\( e \) is transcendental

**Proof.** Suppose \( e \) is algebraic, i.e., there is a polynomial \( p \) with integer coefficients \( a_0, a_1, \ldots, a_n \) such that

\[
p(e) = a_n e^n + \cdots + a_1 e + a_0 = 0,
\]

where the degree of the polynomial \( p \) is \( n \) \((a_n \neq 0)\) and we may assume \( a_0 \neq 0 \).

Step 1: Let \( f \) be a polynomial with degree \( m \) and

\[
f(x) = \frac{1}{(N-1)!}x^{N-1}(x-1)^N(x-2)^N \cdots (x-n)^N,
\]

where \( N \) is a sufficiently large prime number with \((n+1)N-1 = m \) and \( N > |a_0|n \). Then (repeated) integrations by parts gives

\[
\int_0^k f(x)e^{-x} \, dx = -\int_0^k f(x) \, de^{-x} \\
= -f(x)e^{-x}|_0^k + \int_0^k f'(x)e^{-x} \, dx \\
= -\left(f(x) + f'(x) + \cdots + f^{(m)}(x) \right) e^{-x}|_0^k \\
= -\left(f(k) + f'(k) + \cdots + f^{(m)}(k) \right) e^{-k} + \left(f(0) + f'(0) + \cdots + f^{(m)}(0) \right).
\]

Multiply by \( a_k e^k \) and add up: Then

\[
\sum_{k=0}^n a_k e^k \int_0^k f(x)e^{-x} \, dx = -\sum_{k=0}^n a_k (f(k) + f'(k) + \cdots + f^{(m)}(k)) \tag{2}
\]

since \( p(e) = \sum_{k=0}^n a_k e^k = 0 \) from the equation (1). Our goal is to obtain a contradiction by showing really there is an \( f \) such that the LHS is small and the RHS is a nonzero integer in the equation (2).

Step 2: Claim that the LHS of the equation (2) is small. For \( 0 \leq x \leq n \), we have

\[
|f(x)| \leq \frac{1}{(N-1)!}n^N n^N \cdots n^N = \frac{n^{(n+1)N}}{(N-1)!} = \frac{\alpha^N}{(N-1)!},
\]

where \( \alpha = n^{(n+1)} \). Therefore, the LHS of the equation (2) is bounded by

\[
|\text{LHS}| = \left| \sum_{k=0}^n a_k e^k \int_0^k f(x)e^{-x} \, dx \right| \leq (n+1) \max_k |a_k| e^n \cdot n \cdot \frac{\alpha^N}{(N-1)!} = \frac{\rho \alpha^N}{(N-1)!},
\]

where \( \rho = n^{(n+1)} \max_k |a_k| \) is independent of \( N \). It implies that \( \frac{\rho \alpha^N}{(N-1)!} \to 0 \) as \( N \to \infty \). Hence, we have \( |\text{LHS}| < 1 \) as \( N \) large enough.

Step 3: Claim that the RHS of the equation (2) is a nonzero integer.

\[
|\text{RHS}| = -\sum_{k=0}^n a_k (f(k) + f'(k) + \cdots + f^{(m)}(k)) \\
= -a_0 (f(0) + f'(0) + \cdots + f^{(m)}(0)) - \sum_{k=1}^n a_k (f(k) + f'(k) + \cdots + f^{(m)}(k)).
\]
Since $f$ is the polynomial with the degree $m$ and the roots, $0, 1, \ldots, n$,
\[
f^{(i)}(0) = \begin{cases} 
0, & \text{if } 0 \leq i \leq N - 2, \\
(-1)^{mN}(n!)^N, & \text{if } i = N - 1, \\
I_iN, & \text{if } N \leq i \leq m,
\end{cases}
\]
where $I_i = \frac{f^{(i)}(0)}{N}$ for $N \leq i \leq m$ and it is easily to check that $I_i$ is a nonzero integer; and
for $1 \leq k \leq n$
\[
f^{(i)}(k) = \begin{cases} 
0, & \text{if } 0 \leq i \leq N - 1, \\
J_{k,i}N, & \text{if } N \leq i \leq m,
\end{cases}
\]
where $J_{k,i} = \frac{f^{(i)}(k)}{N}$ for $N \leq i \leq m$ and it is easily to check that $J_{k,i}$ is a nonzero integer. Therefore,
\[
|RHS| = -a_0(-1)^{nN}(n!)^N - a_0 \left( f^{(N)}(0) + f^{(N+1)}(0) + \cdots + f^{(m)}(0) \right) \\
- \sum_{k=1}^{n} a_k \left( f^{(N)}(k) + f^{(N+1)}(k) + \cdots + f^{(m)}(k) \right) \\
= -(-1)^{nN}a_0(n!)^N - a_0 \left( I_NN + I_{N+1}N + \cdots + I_mN \right) \\
- \sum_{k=1}^{n} a_k \left( J_{k,N}N + J_{k,N+1}N + \cdots + J_{k,m}N \right) \\
= -(-1)^{nN}a_0(n!)^N + \beta N,
\]
where $\beta$ is a nonzero integer ($a_0, a_1, \ldots, a_n, I_N, I_{N+1}, \ldots, I_m$ and $J_{k,N}, J_{k,N+1}, \ldots, J_{k,m}$ are all integers). We have that $N$ is a prime and $N > |a_0|N$, therefore, $a_0(n!)^N$ is not divisible by $N$. Hence, the RHS is a nonzero integer, so $|RHS| \geq 1$.

Thus, we may choose $N$ such that $|LHS| < 1$ and $|RHS| \geq 1$. It is a contradiction to prove that $e$ is not algebraic. Hence, $e$ is transcendental.