-Yin: Some version of local QUE holds for eigenvectors of Wigner matrices near the spectral edge. In the bulk, same property holds if the first four moments of the matrix elements matching those of the standard Gaussian.

In the bulk this was also proved by Tao-Vu under similar assumption (i.e., four or five moment matching).

Why study e vector distributions of Wigner matrices?

1. QUE for a random graph is an important model for Laplacian in a domain or in a random potential.

2. QUE is an important tool in the study of eigenvalue statistics of non-mean field matrix models such as band matrices.

3. E-vectors are important in statistical analysis of large data sets.
Matrix Dyson Brownian Motion (Matrix-DBM)

Evolve the matrix with a matrix Ornstein-Uhlenbeck process:

\[ dH_t = \frac{1}{\sqrt{N}} dB_t - \frac{1}{2} H_t dt \quad B_{ij} : \text{symm. indep. BM} \]

The distribution of \( H_t \sim e^{-t/2} H_0 + \sqrt{1 - e^{-t}} V \) where \( V \) is a GOE.

\[ d\lambda_k = \frac{dB_{kk}}{\sqrt{N}} + \left( \frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell - \frac{1}{2} \lambda_k} \right) dt \]

Dyson Brownian Motion (1962) E-value equations are autonomous!
Dyson: The classical Coulomb gas is invariant under the DBM:

\[ \mu_\beta \sim e^{-\beta N \mathcal{H}(\lambda)}, \quad \mathcal{H} = \sum_i \frac{\lambda_i^2}{4} - \frac{1}{N} \sum_{i<j} \log(\lambda_j - \lambda_i) \]

Prob. density for the classical ensembles with \( \beta = 1 \) for the GOE.

**Dyson’s conjecture:** Time to “local equilibrium” for DBM is \( \gtrsim N^{-1} \).

**Erdős-Schlein-Y-Yin, 2009-10:** Dyson conjecture holds for \( \beta > 0 \).

*The Coulomb interactions drive the system locally to equilibrium very fast!*

\[ \implies \text{Wigner-Dyson-Mehta conjecture holds for } H_t \text{ with } t \gtrsim N^{-1}. \]
Question: How to connect to $t = 0$?

Theorem* [Continuity of matrix-DBM]

$F$: function of matrix elements with uniformly bounded 3rd derivatives. If $t \ll N^{-1/2}$ then

$$\mathbb{E}F(H_t) - \mathbb{E}F(H_0) \to 0$$

Proof: By Ito’s formula and local semicircle laws.

Corollary: Continuity of eigenvalues and eigenfunctions for $t \ll N^{-1/2}$.

* [Bourgade-Y 2013 (see also Erdos-Knowles-Y-Yin)]
Continuity of matrix DBM

Equilibration of DBM

No contradiction since we use the matrix structure.
How to study eigenvectors: 1. **Perturbation formula:**

\[
\frac{\partial \lambda_k}{\partial h_{ab}} = u_k(a)u_k(b), \quad \frac{\partial u_k}{\partial h_{ab}} = \sum_{\ell \neq k} \frac{u_\ell(a)u_k(b)}{\lambda_k - \lambda_\ell} u_\ell.
\]

Perturbation for eigenvectors are much more singular as eigenvalues can be extremely close to each other. In fact, perturbation formula cannot be applied in general.

2. **Green functions.** Let

\[
G_{aa}(z) := \left( \frac{1}{H - E - i\eta} \right)_{aa} = \sum_j \frac{|u_j(a)|^2}{\lambda_j - E - i\eta}
\]

As \( \eta = N^{-1+\varepsilon} \), \( G_{aa} \) is some averaged behavior of \( |u_j(a)|^2 \). To get behavior of individual eigenvector, we need \( \eta \ll 1/N \). But \( G_{aa}(z) \) will be very singular since there are eigenvalues near \( E \) in the bulk.

But eigenvector distributions are expected to be like \( O(N) \)—a stable object.
Dyson e-vector flow  E-vector equations depend on the e-values.

\[
du_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{u_\ell dB_{k\ell}}{\lambda_k - \lambda_\ell} - \frac{1}{2N} \sum_{\ell \neq k} \frac{u_k dt}{(\lambda_k - \lambda_\ell)^2}
\]

(Very complicated and singular equations).

[Bourgade-Y 2013] For any unit vector \( q \) fixed, define the conditional expectation given the eigenvalue trajectories

\[
f(t, j) = f(t, j, \lambda(\cdot)) = N \mathbb{E}
\left[
|q \cdot u_j(t)|^2 |\lambda(\cdot)|
\right].
\]

Then we have the eigenvector moment flow

\[
\partial_t f(t, j) = N \sum_{k \neq j} \frac{f(t, k) - f(t, j)}{(\lambda_j - \lambda_k)^2(t)}
\]

Random walk (in the index \( j \)) in random environments.
There are analogue equations for higher moments:

\[
f(t, j) = N^k \mathbb{E}
\left[
|q \cdot u_{j_1}(t)|^2 \ldots |q_k \cdot u_{j_k}(t)|^2 |\lambda(\cdot)|
\right].
\]
Detour to single gap universality:

**Theorem** [Erdos-Y 2012] (for a similar equation) With high probability, for any $|i - j| \ll Nt$,

$$|f(t, k) - f(t, j)| \leq C\|f(t)\|_\infty (Nt)^{-\varepsilon}$$

$\varepsilon$ is a Hölder regularity exponent from De Giorgi-Nash-Moser (Caffarelli-Chan-Vasseur) theory!

One gains regularity as long as $t \gg N^{-1}$.

$\implies$ **Single gap universality**, i.e., the distribution of any eigenvalue gap (up to normalization) of a generalized Wigner matrix is independent of the distributions of the matrix elements.

**Key difficulty:** The jump rate $\frac{1}{(\lambda_j - \lambda_k)^2(t)}$ can be very singular. We are in a situation that

$$\partial_t u(t, x) = \nabla[D(x, t) \nabla u(x, t)]$$

with $D$ very singular (our dynamics is also nonlocal).
Theorem [Bourgade-Y 2013] Let $H_t$ denote the DBM with a symmetric generalized Wigner matrix as the initial condition. Suppose that $f(t,j)$ is the solution to the eigenvector moment flow equation. Then for $t \geq N^{-1/4}$

$$|f(t,k) - 1| \leq C (Nt)^{-\epsilon}.$$ 

Proof: By maximum principle and (isotropic) local semicircle law (i.e., semicircle law holds in small scales and in every direction).

Together with extension to higher moments $\Rightarrow$

Corollary. Probabilistic local QUE holds for $H_t; \quad N|\mathbf{q} \cdot u_j(t)|^2 \rightarrow \text{Gaussian}$.

By continuity of matrix-DBM $\Rightarrow$ probabilistic local QUE holds for $H_0$. 


**Local semicircle law** (Erdos-Schlein-Y-Yin): Define the Green function

\[
G_{aa}(z) := \left( \frac{1}{H - z} \right)_{a,a} = \sum_j \frac{|u_j(a)|^2}{\lambda_j - E - i\eta}, \quad z = E + i\eta
\]

Stieltjes transform of semicircle law:  
\[
m_{sc}(z) = \int \frac{\rho_{sc}(x)dx}{x - z}
\]

For any \( z = E + i\eta \) with \( \eta \gg N^{-1} \) the following holds with exponentially high probability:

\[
\max_i |G_{ii}(z) - m_{sc}(z)| + \max_{i \neq j} |G_{ij}(z)| \leq \frac{1}{\sqrt{N\eta}}
\]

Isotropic law (knowles-Yin, Erdos-Knowles-Y-Yin): For any fixed unit vector \( v \) the following holds with exponential high prob:

\[
\left| \sum_{a,b=1}^N v_a G_{ab}(z) v_b - m \right| \leq \frac{1}{\sqrt{N\eta}}
\]

Require knowledge of cancellations due to summation.
The underlying mechanism of the universality and QUE for random matrices: Dyson Brownian motion (A dynamical idea)

Universality: *Fast relaxation of DBM* + continuity of matrix-DBM

QUE: *Holder regularity of e-vector moment flow* + continuity of matrix-DBM

- Dynamical idea to solve time independent problems: *study the limits of the flow as* \( t \to \infty \).
- Dyson Brownian Motion: *study the initial layer to understand* \( t = 0 \).
If you admit that the Wigner ensemble gives a completely wrong answer for the level density, why do you believe any of the other predictions of random-matrix theory?

George Uhlenbeck

- **Local theory** for universality was developed.
- Random matrix is a mean-field model. Going **beyond mean-field** is a key question. One key example: band matrices.