Introduction to Distance-Regular Graphs

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Let $\Gamma = (X, R)$ denote a graph (not necessary distance-regular for this moment) with diameter $D$. 
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For two vertices $x, y \in X$, a walk of length $t$ from $x$ to $y$ is a sequence of vertices $x = u_0, u_1, \ldots, u_t = y$ such that $u_i u_{i+1} \in R$. 
Let $A = A_1$ denote the adjacency matrix of $\Gamma$.

**Lemma**

$A_{xy}^t$ is the number of walks of length $t$ from $x$ to $y$. 
Number of walks

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\( A_{xy}^t \) is the number of walks of length \( t \) from \( x \) to \( y \).

**Proof.**

\[
A_{xy}^t = \sum_{u_1, u_2, \ldots, u_{t-1} \in X} A_{xu_1} A_{u_1 u_2} \cdots A_{u_{t-2} u_{t-1}} A_{u_{t-1} y} \\
= \sum_{u_1, u_2, \ldots, u_{t-1} \in X} 1.
\]
Lemma

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Proof.

Suppose $g(A) = 0$ for some polynomial of degree $k \leq D$. Pick two vertices $x, y \in X$ with distance $k$. Then $g(A)_{xy} \neq 0$, a contradiction.
Find all graphs of diameter $D$ whose adjacency matrices with minimal polynomials of degree $D + 1$. 
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From now on let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. We have seen earlier that $A_i = f_i(A)$, $f_{D+1}(A) = A_{D+1} = 0$, and $f_i(x)$ is monic with degree $i$, where $f_0(x) := 1$, $f_1(x) := x$ and $f_i(x)$ is defined recursively using

$$xf_i(x) = b_{i-1}f_{i-1}(x) + a_if_i(x) + c_{i+1}f_{i+1}(x) \quad 2 \leq i \leq D.$$
Recall

From now on let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. We have seen earlier that $A_i = f_i(A)$, $f_{D+1}(A) = A_{D+1} = 0$, and $f_i(x)$ is monic with degree $i$, where $f_0(x) := 1$, $f_1(x) := x$ and $f_i(x)$ is defined recursively using

$$xf_i(x) = b_{i-1}f_{i-1}(x) + a_if_i(x) + c_{i+1}f_{i+1}(x) \quad 2 \leq i \leq D.$$ 

In particular, the adjacency matrix of a distance-regular graph of diameter $D$ has minimal polynomial of degree $D+1$. 
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2. \( k = b_0 \) is an eigenvalue of \( A \) with eigenvector \( j := (1, 1, \ldots, 1)^t \), i.e. \( A j = k j \).
3. \( A \) has \( D + 1 \) distinct eigenvalues \( \theta_0 = k, \theta_1, \ldots, \theta_D \) since its minimal polynomial has degree \( D + 1 \).
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4. Let $V_0, V_1, \ldots, V_D$ be corresponding orthogonal eigenspaces.
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5. Let $E_i$ be a matrix to present the projection of $\mathbb{R}^X$ into $V_i$ in standard basis. $E_i$ are called the primitive idempotents of $\Gamma$. 
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8. $A^i E_j u = \theta_j^i E_j u$ for $u \in \mathbb{R}^x$. 
Eigenvalues of distance-regular graphs

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7. $A E_j u = \theta_j E_j u$ for $u \in \mathbb{R}^X$.
8. $A^i E_j u = \theta_j^i E_j u$ for $u \in \mathbb{R}^X$.
9. $A_i E_j u = f_i(A) E_j u = f_i(\theta_j) E_j u$ for $u \in \mathbb{R}^X$. 

Invertible Vandermonde matrix

In previous page we show

\[ A^i E_j u = \theta^i_j E_j u \quad \text{for} \quad u \in \mathbb{R}^x. \]
Invertible Vandermonde matrix

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\[ A^i E_j u = \theta_j^i E_j u \quad \text{for} \quad u \in \mathbb{R}^x. \]

Then

\[
\begin{pmatrix}
I \\
A \\
A^2 \\
\vdots \\
A^D
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\theta_0 & \theta_1 & \theta_2 & \cdots & \theta_D \\
\theta_0^2 & \theta_1^2 & \theta_2^2 & \cdots & \theta_D^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta_0^D & \theta_1^D & \theta_2^D & \cdots & \theta_D^D
\end{pmatrix}
\begin{pmatrix}
E_0 \\
E_1 \\
E_2 \\
\vdots \\
E_D
\end{pmatrix}
\]
Invertible Vandermonde matrix

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\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta_0^D & \theta_1^D & \theta_2^D & \cdots & \theta_D^D
\end{pmatrix}
\begin{pmatrix}
E_0 \\
E_1 \\
E_2 \\
\vdots \\
E_D
\end{pmatrix}
\]

In particular \( E_i \in M = \langle A \rangle \) and is symmetric.
Bose Mesner algebra

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Bose Mesner algebra

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$= \text{Span}\{A_0, A_1, A_2, \ldots, A_D\}$

$= \text{Span}\{E_0, E_1, E_2, \ldots, E_D\}$. 
The eigenmatrices $P$ and $Q$

Define the matrices $P$ and $Q$ to satisfy

\[
\begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_D \\
\end{pmatrix}
= 
\begin{pmatrix}
E_0 \\
E_1 \\
\vdots \\
E_D \\
\end{pmatrix}
= 
|X|^{-1}Q
\begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_D \\
\end{pmatrix}.
\]
The polynomials

\[ \mathcal{K}_k(x; n, q) := \sum_{i=0}^{k} \binom{x}{i} \binom{n-x}{k-i} (-1)^i (q - 1)^{k-i} \]

are called the Krawtchouk polynomials.
The Hamming graph $H(D, 2)$

It turns out the Hamming graph $H(D, 2)$ has entries

$$P_{ij} = Q_{ij} = (K_i(j; D, 2))_{ij}$$

of the eigenmatrices $P$ and $Q$ for $0 \leq i, j \leq D$. 
For a subset $Y \subseteq X$ and $0 \leq i \leq D$, define

$$d_i(Y) = \frac{1}{|Y|} \sum_{y \in Y} |\Gamma_i(y) \cap Y| = \frac{Y^t A_i Y}{|Y|},$$

where $Y$ is the characteristic vector of $Y$. 
Distribution vectors

For a subset $Y \subseteq X$ and $0 \leq i \leq D$, define

$$d_i(Y) = \frac{1}{|Y|} \sum_{y \in Y} |\Gamma_i(y) \cap Y| = \frac{Y^t A_i Y}{|Y|},$$

where $Y$ is the characteristic vector of $Y$. The column vector

$$d(Y) = (d_0(Y), d_1(Y), \ldots, d_D(Y))^t$$

is called the distribution vector of $Y$. 

1 $d_i(Y)$ is the average number of elements in $Y$ with distance $i$ to a vertex in $Y$. 
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2. $d_0(Y) + d_1(Y) + \cdots + d_D(Y) = |Y|$. 

Remarks on distribution vectors

1. $d_i(Y)$ is the average number of elements in $Y$ with distance $i$ to a vertex in $Y$.

2. $d_0(Y) + d_1(Y) + \cdots + d_D(Y) = |Y|$.

3. The minimum distance of $Y$ is $\min\{i \mid 1 \leq i \leq D, d_i(Y) \neq 0\}$. 
1. \( d_i(Y) \) is the average number of elements in \( Y \) with distance \( i \) to a vertex in \( Y \).

2. \( d_0(Y) + d_1(Y) + \cdots + d_D(Y) = |Y| \).

3. The minimum distance of \( Y \) is \( \min\{i \mid 1 \leq i \leq D, d_i(Y) \neq 0\} \).

4. The study \( \mathbf{d}(Y) \) is important in coding theory. For example in \( H(D, 2) \), MacWilliams identity in coding theory is a description of \( \mathbf{d}(Y) \) and \( \mathbf{d}(Y^\perp) \), where \( Y \subseteq X = F_2^D \) is a subspace of \( F_2^D \) and \( Y^\perp \) consists of vectors orthogonal to each vector of \( Y \).
Theorem

(Delsarte’s thesis) For any $Y \subseteq X$,

$$Q_d(Y) \geq 0.$$
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Proof.

\[ (Q_d(Y))_i = \sum_{j=0}^{D} Q_{ij} d_j(Y) \]
Theorem

(Delsarte’s thesis) For any $Y \subseteq X$, 

$$Q_d(Y) \geq 0.$$ 

Proof.

$$\left(Q_d(Y)\right)_i = \sum_{j=0}^{D} Q_{ij} d_j(Y) = \sum_{j=0}^{D} Q_{ij} \frac{Y^t A_j Y}{|Y|}$$
Theorem

(Delsarte’s thesis) For any $Y \subseteq X$,

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Proof.

$$\begin{align*}
(Q_d(Y))_i &= \sum_{j=0}^{D} Q_{ij}d_j(Y) = \sum_{j=0}^{D} Q_{ij} \frac{Y^t A_j Y}{|Y|} \\
&= Y^t \left( \sum_{j=0}^{D} Q_{ij} \frac{A_j}{|Y|} \right) Y
\end{align*}$$
Theorem

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Proof.

$$\left(Q_d(Y)\right)_i = \sum_{j=0}^{D} Q_{ij} d_j(Y) = \sum_{j=0}^{D} Q_{ij} \frac{Y^t A_j Y}{|Y|}$$  

$$= Y^t \left( \sum_{j=0}^{D} Q_{ij} \frac{A_j}{|Y|} \right) Y$$  

$$= \frac{|X|}{|Y|} Y^t E_i Y$$
Linear programming bound

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(Delsarte’s thesis) For any \( Y \subseteq X \),

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\[
(Q_d(Y))_i = \sum_{j=0}^{D} Q_{ij} d_j(Y) = \sum_{j=0}^{D} Q_{ij} \frac{\mathbf{Y}^t A_j \mathbf{Y}}{|Y|}
\]

\[
= \mathbf{Y}^t \left( \sum_{j=0}^{D} Q_{ij} \frac{A_j}{|Y|} \right) \mathbf{Y}
\]

\[
= \frac{|X|}{|Y|} \mathbf{Y}^t E_i \mathbf{Y} = (E_i \mathbf{Y})^t E_i \mathbf{Y}
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**Theorem**

(Delsarte’s thesis) For any $Y \subseteq X$,

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**Proof.**

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(Q_d(Y))_i = \sum_{j=0}^{D} Q_{ij}d_j(Y) = \sum_{j=0}^{D} Q_{ij} \frac{Y^t A_j Y}{|Y|}
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= Y^t (\sum_{j=0}^{D} Q_{ij} \frac{A_j}{|Y|}) Y
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= \frac{|X|}{|Y|} Y^t E_i Y = (E_i Y)^t E_i Y \geq 0.
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Suppose $Y \subseteq X$ with minimum distance $k$. 
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**Problem** Maximize

$$1 + d_k + d_{k+1} + d_{k+2} + \cdots + d_D,$$

with variables $d_i$ subject to $d_i \geq 0$ for all $i$, $d_k > 0$, and $Qd \geq 0$. 
Suppose $Y \subseteq X$ with minimum distance $k$. Then $|Y| = 1 + d_k(Y) + d_{k+1}(Y) + \cdots + d_D(Y)$.

**Problem** Maximize

$$1 + d_k + d_{k+1} + d_{k+2} + \cdots + d_D,$$

with variables $d_i$ subject to $d_i \geq 0$ for all $i$, $d_k > 0$, and $Qd \geq 0$.

The answer of the above problem is an upper bound of the size of all $Y \subseteq X$ with minimum distance $k$. 
Application of Delsarte’s linear programming bound


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For $T \subseteq \{1, \ldots, D\}$, $Y \subseteq X$ is a $T$-design if $(Qd(Y))_i = 0$ for all $i \in T$. 
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For $T \subseteq \{1, \ldots, D\}$, $Y \subseteq X$ is a $T$-design if $(Qd(Y))_i = 0$ for all $i \in T$.

Theorem (Delsarte's thesis) In $J(n, D)$ and $Y \subseteq X = \left(\begin{matrix} n \\ D \end{matrix}\right)$, $Y$ is a $\{1, 2, \ldots, t\}$-design if and only if $(\bigcup_{y \in Y} y, Y)$ is a $t$-$(\big| \bigcup_{y \in Y} y \big|, D, \lambda)$ design for some $\lambda$. 
We raised the question to define a design in a distance-regular graph.

For $T \subseteq \{1, \ldots, D\}$, $Y \subseteq X$ is a $T$-design if $(Qd(Y))_i = 0$ for all $i \in T$.

**Theorem (Delsarte’s thesis)** In $J(n, D)$ and $Y \subseteq X = \binom{[n]}{D}$, $Y$ is a $\{1, 2, \ldots, t\}$-design if and only if $(\bigcup_{y \in Y} y, Y)$ is a $t$-$(|\bigcup_{y \in Y} y|, D, \lambda)$ design for some $\lambda$.

**Proof.** Skip.