The Onset Problem for a Thin Superconducting Loop in a Large Magnetic Field

Tien-Tsan Shieh
Peter Sternberg

Department of Mathematics
Indiana University
Bloomington, IN 47405
Ginzburg-Landau Model

\[ \mathcal{E}(\psi, A) = \int_U \left( \left| (\nabla - iA)\psi \right|^2 + \frac{1}{2} \left( |\psi|^2 - \mu^2 \right)^2 \right) dx \]

\[ + \int_{\mathbb{R}^3} \left| \nabla \times A - H^e \right|^2 dx. \]

Here \( \psi : U \rightarrow \mathbb{C} \), with
\[ |\psi|^2 = \text{density of supercond. charge carriers}, \]
\( A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is the effective mag. potential,
\( \mu^2 \propto T_c - T \),
\( H^e : \mathbb{R}^3 \rightarrow \mathbb{R}^3 = \text{given external magnetic field.} \)
Normal state: $(\psi, A) = (0, A^e)$

where $\nabla \times A^e = H^e$.

The phase transition associated with onset of superconductivity is characterized by the value $\mu^2 \propto (T_c - T)$ at which this normal state loses stability.
Rayleigh quotient problem

Second Variation:

\[ \delta^2 \mathcal{E}(0, A_e; \psi, A) = \int_U \left( |(\nabla - iA_e)\psi|^2 - \mu^2 |\psi|^2 \right) \, dx \]

\[ + 2 \int_{\mathbb{R}^3} |\nabla \times A|^2 \, dx. \]

This leads us to consider the following Rayleigh quotient problem:

\[ \mu_c^2(H^e) := \inf_{\psi} \frac{\int_U |(\nabla - iA_e)\psi|^2 \, dx}{\int_U |\psi|^2 \, dx}. \]
Phase Transition Curve

From the second variation, we know

\[ \mu_c^2(\mathbf{H}^e) > \mu^2(T) \quad \text{Normal state is stable} \]
\[ \mu_c^2(\mathbf{H}^e) < \mu^2(T) \quad \text{Normal state is unstable.} \]

Phase Transition curve:

\[ \mu_c^2(\mathbf{H}_e) = \mu^2(T) = \alpha(T_c - T) = \alpha \Delta T \]
Question

Here we ask:

- Can one (rigorously) derive a model of an onset problem for a thin superconducting loop in a presence of large magnetic field starting from three-dimensional Ginzburg-Landau model?

This case was first treated by Richardson and Rubinstein using formal asymptotic expansion.
Model our superconductor using GL

- Two Assumptions in our study:

1. The sample domain $U$ is a sequence of domains $\{U_\varepsilon\}$ consisting of $\varepsilon$-neighborhoods of a limiting simple closed curve.

2. The given applied field $H_\varepsilon$ take the form

$$H^e_\varepsilon = \frac{H^e}{\varepsilon}$$

where $H^e$ is a given smooth magnetic field independent of $\varepsilon$. 
GL Energy in this setting

- The Ginzburg-Landau energy in this setting is

\[
\mathcal{E}_\varepsilon(\psi, A) = \int_{U_\varepsilon} \left( |(\nabla - iA)\psi|^2 + \frac{1}{2} \left( |\psi|^2 - \mu^2 \right)^2 \right) \, dx \\
+ \int_{\mathbb{R}^3} \left| \nabla \times A - \frac{H^e}{\varepsilon} \right|^2 \, dx.
\]

- The corresponding Rayleigh quotient becomes

\[
E_\varepsilon(\psi) := \frac{\int_{U_\varepsilon} |(\nabla - i\frac{A^e}{\varepsilon})\psi|^2 \, dx}{\int_{U_\varepsilon} |\psi|^2 \, dx}.
\]
Let \( r : [0, L] \to \mathbb{R}^3 \) be a simple, closed \( C^2 \) curve. The triple \( \{t, n, b\} \) forms a Frenet Frame for the curve. The \( \varepsilon \)-neighborhood \( U_{\varepsilon} \) is the image of the cylinder

\[
\Omega = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : 0 \leq y_1 \leq L, y_2^2 + y_3^2 < 1\}
\]

under the mapping

\[
T_{\varepsilon}(y) = r(y_1) + \varepsilon y_2 \mathbf{n}(y_1) + \varepsilon y_3 \mathbf{b}(y_1).
\]
Main Idea and Observation

- **Main Idea:** Identify a limiting energy such that if the minimizers $\psi_\varepsilon$ of $E_\varepsilon$ converge to some limit $\psi_0$ defined on the limiting curve, then $\psi_0$ minimizes this limiting energy.

- **Observation:** The phase of the minimizer $\psi_\varepsilon$ oscillates rapidly as $\varepsilon \to 0$. Without shifting the phase, we can’t attain compactness of minimizers $\psi_\varepsilon$ of $E_\varepsilon$. 
Finding an equivalent functional

We shift the phase function by special phase functions \( \phi_\varepsilon \).

\[
\frac{A_\varepsilon}{\varepsilon} - \nabla \phi_\varepsilon \approx \mathcal{O}(1) \quad \text{in } U_\varepsilon.
\]

This lead us to find an equivalent energy functional to \( E_\varepsilon \).

\[
F_\varepsilon(\psi) := E_\varepsilon(\psi e^{i\phi_\varepsilon}) = \frac{\int_{U_\varepsilon} \left| (\nabla - i(\frac{A^e}{\varepsilon} - \nabla \phi_\varepsilon)) \psi \right|^2 \, dx}{\int_{U_\varepsilon} \left| \psi \right|^2 \, dx}
\]
Effective Magnetic Flux $\beta_\varepsilon$

1. We decompose $A^e$ into components $A_1^e$, $A_2^e$ and $A_3^e$ lying along the tangent, normal and bi-normal to the limiting curve $\mathbf{r}$.

2. We find quantum number of magnetic flux $k_\varepsilon$ which is the closest integer to the number

$$\left( \frac{1}{L} \int_0^L \frac{A_1^e(y_1, 0, 0)}{\varepsilon} dy_1 \right)$$

and set the effective magnetic flux

$$\beta_\varepsilon = \left( \frac{1}{L} \int_0^L \frac{A_1(t, 0, 0)}{\varepsilon} dt \right) - \frac{2\pi}{L} k_\varepsilon.$$
The special phase function $\phi_\varepsilon$ is defined as:

$$\phi_\varepsilon(y_1, y_2, y_3) := \int_0^{y_1} \left( \frac{A_1(t, 0, 0)}{\varepsilon} - \beta_\varepsilon \right) dt$$

$$+ \frac{1}{\varepsilon} \left( y_2 A_2(y_1, 0, 0) + y_3 A_3(y_1, 0, 0) \right)$$

$$+ \frac{y_2^2}{2} \partial_{y_2} A_2(y_1, 0, 0) + \frac{y_3^2}{2} \partial_{y_3} A_3(y_1, 0, 0)$$

$$+ \frac{1}{2} y_2 y_3 \partial_{y_3} A_2(y_1, 0, 0) + \frac{1}{2} y_2 y_3 \partial_{y_2} A_3(y_1, 0, 0)$$
The equivalent functional $F_\varepsilon$

$$F_\varepsilon(\psi) = \frac{\int_{\Omega} \left| \frac{1}{\eta_\varepsilon} \left( \partial_{y_1} \psi + \tau y_3 \partial_{y_2} \psi - \tau y_2 \partial_{y_3} \psi \right) \right|^2 \eta_\varepsilon \, dy}{\int_{\Omega} |\psi|^2 \eta_\varepsilon \, dy}$$

$$- \frac{i \left( \frac{\beta_\varepsilon}{\eta_\varepsilon} - y_2 H_3^\varepsilon + y_3 H_2^\varepsilon \right) \psi - \frac{1}{\varepsilon} R^\varepsilon \psi \right|^2 \eta_\varepsilon \, dy}{\int_{\Omega} |\psi|^2 \eta_\varepsilon \, dy}$$

$$+ \frac{\int_{\Omega} \left| \frac{1}{\varepsilon} \partial_{y_2} \psi + i \left( \frac{1}{2} y_3 \right) H_1^\varepsilon \psi - \left( \frac{i}{\varepsilon} R_2^\varepsilon \right) \psi \right|^2 \eta_\varepsilon \, dy}{\int_{\Omega} |\psi|^2 \eta_\varepsilon \, dy}$$

$$+ \frac{\int_{\Omega} \left| \frac{1}{\varepsilon} \partial_{y_3} \psi - i \left( \frac{1}{2} y_2 \right) H_1^\varepsilon \psi - \left( \frac{i}{\varepsilon} R_3^\varepsilon \right) \psi \right|^2 \eta_\varepsilon \, dy}{\int_{\Omega} |\psi|^2 \eta_\varepsilon \, dy}$$

where $\eta_\varepsilon = 1 - \varepsilon \kappa y_2$ and $\kappa$ is the curvature of the curve $\mathbf{r}$. 

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We guess limiting energy should take the following form:

\[
G_\beta(\psi) := \frac{\int_0^L \left| \left( \frac{d}{dy_1} - i\beta \right) \psi \right|^2 + W(y_1) |\psi|^2 dy_1}{\int_0^L |\psi|^2 dy_1}.
\]

Here

\[
W(y_1) := \frac{1}{8} (H_1^e)^2 + \frac{1}{4} (H_2^e)^2 + \frac{1}{4} (H_3^e)^2.
\]
Theorem 1

We obtain a compactness result

**Theorem 1** Let \( \psi_\varepsilon \) is the minimizer of \( F_\varepsilon \) in \( H^1(\Omega) \). There exists a subsequence \( \{ \psi_\varepsilon_j \} \) and \( \psi_0 \in H^1(\Omega) \) and \( \beta_0 \in \left[ -\frac{\pi}{L}, \frac{\pi}{L} \right] \) such that

\[
\psi_\varepsilon_j \rightharpoonup \psi_0 \quad \text{weakly in } H^1(\Omega), \\
\psi_\varepsilon_j \rightarrow \psi_0 \quad \text{strongly in } L^q(\Omega), 1 \leq q < 6 \\
\beta_\varepsilon_j \rightarrow \beta_0
\]

Note that \( \psi_0 \) is a function of \( y_1 \) only.
Theorem 2

Involving techniques of dimension reduction and $\Gamma$-convergence. We obtain

**Theorem 2** The limiting function $\psi_0 \in H^1((0, L))$ minimizes $G_{\beta_0}$. 
Comparing the minimum of $F_{\varepsilon}$ and the minimum of $G_{\beta \varepsilon}$. This gives us

**Theorem 3** Let $\lambda_{\varepsilon}$ be the minimum of $F_{\varepsilon}$ and let $\sigma_{\varepsilon}$ be the minimum of $G_{\beta \varepsilon}$. Then

$$(\lambda_{\varepsilon} - \sigma_{\varepsilon}) = O(\varepsilon).$$
Theorem 4

Asymptotic relationship between the first eigenspaces of functionals $F_\varepsilon$ and $G_{\beta_\varepsilon}$:

**Theorem 4** Let $\varepsilon_j \to 0$ be any sequence such that

$$\frac{-\pi}{L} < \liminf_{j \to \infty} \beta_{\varepsilon_j} \leq \limsup_{j \to \infty} \beta_{\varepsilon_j} < \frac{\pi}{L}.$$

and let $\psi_{\varepsilon_j}$ be a minimizer of $F_{\varepsilon_j}$ in $H^1(\Omega)$ with $\|\psi_{\varepsilon_j}\|_{L^2(\Omega)} = 1$. Then there exists a sequence $\{\psi_0^{\varepsilon_j}\}$ minimizing $\{G_{\beta_{\varepsilon_j}}\}$ in $H^1_{\text{per}}((0, L))$ with $\|\psi_0^{\varepsilon_j}\|_{L^2(\Omega)} = 1$ such that

$$\psi_{\varepsilon_j} - \psi_0^{\varepsilon_j} \to 0 \quad \text{strongly in } H^1(\Omega).$$
Little-Parks Experiment

In 1961, Little and Parks observed that the phase transition temperature in thin ring is essentially a periodic function of the axial magnetic flux through the ring with a parabolic background.
Our model indeed coincides with the result of the Little-Parks experiment.

Through the relation

$$\mu_c^2(h H_e) = \mu^2(T) = \alpha(T_c - T) = \alpha \Delta T,$$

we obtain

$$\beta_0(h) + \frac{1}{4}(H_3^e)^2 h^2 = \mu_c(h H_e) = \alpha \Delta T.$$