Research Statement

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My research focuses on the calculus of variations and nonlinear partial differential equations which model various natural phenomena. In particular, I am interested in studying mathematical problems arising from many physical systems which are governed by the minimization of some appropriate free energy. Examples of such systems include the Ginzburg-Landau system of superconductivity, the Cross-Newell energy of pattern formation, the Cahn-Hilliard energy of phase seperation, the Ohta-Kawasaki functional of diblock copolymers, liquid crystals, ferromagnetism and many other systems in condensed matter physics. Changing parameters in these systems, like temperature, often leads to phase transitions and a variety of singularities, including vortices, line singularities, microstructure and domain wall branching and multiple scale oscillations. Most of my work is to better understand phase transitions and formation of singularities in a rigorous mathematical framework.

1 Non-convex Variational Problems

A variational problem for a functional $\mathcal{E} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is to show the existence of a minimizer u^* of $\mathcal{E}$ within the admissible set $\mathcal{A}$. The prototypes for many interesting free energies in physical systems are in the following two forms:

$$\mathcal{E}^\varepsilon[u] = \int_\Omega \big(W(u) + \varepsilon^2 F(Du)\big)\; dx$$

and

$$\mathcal{E}^\varepsilon[u] = \int_\Omega \big(W(Du) + \varepsilon^2 F(D^2u)\big)\; dx$$

where $W[.]$ is non-convex, $F[.]$ is convex and ε is a small parameter. The small parameter $\varepsilon > 0$ introduces a regularizing term in the energy which penalizes interfaces and rapid transitions thereby leading to "physically" relevant minimizers. The $\varepsilon \to 0$ limit of minimizers of the non-convex variational problem of singular perturbation is analogous to the viscosity solutions of the scalar conservation laws.

I am interested in finding the singular limit of energy functionals $\mathcal{E}^\varepsilon$ as $\varepsilon \to 0$. The variational problem for the singular limit will reveal the asymptotic behavior of the minimizers for the energy functional $\mathcal{E}^\varepsilon$. To achieve this goal, we have to address the following questions:

1. **Scaling laws:** How does the scale of the energy $\mathcal{E}^\varepsilon(u^\varepsilon)$ for the minimizer u^ε depend on the small parameter ε? This requires proving suitable upper and lower bounds for the energy E^ε.

2. **Variational convergence:** Does there exist a limiting energy $\mathcal{E}$ such that its minimum determines $\lim_{\varepsilon\to 0} u^\varepsilon$ in some appropriate sense? This usually can be answered through the techniques of Γ-convergence of the energy functional $\{E^\varepsilon\}$ introduced by De Giorgi.

My work on variational problems in pattern formation and thin superconducting loops in large magnetic fields are in this framework.

2 Variational Problems in Pattern Formation

I am interested in the general principles that underlie pattern formation. Stripe patterns are seen in wildly different contexts: whorls in fingerprints, ripples in sandy deserts, zebra stripes and convection rolls. The stripes in all these distinct sytems share many common features including the typical defects that are present in the patterns. A pattern-forming model that is derived from a high Prandtl number Rayleigh-Bénard convection problem is the L^2 gradient flow for the Swift-Hohenberg energy functional

$$\mathcal{F}[\psi] = \int_\Omega \frac{|(\nabla^2+1)\psi|^2}{2} + \left(\frac{\psi^4}{4} - \frac{R\psi^2}{2}\right) dx$$

where ψ is a real order parameter, $\Omega \subset \mathbb{R}^2$ is in a $2D$ domain, $R > 0$ is the Rayleigh number that is not necessarily small. Periodic stripe patterns are among the critical points for the energy. The Swift-Hohenberg model is a microscopic model in the sense that features are resolved on the scale of the wavelength.

Cross and Newell derived a model of a large aspect ratio system, far from onset, in terms of a phase function θ of the pattern. This was done via modulation of the stable critical points $\psi = f(\theta)$ of the Swift-Hohenberg energy, where f is an even, 2π-periodic function of θ. In a time scale $\mathcal{O}(\varepsilon^2)$, the dynamics of such patterns is driven by the gradient ow of the regularized Cross-Newell energy

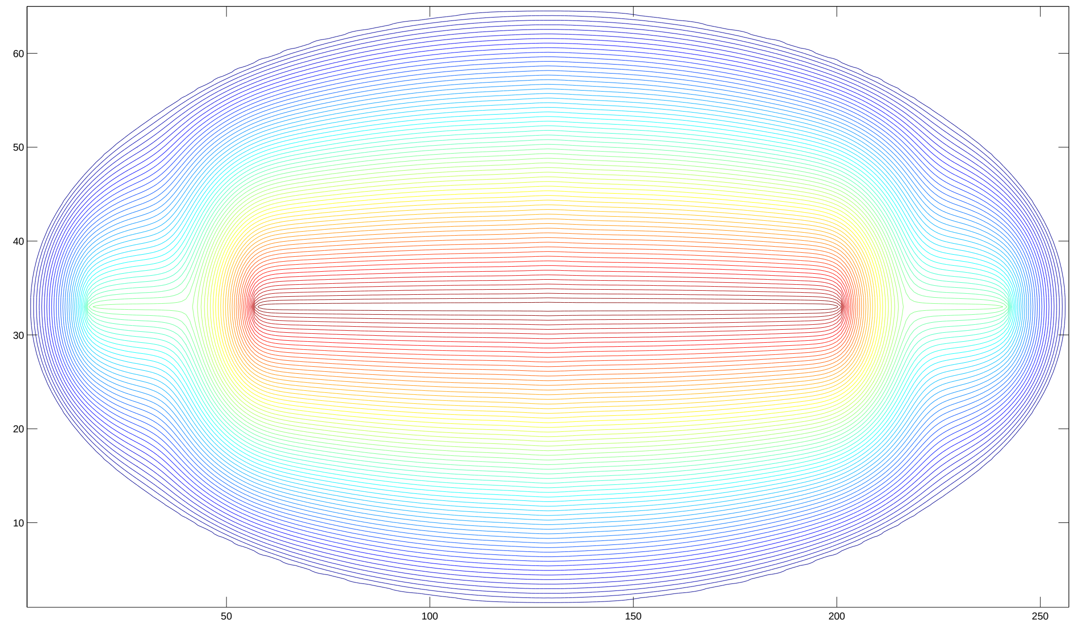
In a rescaling, in terms of the slow variables $\Theta = \varepsilon\theta$, $X = \varepsilon x$, $Y = \varepsilon y$, the Cross-Newell energy takes a standard form, also known as the Aviles-Giga energy, whose minimizers have only sharp interfaces along one dimensional arcs. However, in natural pattern-forming systems, zero dimensional defects coexist with the one dimensional ones. The difference is due to the structure of all possible phase functions and how they are related to the real order parameter ψ of the Swift-Hohenberg energy by the relation $\psi = f(\theta)$ through an even 2π-periodic function f. This implies that the phase functions θ should be multi-valued functions mapping $\Omega \to \mathcal{Q}$, where $\mathcal{Q}$ is a real line $\mathbb{R}$ modulo by two identifications $\theta \to (\theta + 2\pi)$ and $\theta \to -\theta$. For purely topological reasons, the multiple-valued phase Cross-Newell model can capture point defects like concave disclinations and convex disclinations which are often seen in the stripe pattern-forming systems. This motivatates an interesting variation of the Cross-Newell energy among an appropriate multi-valued function space $H^2(\Omega, \mathcal{Q})$.

I have been involved in both analytical and the numerical work on this problem.

1. **Numerical simulation supports the multi-phase minimizer of the regular Cross-Newell energy on a ellipse:**
   I numerically minimize the regular Cross-Newell energy. The minimizer of the energy supports the observation of convex and concave disclinations found in the experiment and other similar pattern-formation systems.

2. **Existence of minimizers of the regularized Cross-Newell energy among multi-valued functions on a ellipse:**
   Along with Shankar Venkataramani and Nick Ercolani, we have shown the existence of minimizers of the regularized Cross-Newell energy among the class of multi-valued functions on an ellipse, which have a lower energy than the energy of minimizers among the class of single-valued functions, as in Kohn and Jin's paper [5].

3. **Find an upper bound of the regularized Cross-Newell energy among multi-valued functions on a ellips:**
   We construct an upper bound of the rularized Cross-Newell energy among multi-valued functions on ellipse. This upper bound indicates that the multiple scaling behaviour of the minimizer. One scaling is due to dislocation of patterns and another is is due to point defect of patterns.



2.1 Summary of Results

For more detail, we consider the variational problem

$$\inf_{\theta \in H^2(\Omega_{\varepsilon,a},\mathcal{Q})} \mathcal{E}(\theta;\Omega_{\varepsilon,a})$$

of the Cross-Newell energy

$$\mathcal{E}(\theta;\Omega_{\varepsilon,a}) = \int_{\Omega_{\varepsilon,a}} (\Delta\theta)^2 + \left(|\nabla\theta|^2 - 1\right)^2 \, dx.$$

on the flat ellipse $\Omega_{\varepsilon,a} = \{(x,y)|\, \varepsilon^2 x^2 + y^2 < a^2\}$ in the regime $\varepsilon \ll 1$ and $a \gg 1$. The numerical results suggest that in symmetrical geometries the multi-valued variational problem could be reduced to a single-valued variational problem by using the symmetry. In the ellipse case, the symmetry of the boundary condition between the upper and lower half-ellipse induces a natural equivalent single-valued variational problem of the regularized Cross-Newell energy on the upper right ellipse Ω^+ with mixed Dirichlet and Neumann boundary conditions on the middle horizontal axis. This is

$$\inf_{n\in\mathbb{N}} \inf_{A_n,B_n} \inf_{\theta\in H^2_A(\Omega^+)} \mathcal{E}(\theta;\Omega^+)$$

where $A_n = \{0 = a_0 < a_1 < a_2 < a_3 \cdots < a_{2n-1} < a_{2n} = \frac{a}{\varepsilon}\}$ is a partition of the interval $[0, \frac{a}{\varepsilon}]$ and $H^2_A(\Omega^+)$ is the Sobolev space containing all the $H^2(\Omega^+)$-functions with boundary conditions: the Dirichlet condition $\theta(x,y) = 0$ on the top boundary of the ellipse $\partial\Omega^+ \cap \{y > 0\}$, the Neumann condition $\theta_x(0,y) = 0$ for $0 \le y \le a$, and the mixed Dirichlet-Neumann boundary condition on the x-axis:

$$\begin{aligned} \theta(x,0) &= b_k\pi \qquad a_{2k} \le x \le a_{2k+1} \quad \text{for } k = 0,1,2,\ldots,n \\ \theta_y(x,0) &= 0 \qquad a_{2k+1} \le x \le a_{2k+2} \quad \text{for } k = 0,1,2,\ldots,n-1 \end{aligned}$$

where $B_n = \{b_k \in \mathbb{Z}|\, k = 0,1,2,\ldots,n-1\}$.

The existence of the minimizer is obtained by the standard direct method of calculus of variations. For each ε and a, we construct a test function $\theta_{\varepsilon,a}$ such that

$$\mathcal{E}(\theta_{\varepsilon,a}) \le \frac{C_1}{\varepsilon} + C_2 a$$

for some constants C_1 and C_2. However, the minimum energy of the usual Aviles-Giga functional in the single-valued function space is of the order $\mathcal{O}(\frac{a}{\varepsilon})$. Thus in the regime $\varepsilon \ll 1$ and $a \gg 1$, the multi-valued minimizer of the Cross-Newell energy has lower energy than the energy of minimizers among a class of single-valued functions in Kohn and Jin's paper [5].

2.2 Future directions

My future planned research on this problem includes:

1. Characterize the Sobolev space $H^2(\Omega, \mathcal{Q})$. Since the space $\mathcal{Q}$ is not a manifold, this is a challenging and interesting problem.
2. Give a rigorous construction of the first variation for the energy functional defined on the multi-valued space $H^2(\Omega, \mathcal{Q})$.
3. Show that the regularized Cross-Newell energy is a variational limit for the Swift-Hohenberg energy rigorously in some appropriate sense.
4. Study the bifurcation behaviour between line defects and point defects of the system.
5. Find the lower bound of the regular Cross-Newell energy on ellipse.

3 Ginzburg-Landau System

I study the Ginzburg-Landau system of superconductivity. A superconductor is a material which undergoes a phase transition from the normal state, like conventional metal, to the superconducting state, in which it can support electric currents without resistance and the applied magnetic field is expelled when cooled below a certain critical temperature. In the presence of an applied magnetic field acting on the superconducting material, the critical temperature associated with this phase transition decreased. It is proved in [7] that superconductivity breaks down when subjected to sufficiently large magnetic field. The problem I study is the phase transition between normal and superconducting states for a thin domain in a large magnetic field. It was considered in the presence of a large magnetic field on a fixed domain in [6, 8, 9, 10]. The problem also has been treated in the case of a very thin domain for a fixed magnetic field in [12]. Richardson and Rubinstein considered the problem with the two different regimes together, namely a thin domain and a large magnetic field in [11]. The authors used formal asymptotic expansions to analyze the Ginzburg-Landau system in this setting and my Ph.D thesis, supervised by Peter Sternberg, focused on making rigorous some of the results in Richardson and Rubinstein's paper. My work can be summarized as follows:

3.1 Onset of a thin superconducting loop in a large magnetic field

I consider the dimensionless Ginzburg-Landau energy

$$G_\varepsilon(\psi, \mathbf{A}) = \int_{\Omega_\varepsilon} \Big(\frac{1}{2}\big|(\nabla - i\mathbf{A})\psi\big|^2 + \frac{\mu}{4}\big(|\psi|^2 - 1)^2\Big)\, dx + \frac{\kappa^2}{2\mu}\int_{\mathbb{R}^3} \big|\nabla \times \mathbf{A} - \frac{1}{\varepsilon}\mathbf{H}_e\big|^2\, dx \qquad (1)$$

where Ω_ε is the loop consisting of an ε-neighborhood of a given simple closed curve C, $\psi : \Omega_\varepsilon \to \mathbb{C}$ is a complex order parameter (where $|\psi|^2$ represents the density of superconducting electron pairs), $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential corresponding to the magnetic field $\mathbf{B}$ via $\nabla \times \mathbf{A} = \mathbf{B}$ and $\mathbf{A}_e$ is the applied magnetic potential corresponding to the given applied magnetic field $\mathbf{H}_e$. The parameter κ is the dimensionless Ginzburg-Landau parameter and μ is proportional to $(T_c - T)$

where T is the temperature and T_c is the critical temperature in the absence of any applied field. The second variation of the functional G_ε evaluated at the trivial solution $(0, \frac{\mathbf{A}_e}{\varepsilon})$, the normal state, takes the form:

$$\delta^2 G\big(0, \frac{\mathbf{A}_e}{\varepsilon}; \psi, \mathbf{A}\big) = \int_{\Omega_\varepsilon} \Big(\big|(\nabla - i\frac{\mathbf{A}_e}{\varepsilon})\psi\big|^2 - \mu|\psi|^2 \Big)\, dx + \frac{\kappa^2}{\mu} \int_{\mathbb{R}^3} \big|\nabla \times \mathbf{A}\big|^2\, dx \tag{2}$$

The normal state first loses stability when the temperature-related parameter μ drops below the infimum of the following Rayleigh quotient

$$\lambda_\varepsilon = \inf_{\psi \in H^1(\Omega_\varepsilon)} \frac{\int_{\Omega_\varepsilon} \big|(\nabla - i\frac{\mathbf{A}_e}{\varepsilon})\psi\big|^2\, dx}{\int_{\Omega_\varepsilon} |\psi|^2\, dx} \,. \tag{3}$$

As ε becomes very small, I find that the problem (3) is close, in the sense of Γ-convergence, to the following one-dimensional Rayleigh quotient problem

$$\widetilde{\lambda}_\varepsilon = \inf_{u \in H^1_{per}(0,L)} \frac{\int_0^L |\frac{\partial u}{\partial s} - i\beta_\varepsilon u|^2 + g(s)|u|^2 ds}{\int_0^L |u|^2 ds} \tag{4}$$

where L is the length of the curve C, $\beta_\varepsilon \in [-\frac{\pi}{L}, \frac{\pi}{L}]$ is a constant related to the given applied potential and g is an explicit non-negative function. This means the difference between the first eigenvalue λ_ε of the full problem and the first eigenvalue $\widetilde{\lambda}_\varepsilon$ of the limiting problem approaches zero as $\varepsilon \to 0$, as does the difference between the first eigenfunction of the problems (3) and (4).

3.2 Γ-Convergence of the full Ginzburg-Landau Energy

Having already analyzed the onset problem for a thin domain with a large magnetic field, I turned to the more general question of Γ-convergence for the full Ginzburg-Landau functional (1).

I define a new equivalent functional $F_\varepsilon(\psi, \mathbf{A}) = G_\varepsilon(\psi e^{i\phi_\varepsilon}, \mathbf{A} + \frac{\mathbf{A}_e}{\varepsilon})$ by introducing a special phase function $\phi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ such that

$$\frac{\mathbf{A}_e}{\varepsilon} - \nabla\phi_\varepsilon \approx \mathcal{O}(1) \qquad \text{on } \Omega_\varepsilon \,.$$

Applying the dimension reduction argument, I show that for given sequence $\{\varepsilon_j\} \to 0$ along which β_{ε_j} converge to β, the Γ-limit of the seauence $\{\frac{1}{{\varepsilon_j}^2} F_{\varepsilon_j}\}$ is

$$F_0(u, \mathbf{A}) = \begin{cases} \int_0^L \frac{1}{2}\big|\frac{\partial u}{\partial s} - i\beta u\big|^2 + \frac{g(s)}{2}|u|^2 + \frac{\mu}{4}(|u|^2 - 1)^2\, ds & \text{if } \mathbf{A} = 0 \text{ a.e.} \\ +\infty & \text{otherwise .} \end{cases} \tag{5}$$

Because of the Little-Parks effect arising from the non-simply-connected nature of the domain, it is not possible to obtain the Γ-limit for all $\varepsilon \to 0$. One obstacle is that the limit still has order 1 ε-dependence. Another main difficulty arises from non-uniform cross-section. By introducing this phase function ϕ_ε, we attain the compactness for minimizers of this new equivalent energy functional F_ε and resolve the difficulty due to the domain with non-uniform cross-section.

3.3 Future Directions.

My future planned research on this problem includes:

1. **Spectrum of a thin superconducting loop in a large magnetic field :**
   Having already shown the convergence of the first eigenvalue in the onset problem for a thin domain with a large magnetic field, I am turning my attention to the convergence of all eigenvalues and all eigenvectors of the Rayleigh quotient (3). By using a Γ-convergence argument with the min-max method, I hope to be able to prove the convergence of all eigenvalues.

2. **Derive models of thin liquid crystals by applying the dimension reduction technique:**
Many liquid crystal devices are made very thin. Thus, by using the same dimension reduction technique on the Frank-Ossen energy and the Landau-de Gennes energy, it is possible to identify the limiting energies and to understand the responses of thin liquid crystals to electric fields.

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