Chapter 5

METHOD OF LYAPUNOV FUNCTIONS

5.1 An Introduction to Dynamical Systems

Example 5.1.1 Consider the autonomous system

\[ x' = f(x), \quad f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n. \quad (5.1) \]

Let \( \varphi(t, x_0) \) be the solution of (5.1) with initial condition \( x(0) = x_0 \). Then it follows that

(i) \( \varphi(0, x_0) = x_0 \)

(ii) \( \varphi(t, x_0) \) is a continuous function of \( t \) and \( x_0 \)

(iii) \( \varphi(t + s, x_0) = \varphi(s, \varphi(t, x_0)) \).

Property (i) is obvious; property (ii) follows directly from the property of continuous dependence on initial conditions. Property (iii) which is called group property, follows from uniqueness of ODE and the fact that (5.1) is an autonomous system. We call \( \varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) the flow induced by (5.1).

Remark 5.1.1 For nonautonomous system \( x' = f(t, x) \), property (iii) does not hold.

For generality we shall consider flows defined on the metric space \( M \). In Example 5.1.1, \( M = \mathbb{R}^n \). In fact, in many applications of dynamical
systems to partial differential equations, functional differential equations, the metric space $M$ is a Banach space and the flow $\varphi$ is a semiflow, i.e., $\varphi : R^+ \times M \to M$ satisfying (i), (ii), (iii). Now let $(M, \rho)$ be a metric space. In the rest of section 5.1, we refer to the book in [NS].

**Definition 5.1.1** We call a map $\pi : R \times M \to M$ a **continuous** dynamical system if

(i) $\pi(0, x) = x$

(ii) $\pi(t, x)$ is continuous in $x$ and $t$

(iii) $\pi(t + s, x) = \pi(t, \pi(s, x))$, $x \in M$, $t, s \in R$.

We may interpret $\pi(t, x)$ as the position of a particle at time $t$ when the initial (i.e. time = 0) position is $x$.

**Remark 5.1.2** A discrete dynamical system is defined as a continuous map $\pi : Z \times M \to M$ satisfying (i), (ii), (iii) where $Z = \{ n : n$ is an integer$\}$. The typical examples of discrete dynamical systems are $x_{n+1} = f(x_n)$, $x_n \in R^d$ and the Poincaré map $x_0 \mapsto \varphi(w, x_0)$ where $\varphi(t, x_0)$ is the solution of a periodic system $x' = f(t, x)$, $x(0) = x_0$ where $f(t, x) = f(t + w, x)$.

The next lemma says that property (ii) implies the property of continuous dependence on initial data.

**Lemma 5.1.1** Given $T > 0$ and $p \in M$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\rho(\pi(t, p), \pi(t, q)) < \varepsilon \text{ for } 0 \leq t \leq T$$

whenever $\rho(p, q) < \delta$.

**Proof.** If not, then there exists $\{ q_n \}$, $q_n \to p$ and $\{ t_n \}$, $|t_n| \leq T$ and $\alpha > 0$ with

$$\rho(\pi(t_n, p), \pi(t_n, q_n)) \geq \alpha.$$  \hspace{1cm} (5.2)

Without loss of generality, we may assume $t_n \to t_0$. Then

$$0 \ < \ \alpha \leq \rho(\pi(t_n, p), \pi(t_n, q_n)) \ \leq \ \rho(\pi(t_n, p), \pi(t_0, p)) + \rho(\pi(t_0, p), \pi(t_n, q_n)).$$

From (ii), the right hand side of (5.2) approaches zero as $n \to \infty$. This is a contradiction.
Notations:
\[
\gamma^+(x) = \{\pi(t,x) : t \geq 0\} \text{ is the positive orbit through } x.
\]
\[
\gamma^-(x) = \{\pi(t,x) : t \leq 0\} \text{ is the negative orbit through } x.
\]
\[
\gamma(x) = \{\pi(t,x), -\infty < t < \infty\} \text{ is the orbit through } x.
\]

Definition 5.1.2 We say that a set \(S \subseteq M\) is positively (negatively) invariant under the flow \(\pi\) if for any \(x \in S\), \(\pi(t,x) \in S\) for all \(t \geq 0\) \((t \leq 0)\), i.e., \(\pi(t,S) \subseteq S\) for all \(t \geq 0\) \((t \leq 0)\). \(S\) is invariant if \(S\) is both positively and negatively invariant, i.e., \(\pi(t,S) = S\) for \(-\infty < t < \infty\).

Lemma 5.1.2 The closure of an invariant set is invariant.

Proof. Let \(S\) be an invariant set and \(\bar{S}\) be its closure. If \(p \in S\), then \(\pi(t,p) \in S \subseteq \bar{S}\). If \(p \in \bar{S} \setminus S\), then there exists \(\{p_n\} \subseteq S\) such that \(p_n \to p\). Then for each \(t \in \mathbb{R}\), we have \(\lim_{n \to \infty} \pi(t,p_n) = \pi(t,p)\). Since \(\pi(t,p_n) \in S\), it follows that \(\pi(t,p) \in \bar{S}\). Hence \(\pi(t,\bar{S}) \subseteq \bar{S}\).

Definition 5.1.3 We say a point \(p \in M\) is an equilibrium or a rest point of the flow \(\pi\) if \(\pi(t,p) \equiv p\) for all \(t\). If \(\pi(T,p) = p\) for some \(T > 0\) and \(\pi(t,p) \neq p\) for all \(0 < t < T\), then we say \(\{\pi(t,p) : 0 \leq t \leq T\}\) is a periodic orbit.

Example 5.1.2 Rest points and periodic orbits are (fully) invariant. We note that positively invariance does not necessarily imply negatively invariance. For instance, if an autonomous system \(x' = f(x)\) satisfies \(f(x) \cdot n(x) < 0\) for all \(x \in \partial\Omega\) where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), then \(\Omega\) is positively invariant, but not negatively invariant.

Lemma 5.1.3

(i) The set of rest points is a closed set.
(ii) No trajectory enters a rest point in finite time.

Proof.

(i) Trivial

(ii) Suppose \(\pi(T,p) = p^*\) where \(p^*\) is a rest point and \(p \neq p^*\). Then \(p = \pi(-T,p^*) = p^*\). This is a contradiction.
Lemma 5.1.4

(i) If for any \( \delta > 0 \), there exists \( p \in B(q, \delta) \) such that \( \gamma^+(p) \subset B(q, \delta) \), then \( q \) is a rest point.

(ii) If \( \lim_{t \to \infty} \pi(t, p) = q \) then \( q \) is a rest point.

Proof. We shall prove (i). (ii) follows directly from (i). Suppose \( q \) is not a rest point, then \( \pi(t_0, q) \neq q \) for some \( t_0 > 0 \). Let \( \rho(q, \pi(t_0, q)) = d > 0 \).
From continuity of the flow \( \pi \), there exists \( \delta > 0 \) such that if \( \rho(p, q) < \delta < d/2 \) then \( \rho(\pi(t, q), \pi(t, p)) < d/2 \) for all \( t, |t| \leq t_0 \). By hypothesis of (i) there exists \( p \in B(q, \delta) \) such that \( \gamma^+(p) \subset B(q, \delta) \). Then

\[
d = \rho(q, \pi(t_0, q)) < \rho(q, \pi(t_0, p)) + \rho(\pi(t_0, p), \pi(t_0, q)) < \delta + d/2 < d/2 + d/2 = d.
\]

This is a contradiction.

Next we introduce the notion of limit sets.

Definition 5.1.4 We say that \( p \) is an omega limit point of \( x \) if there exists a sequence \( \{t_n\}, t_n \to +\infty \) such that \( \pi(t_n, x) \to p \). The set

\[\omega(x) = \{p : p \text{ is an omega limit point of } x\}\]

is called the \( \omega \)-limit set of \( x \).

Similarly we say that \( p \) is an alpha limit point of \( x \) if there exists a sequence \( \{t_n\}, t_n \to -\infty \) such that \( \pi(t_n, x) \to p \). The set

\[\alpha(x) = \{p : p \text{ is an alpha limit point of } x\}\]

is called the \( \alpha \)-limit set of \( x \).

Remark 5.1.3 Note that \( \omega(x) \) represents where the positive orbit \( \gamma^+(x) \) ends up, while \( \alpha(x) \) represents where the negative orbit \( \gamma^-(x) \) started. We note that \( \alpha, \omega \) are the initial and final alphabets of Greek characters.

Remark 5.1.4

\[
\omega(x) = \bigcap_{t \geq 0} \text{cl} \left( \bigcup_{s \geq t} \pi(s, x) \right)
\]

\[
\alpha(x) = \bigcap_{t \leq 0} \text{cl} \left( \bigcup_{s \leq t} \pi(s, x) \right).
\]
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**Theorem 5.1.1** \( \omega(x) \) and \( \alpha(x) \) are closed, invariant sets.

**Proof.** We shall prove the case of \( \omega(x) \) only. First we show that \( \omega(x) \) is invariant. Let \( q \in \omega(x) \) and \( \tau \in R \). We want to show \( \pi(\tau, q) \in \omega(x) \). Let \( \pi(t_n, x) \to q \) as \( t_n \to +\infty \). By continuity of \( \pi \), it follows that

\[
\pi(t_n + \tau, x) = \pi(\tau, \pi(t_n, x)) \to \pi(\tau, q),
\]
as \( t_n \to +\infty \). Thus \( \pi(\tau, q) \in \omega(x) \) and it follows that \( \omega(x) \) is invariant.

Next we shall prove that a trajectory is asymptotic to its omega limit. We shall prove the case of \( \omega(x) \) only. Let \( \rho \) converge subsequence, \( \rho \) as \( n \to \infty \). The compactness of \( \omega \) implies that \( \rho(\tau_n, x) \to q \) as \( n \to \infty \). Thus we have \( \lim t_n \to \infty \).

Next we show that a trajectory is asymptotic to its omega limit.

Let \( \rho(\pi(\tau_n, x), q) < \rho(\pi(\tau_n, x), q_n) + \rho(q_n, q) \)

\[
< \varepsilon /2 + \varepsilon /2 = \varepsilon.
\]

Thus we have \( \lim_{n \to \infty} \pi(\tau_n, x) = q \) and \( q \in \omega(x) \).

**Theorem 5.1.2** If the closure of \( r^+(p) \), \( Cl(r^+(p)) \), is compact then \( w(p) \) is nonempty, compact and connected. Furthermore \( \lim_{t \to \infty} \rho(\pi(t,p), w(p)) = 0 \).

**Proof.** Let \( p_k = \pi(k, p) \). Since \( Cl(r^+(p)) \) is compact, then there exists a subsequence of \( \{p_{k_j}\}, p_{k_j} \to q \in Cl(r^+(p)) \). Then \( q \in w(p) \) and hence \( w(p) \) is nonempty. The compactness of \( w(p) \) follows directly from the facts that \( Cl(r^+(p)) = r^+(p) \cup w(p) \), \( w(p) \) is closed and \( Cl(r^+(p)) \) is compact.

We shall prove that \( w(p) \) is connected by contradiction. Suppose on the contrary, \( w(p) \) is disconnected. Then \( w(p) = A \cup B \), a disjoint union of two closed subsets \( A, B \) of \( w(p) \). Since \( w(p) \) is compact, then \( A, B \) are compact. Then \( d = \rho(A, B) > 0 \) where \( \rho(A, B) = \inf_{x \in A, y \in B} \rho(x, y) \). Consider the neighborhoods \( N(A, d/3) \) and \( N(B, d/3) \) of \( A, B \) respectively. Then there exist \( t_n \to +\infty \) and \( t_n \to +\infty \), \( t_n \) such that \( \pi(t_n, p) \in N(A, d/3) \) and \( \pi(p, \tau_n) \in N(B, d/3) \).

Since \( \rho(\pi(p, t), A) \) is a continuous function of \( t \), then from the inequalities

\[
\rho(\pi(\tau_n, p), A) > \rho(A, B) - \rho(\pi(\tau_n, p), B) > d - d/3 = 2d/3,
\]
and

\[
\rho(\pi(t_n, p), A) < d/3
\]
we have \( \rho(\pi(t_n, p), A) = d/2 \) for some \( t_n \in (t_n, \tau_n) \). \( \{\pi(t_n^*, p)\} \) contains a convergent subsequence, \( \pi(t_n^*, p) \to q \in w(p) \). However \( q \notin A \) and \( q \notin B \) and we obtain a desired contradiction.

Next we shall prove that a trajectory is asymptotic to its omega limit.
set. If not, then there exists a sequence \( \{ t_n \} \), \( t_n \to +\infty \) and \( \alpha > 0 \) such that \( \rho(\pi(t_n, p), w(p)) \geq \alpha \). Then there exists a subsequence of \( \{ \pi(t_n, p) \} \), \( \pi(t_{n_k}, p) \to q \in w(p) \) and we obtain the following contradiction:

\[
0 = \rho(q, w(p)) = \lim_{n_k \to \infty} \rho(\pi(t_{n_k}, p), w(p)) \geq \alpha.
\]

**Example 5.1.3** Consider

\[
\begin{align*}
x' &= x + y - x(x^2 + y^2) \\
y' &= -x + y - y(x^2 + y^2).
\end{align*}
\]

Using the polar coordinate \((r, \theta)\), we have

\[
\begin{align*}
r' &= r(1 - r^2) \\
\theta' &= -1.
\end{align*}
\]

If \( r(0) = 0 \), then \((0,0) = w((0,0))\).
If \( 0 < r(0) < 1 \), then \( w(x_0, y_0) = \text{unit circle} \), \( \alpha(x_0, y_0) = (0,0) \).
If \( r(0) = 1 \), then \( w(x_0, y_0) = \alpha(x_0, y_0) = \text{unit circle} \).
If \( r(0) > 1 \), then \( w(x_0, y_0) = \text{unit circle} \), \( \alpha(x_0, y_0) = \emptyset \).

**Example 5.1.4** Consider

\[
\begin{align*}
x' &= \frac{x - y}{1 + (x^2 + y^2)^{1/2}} \\
y' &= \frac{x + y}{1 + (x^2 + y^2)^{1/2}}.
\end{align*}
\]

Then, in terms of polar coordinate, we have

\[
\begin{align*}
\frac{dr}{dt} &= \frac{r}{1 + r^2}, \\
\frac{d\theta}{dt} &= \frac{1}{1 + r^2}.
\end{align*}
\]

![Fig. 5.1](image_url)
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From Fig. 5.1 it follows that $\omega(x) = \phi$ and $\alpha(x) = \{(0, 0)\}.$

The following examples show that an unbounded trajectory may have disconnected or noncompact $\omega$-limit set.

**Example 5.1.5** Let $X = \frac{x}{1-x^2},$ $Y = y,$ where $(X(t), Y(t))$ satisfies (5.3). Then

$$x' = \frac{x(1-x^2) - y(1-x^2)^2}{(1 + x^2) \left(1 + \left(\frac{x}{1-x^2}^2 + y^2\right)^{1/2}\right)} = f(x, y)$$

$$y' = \frac{y}{1 + \left(\frac{x}{1-x^2}^2 + y^2\right)^{1/2}} + \frac{x}{1 - x^2} \frac{1}{1 + \left(\frac{x}{1-x^2}^2 + y^2\right)^{1/2}} = g(x, y).$$

Then $\lim_{x \to \pm 1} f(x, y) = 0,$ $\omega((x_0, y_0)) = \{x = 1\} \cup \{x = -1\}$ is not connected.

![Fig. 5.2](image-url)

**Example 5.1.6** Let $X = \log(1+x), Y = y,$ $-1 < x < +\infty,$ $y \in R.$ Where $(X(t), Y(t))$ satisfies (5.3). Then the equation

$$x' = f(x, y) = (1 + x) \frac{\log(1 + x) - y}{1 + \{(\log(1 + x))^2 + y^2\}^{1/2}}$$

$$y' = g(x, y) = \frac{y + \log(1 + x)}{1 + \{(\log(1 + x))^2 + y^2\}^{1/2}}$$

$f(x, y), g(x, y)$ satisfies $\lim_{x \to +1} f(x, y) = 0,$ $\lim_{x \to -1} g(x, y) = 1.$ The $\omega$-limit set $\omega(x) = \{(x, y) : x = -1\}$ is not compact.
5.2 Lyapunov Functions [H]

Let \( V : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, 0 \in \Omega, \Omega \) is an open set in \( \mathbb{R}^n \).

**Definition 5.2.1** \( V : \Omega \subseteq \mathbb{R}^n \to \mathbb{R} \) is said to be positive definite (negative definite) on \( \Omega \) if \( V(x) > 0 \) for all \( x \neq 0 \) (\( V(x) < 0 \) for all \( x \neq 0 \)) and \( V(0) = 0 \).

\( V \) is said to be semi-positive definite (semi-negative definite) on \( \Omega \) if \( V(x) \geq 0 \) (\( V(x) \leq 0 \)) for all \( x \in \Omega \).

**Remark 5.2.1** In applications, we consider the autonomous system \( x' = f(x) \) with \( x^* \) as an equilibrium. Let \( y = x - x^* \), then \( y' = g(y) = f(y + x^*) \) and 0 is an equilibrium of the system \( y' = g(y) \). In general, \( V(x) \) satisfies \( V(x^*) = 0, \ V(x) > 0 \) for all, \( x \in \Omega, x \neq x^* \).

For the initial-value problem:

\[
x' = f(x), \quad x(0) = x_0,
\]

we assume that the solution \( \varphi(t, x_0) \) exists for all \( t \geq 0 \). We introduce “Lyapunov function” \( V(x) \) to locate the \( \omega \)-limit set \( \omega(x_0) \). The function \( V(x) \) satisfies the following: \( V(0) = 0 \), \( V(x) > 0 \) for all \( x \in \Omega \) and \( V(x) \to +\infty \) as \( |x| \to \infty \). The level sets \( \{ x \in \Omega : V(x) = c \} \) is a closed surface. The surface \( \{ x \in \Omega : V(x) = c_1 \} \) encloses the surface \( \{ x \in \Omega : V(x) = c_2 \} \) if \( c_1 > c_2 \). We want to prove \( \lim_{t \to \infty} \varphi(t, x_0) = 0 \), even though we don’t have to know the exact location of the solution \( \varphi(t, x_0) \). If we are able to construct a suitable Lyapunov function \( V(x) \) such that \( \frac{d}{dt} V(\varphi(t, x_0)) < 0 \) as Fig. 5.4 shows, then \( \lim_{t \to \infty} \varphi(t, x_0) = 0 \).
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Example 5.2.1 $m\ddot{x} + kx = 0, \quad x(0) = x_0, \quad x'(0) = x'_0$ describes the motion of a spring without friction according to Hooke’s law.

Consider the total energy

$$V(t) = \text{kinetic energy} + \text{potential energy}$$

$$= \frac{1}{2} m(v(t))^2 + \int_0^{x(t)} k s \, ds$$

$$= \frac{1}{2} m(x'(t))^2 + \frac{k}{2} x^2(t).$$

Then, it follows that

$$\frac{d}{dt} V(t) = mx'(t)x''(t) + kx(t)x'(t) \equiv 0,$$

and hence the energy is conserved. In this case the Lyapunov function is $V(x, x') = \frac{m}{2} (x')^2 + \frac{k}{2} x^2$. It is easy to verify that $V(0, 0) = 0$, $V(x, x') > 0$ for all $(x, x') \neq (0, 0)$ and $V(x, x') \to +\infty$ as $|(x, x')| \to +\infty$.

Example 5.2.2 $m\ddot{x} + g(x) = 0, \quad xg(x) > 0, \quad x \neq 0, \quad \int_0^x g(s) ds \to +\infty$ as $|x| \to \infty$. This describes the motion of a spring whose restoring force is a nonlinear function $g(x)$. The energy

$$V(t) = \frac{m}{2} (x'(t))^2 + \int_0^{x(t)} g(s) \, ds \quad (5.5)$$

satisfies $\frac{dV}{dt} \equiv 0$, i.e., the energy is conserved. Hence the solution $(x(t), x'(t))$ is periodic. See Fig. 5.5.
Example 5.2.3 \(mx'' + k(x)x' + g(x) = 0, \ k(x) \geq \delta > 0\) for all \(x\). Use the energy function in (5.5), it follows that

\[
\frac{dV}{dt} = x'(t) [-k(x(t))x'(t) - g(x(t))] + x'(t)g(x(t)) \\
= -k(x(t))(x'(t))^2 < -\delta(x'(t))^2.
\]

Then, we expect \(\lim_{t \to \infty} x(t) = 0\) and \(\lim_{t \to \infty} x'(t) = 0\).

Let \(x(t)\) be the solution of (5.4) and \(V(x)\) be positive definite on \(\Omega\) satisfying \(V(x) \to \infty\) as \(|x| \to \infty\). Compute the derivative of \(V\) along the trajectory of the solution \(x(t)\),

\[
\frac{d}{dt} V(x(t)) = \text{grad} V(x(t)) \cdot x'(t) \\
= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t))f_i(x(t)) \overset{def}{=} \dot{V}(x(t))
\]

Definition 5.2.2 A function \(V : \Omega \to R, \ V \in C^1\) is said to be a Lyapunov function for (5.4) if

\[
\dot{V}(x) = \text{grad} V(x) \cdot f(x) \leq 0 \text{ for all } x \in \Omega.
\]

Remark 5.2.2 If \(V\) is merely continuous on \(\Omega\), we replace \(\frac{d}{dt} V(x(t))\) by

\[
\lim_{h \to 0} \frac{1}{h} [V(x(t+h)) - V(x(t))].
\]
Let $\xi = x(t)$, then $x(t + h) = \varphi(h, \xi)$. Then we define

$$\dot{V}(\xi) = \lim_{h \to 0} \frac{1}{h} \left[ V(\varphi(h, \xi)) - V(\xi) \right].$$

**Remark 5.2.3** In applications to many physical systems, $V(x)$ is the total energy. However for a mathematical problem, we may take $V$ as a quadratic form, namely, $V(x) = x^T B x$, $B$ is some suitable positive definite matrix.

**Example 5.2.4** Consider the Lotka-Volterra predator-prey system,

$$\begin{align*}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= cxy - dy
\end{align*}$$

$a, b, c, d > 0$ (5.6)

where $x = x(t)$, $y = y(t)$ are the densities of prey and predator species respectively. Let $x^* = \frac{d}{c}$, $y^* = \frac{a}{b}$. Then (5.6) can be rewritten as

$$\begin{align*}
\frac{dx}{dt} &= -bx(y - y^*) \\
\frac{dy}{dt} &= cy(x - x^*).
\end{align*}$$

Consider the trajectory in phase plane, we have

$$\frac{dy}{dx} = \frac{cy(x - x^*)}{-bx(y - y^*)}.$$  

(5.7)

Using separation of variables, from (5.7) it follows that

$$\frac{y - y^*}{y} dy + \frac{c}{b} \frac{x - x^*}{x} dx = 0$$

and

$$\int_{y(0)}^{y(t)} \frac{\eta - y^*}{\eta} d\eta + \frac{c}{b} \int_{x(0)}^{x(t)} \frac{\eta - x^*}{\eta} d\eta \equiv \text{const.}$$

Hence we define a Lyapunov function

$$V(x, y) = \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta + \frac{c}{b} \int_{x^*}^{x} \frac{\eta - x^*}{\eta} d\eta$$

(5.8)

$$= \left[ y - y^* - y^* \ln \left( \frac{y}{y^*} \right) \right] + \frac{c}{b} \left[ x - x^* - x^* \ln \left( \frac{x}{x^*} \right) \right].$$
Then it is straightforward to verify that \( \dot{V}(x, y) \equiv 0 \) and hence the system (5.6) is a conservative system. Each solution \((x(t, x_0, y_0), y(t, x_0, y_0))\) is a periodic solution. From the following Fig. 5.6, the system (5.6) has a family of “neutrally stable” periodic orbits.

Consider the system (5.4) and for simplicity we assume 0 is an equilibrium. To verify the stability property of the equilibrium 0, from previous chapter we can obtain the information regarding stability, from the eigenvalues of the variational matrix \( D_x f(0) \). In the following, we present another method by using Lyapunov functions.

**Theorem 5.2.1 (Stability, Asymptotic Stability) [H]**

If there exists a positive definite function \( V(x) \) on a neighborhood \( \Omega \) of 0 such that \( \dot{V} \leq 0 \) on \( \Omega \), then the equilibrium 0 of (5.4) is stable. If, in addition, \( \dot{V} < 0 \) for all \( x \in \Omega \setminus \{0\} \), then 0 is asymptotically stable.

**Proof.** Let \( r > 0 \) such that \( B(0, r) \subseteq \Omega \). Given \( \varepsilon, 0 < \varepsilon < r \) and let \( k = \min_{|x|=\varepsilon} V(x) \). Then \( k > 0 \). From continuity of \( V \) at 0, we choose \( \delta > 0, 0 < \delta \leq \varepsilon \) such that \( V(x) < k \) when \( |x| < \delta \). Then the solution \( \varphi(t, x_0) \) satisfies \( V(\varphi(t, x_0)) \leq V(x_0) \) because of \( \dot{V} \leq 0 \) on \( \Omega \). Hence for \( x_0 \in B(0, \delta) \), \( \varphi(t, x_0) \) stays in \( B(0, \varepsilon) \). Therefore, 0 is stable.

Assume \( \dot{V} < 0 \) for all \( x \in \Omega \setminus \{0\} \). To establish the asymptotic stability of the equilibrium 0, we need to show \( \varphi(t, x_0) \to 0 \) as \( t \to \infty \) if \( x_0 \) is sufficiently close to 0. Since \( V(\varphi(t, x_0)) \) is strictly decreasing in \( t \) for \( |x_0| \leq H \), for some \( H > 0 \). It suffices to show that \( \lim_{t \to \infty} V(\varphi(t, x_0)) = 0 \). If not, we assume \( \lim_{t \to \infty} V(\varphi(t, x_0)) = \eta > 0 \). Then \( V(\varphi(t, x_0)) \geq \eta > 0 \) for all \( t \geq 0 \). By continuity of \( V \) at 0, there exists \( \delta > 0 \) such that \( 0 < V(x) < \eta \) for \( |x| < \delta \). Hence
5.2. LYAPUNOV FUNCTIONS [H]

\[ |\varphi(t, x_0)| \geq \delta \text{ for all } t \geq 0. \]

Set \( S = \{ x : \delta \leq |x| \leq H \} \) and \( \gamma = \min_{x \in S} (-\dot{V}(x)) \).

Then \( \gamma > 0 \) and \(-\frac{d}{dt} V(\varphi(t, x_0)) \geq \gamma \). Integrating both sides from 0 to \( t \) yields \(-[V(\varphi(t, x_0)) - V(x_0)] \geq \gamma t \) or \( 0 < V(\varphi(t, x_0)) \leq V(x_0) - \gamma t \).

Let \( t \to +\infty \), we obtain a contradiction.

**Theorem 5.2.2** If there exists a neighborhood \( \Omega \) of 0 and \( V: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, \ 0 \in \Omega \) such that \( V \) and \( \dot{V} \) are positive definite on \( \Omega \cap \Omega \setminus \{0\} \), then the equilibrium 0 is completely unstable, i.e., 0 is a repeller. More specifically, if \( \Omega_0 \) is any neighborhood of 0, then any solution \( \varphi(t, x_0) \) of (5.4) with \( x_0 \in \Omega \cap \Omega_0 \setminus \{0\} \) leaves \( \Omega_0 \) in finite time.

**Proof.** If not, there exists a neighborhood \( \Omega_0 \) such that \( \varphi(t, x_0) \) stays in \( \Omega \cap \Omega_0 \setminus \{0\} \) for some \( x_0 \in \Omega \cap \Omega_0 \setminus \{0\} \). Then \( V(\varphi(t, x_0)) \geq V(x_0) > 0 \) for all \( t \geq 0 \). Let \( \alpha = \inf \{ V(x) : x \in \Omega \cap \Omega_0, \ V(x) \geq V(x_0) \} > 0 \). Then,

\[
V(\varphi(t, x_0)) = V(x_0) + \int_0^t \dot{V}(\varphi(s, x_0)) ds \\
\geq V(x_0) + \alpha t.
\]

Since \( \varphi(t, x_0) \) remains in \( \Omega \cap \Omega_0 \setminus \{0\} \) for all \( t \geq 0 \), \( V(\varphi(t, x_0)) \) is bounded for \( t \geq 0 \). Thus for \( t \) sufficiently large, we obtain a contradiction from the above inequality.

**Example 5.2.5**

\[
\begin{aligned}
&x' = -x^3 + 2y^3 \\
y' = -2xy^2
\end{aligned}
\quad (5.9)
\]

(0, 0) is an equilibrium. The variational matrix of (5.9) at (0, 0) is \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) whose eigenvalues \( \lambda_1 = \lambda_2 = 0 \) are not hyperbolic. However (0, 0) is indeed asymptotically stable. Use the Lyapunov function \( V(x, y) = \frac{1}{2}(x^2 + y^2) \), then we have

\[
\dot{V} = x(-x^3 + 2y^3) + y(-2xy^2) = -x^4 \leq 0.
\]

From Theorem 5.2.1, (0, 0) is stable. In fact from invariance principle (see Theorem 5.2.4), (0, 0) is globally asymptotically stable.

**Linear Stability by Lyapunov Method [H]**

Consider linear system

\[
x' = Ax, \quad A \in \mathbb{R}^{n \times n}.
\quad (5.10)
\]