1. Richardson’s extrapolation with centered difference formula is used repeatedly to estimate $f'(x)$. Test your program on the following:
   (a) $\ln(x)$ at $x = 3$;
   (b) $\tan(x)$ at $x = \sin^{-1}(0.8)$;
   (c) $\sin(x^2 + \frac{1}{3}x)$ at $x = 0$.

2. Carry out the Romberg Integration for a function $f$ defined on an arbitrary interval $[a,b]$ and test your Romberg program on these three examples:
   (a) $\int \frac{\sin x}{x} \, dx$;
   (b) $\int \frac{\cos x - e^x}{\sin x} \, dx$;
   (c) $\int (xe^x)^{-1} \, dx$.

The routines for these integrals should be written to avoid serious loss of significance due to subtraction. Also, it is customary to defined a function $f$ at any questionable point $x_0$ by the equation $f(x_0) = \lim_{x \to x_0} f(x)$. If the limit exists, this method guarantees continuity of $f$ at $x_0$. For the third example, make a suitable change variable, such as $x=1/t$. Compute seven rows in the Romberg array. Print the array in each case with a format that enables the convergence to be observed.

● 評分：分為兩個部分，程式編撰 與 執行結果報告
   1. 程式編撰時，必須要完成指定的問題進行處理。
   2. 執行結果報告，寫下各個問題的答案並作說明。
作業繳交：
1. 繳交時間：截止日 2008 年 12 月 22 日。
2. 詳細的繳交方式與配分，由助教規定。

配分：佔總成績 10%。

預期效益：
1. 掌握基本的數值微分與數值積分之求解方法。
2. 在撰寫程式的過程中，體認到主程式與副程式（函數）的角色扮演。
Richardson’s Extrapolation

1 Introduction

Suppose ∀ h ≠ 0 we have a formula $N_1(h)$ that approximates an unknown value $M$

$$M - N_1(h) = K_1h + K_2h^2 + K_3h^3 + \cdots,$$

for some unknown constants $K_1, K_2, K_3, \ldots$. Using induction, $M$ can be approximated by

$$M = N_m(h) + O(h^m),$$

where

$$N_m(h) = N_{m-1}(\frac{h}{2}) + \frac{N_{m-1}(h/2) - N_{m-1}(h)}{2^{m-1} - 1}, \quad m = 2, 3, \ldots$$

and $N_1(h)$ is a given formula.

2 Numerical Differentiation $f'(x)$

Richardson’s extrapolation can be used to estimate $f'(x)$. We show that Richardson’s extrapolation with centered difference formula to evaluate $M = f'(x)$ as follows:

$$M = N_j(h) + O(h^2),$$

where

$$N_j(h) = N_{j-1}(\frac{h}{2}) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}, \quad j = 2, 3, \ldots$$

and

$$N_1(h) = \frac{f(x + h) - f(x - h)}{2h}.$$