Chapter 11: Fourier Transforms
Sections 8 & 9
1. Fourier transforms

- Consider a function $f$, which is not necessarily periodic, but absolutely integrable (i.e. $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$) and piecewise continuously differentiable on $(-\infty, \infty)$.

- The Fourier transform of $f$ is defined as

$$\mathcal{F}(f) = \hat{f}, \quad \text{where} \quad \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) \, dx.$$  

- The inverse Fourier transform of $\hat{f}$ is defined as

$$\mathcal{F}^{-1}(\hat{f}) = f, \quad \text{where} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) \exp(ikx) \, dk.$$  

- The relation $f = \mathcal{F}^{-1}(\mathcal{F}(f))$ reads

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\nu) \exp(ik(x - \nu)) \, d\nu \, dk. \quad (1)$$
Properties of the Fourier transform

- As for Fourier series, Equation (1), i.e. \( f(x) = \left( \mathcal{F}^{-1}(\hat{f}) \right)(x) \) is only true at points where \( f \) is continuous.
- At a point of discontinuity \( x_0 \) of \( f \), the inverse Fourier transform of \( f \) converges to the average \( \frac{1}{2} \left[ f^+(x_0) + f^-(x_0) \right] \).
- The Fourier transform is a linear transformation, i.e. if \( f_1 \) and \( f_2 \) are such that their Fourier transforms exist and if \( \alpha \) and \( \beta \) are two arbitrary constants, then
  \[
  \mathcal{F}(\alpha f_1 + \beta f_2) = \alpha \mathcal{F}(f_1) + \beta \mathcal{F}(f_2)
  \]
- Fourier transform of the derivative. If \( f \) and its derivatives are piecewise continuously differentiable and are absolutely integrable on \( \mathbb{R} \), and if \( \lim_{x \to \pm \infty} f(x) = 0 \), then the Fourier transform of the derivative of \( f \) is such that \( \hat{f}'(k) = ik \hat{f}(k) \).
The convolution of two absolutely integrable functions $f$ and $g$ is denoted by $f * g$ and defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t) g(t) \, dt = \int_{-\infty}^{\infty} f(t) g(x - t) \, dt.$$  

Convolution theorem. If $f$ and $g$ are both piecewise continuously differentiable and absolutely integrable on $\mathbb{R}$, then the Fourier transform of the convolution of $f$ and $g$ is given by

$$\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g).$$

Example: Find the Fourier transform of $f * g$ where $f(x) = \exp(-ax^2)$, $a > 0$, and $g$ is such that $g(x) = \exp(-ax)$ if $x > 0$ and $g(x) = 0$ otherwise.
2. Sine and cosine transforms

Consider a piecewise continuously differentiable function $f$, which is absolutely integrable on $\mathbb{R}$.

- **If $f$ is even**, then the Fourier transform of $f$ can be written as a cosine transform, i.e.

$$
\hat{f}(k) = \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(kx) \, dx,
$$

and

$$
f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(k) \cos(kx) \, dk.
$$

- **Similarly, if $f$ is odd**, then the Fourier transform of $f$ is a sine transform, i.e. $\hat{f}(k) = -i \hat{f}_s(k)$, where

$$
\hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(kx) \, dx, \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(k) \sin(kx) \, dk.
$$