Root systems and Coxeter groups

Coxeter groups I

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August 13, 2009
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Root systems and Coxeter groups

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Coxeter groups
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Let $m : S \times S \to \mathbb{N} \cup \{\infty\}$ be a function satisfied $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in $S$. Let $F$ denote the free group on the set $S$. Let $N$ be the normal subgroup generated by all elements

$$(ss')^{m(s,s')}.$$
Coxeter groups

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$$(ss')^{m(s,s')}.$$ 

The group $W := F/N$ is called a Coxeter group, and the pair $(W, S)$ is called a Coxeter system.
Let $\Gamma$ be the undirected graph with vertex set $S$. Join vertices $s$ and $s'$ by an edge labelled $m(s, s')$ whenever the number is at least 3. $\Gamma$ is call the \textbf{Coxeter graph} of $(W, S)$.
Coxeter groups

Let $\Gamma$ be the undirected graph with vertex set $S$. Join vertices $s$ and $s'$ by an edge labelled $m(s, s')$ whenever the number is at least 3. $\Gamma$ is call the **Coxeter graph** of $(W, S)$. We say a Coxeter system $(W, S)$ is **irreducible** if its Coxeter graph $\Gamma$ is connected.
Coxeter groups

Example: Let $S = \{s\}$. Then $N = \{\ldots, s^{-4}, s^{-2}, 1, s^2, s^4, \ldots\}$ and hence $W = \{N, sN\} \cong \{1, -1\}$. 
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sN \leadsto s
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Coxeter groups

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$$
W = \{1, s\}
$$
Question

For \( s \in S \), show that \( s \neq 1 \) in \( W \), in particular \( s \) has order 2 in \( W \).
Universal property:
Let $G$ be a group and let $\phi$ be a map of $S$ into $G$. 

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Let $G$ be a group and let $\phi$ be a map of $S$ into $G$. Then $\phi$ induces the homomorphism $\phi_* : W \to G$ satisfied $\phi_*(s) = \phi(s)$ for $s \in S$ if and only if

$$ (\phi(s)\phi(s'))^{m(s,s')} = 1 $$

for $s, s' \in S$. 
By universal property, there is a homomorphism \( \epsilon : W \to \{1, -1\} \) satisfied \( \phi(s) = -1 \) for \( s \in S \).
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**Question**

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**Proof.**
By universal property, there is a homomorphism $\epsilon : W \to \{1, -1\}$ satisfied $\phi(s) = -1$ for $s \in S$.

**Question**

For $s \in S$, show that $s \neq 1$ in $W$, in particular $s$ has order 2 in $W$.

**Proof.** From the homomorphism $\epsilon$, we have $s \neq 1$ in $W$.  \[\Box\]
Length function
Length function

Since \( s \in S \) has order 2 in \( W \), each \( \omega \in W \) can be written in the form

\[
\omega = s_1 s_2 \cdots s_r
\]

for some \( s_i \) (not necessarily distinct) in \( S \).
Since \( s \in S \) has order 2 in \( W \), each \( \omega \in W \) can be written in the form
\[
\omega = s_1 s_2 \cdots s_r
\]
for some \( s_i \) (not necessarily distinct) in \( S \). If \( r \) is as small as possible, call it the \textbf{length} of \( \omega \), written \( \ell(\omega) \), and call any expression of \( \omega \) as a product of \( r \) elements of \( S \) a \textbf{reduced expression}.
Length function

Basic properties for $\ell(\omega)$:

$\ell(\omega) = \ell(\omega - 1)$.

$\ell(\omega) = 1$ if and only if $\omega \in S$.

$\ell(\omega' \omega) \leq \ell(\omega) + \ell(\omega')$.

$\ell(\omega' \omega) \geq \ell(\omega) - \ell(\omega')$.

$\ell(\omega) - 1 \leq \ell(\omega_s) \leq \ell(\omega) + 1$, for $s \in S$ and $\omega \in W$. 
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- $\ell(\omega \omega') \geq \ell(\omega) - \ell(\omega')$.
- $\ell(\omega) - 1 \leq \ell(\omega s) \leq \ell(\omega) + 1$, for $s \in S$ and $\omega \in W$. 
Length function

Proposition

The homomorphism $\epsilon : W \rightarrow \{1, -1\}$ is given by $\epsilon(\omega) = (-1)^{\ell(\omega)}$. As a result, $\ell(\omega s) = \ell(\omega) \pm 1$, for all $s \in S$, $\omega \in W$, and similarly for $\ell(s\omega)$.

Proof.
**Proposition**

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**Proof.** Write a reduced expression $\omega = s_1 \cdots s_r$. Then

$$\epsilon(\omega) = \epsilon(s_1) \cdots \epsilon(s_r) = (-1)^r = (-1)^{\ell(\omega)},$$

as required.
Proposition

The homomorphism $\epsilon : W \to \{1, -1\}$ is given by $\epsilon(\omega) = (-1)^{\ell(\omega)}$. As a result, $\ell(\omega s) = \ell(\omega) \pm 1$, for all $s \in S$, $\omega \in W$, and similarly for $\ell(s \omega)$.

Proof. Write a reduced expression $\omega = s_1 \cdots s_r$. Then

$$\epsilon(\omega) = \epsilon(s_1) \cdots \epsilon(s_r) = (-1)^r = (-1)^{\ell(\omega)},$$

as required. Now $\epsilon(\omega s) = -\epsilon(\omega)$ implies that $\ell(\omega s) \neq \ell(\omega)$. Hence the length must differ by precisely 1. \qed
Geometric representation of $W$
Geometric representation of $W$

**Question**
For $s \neq s'$ in $S$, show that $s \neq s'$ in $W$.

**Question**
What is the order of $ss'$ in $W$ for $s \neq s'$ in $S$?
Geometric representation of $W$

Let $V$ denote a vector space over $\mathbb{R}$ having a basis $\{\alpha_s \mid s \in S\}$ of cardinality $|S|$. Define a symmetric bilinear form $B$ on $V$ by requiring:

$$B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')}.$$
Geometric representation of $W$

Basic properties for $B$:

- $B(\alpha_s, \alpha_s) = 1$
- $B(\alpha_s, \alpha_s') \leq 0$ if $s \neq s'$.
- The subspace $H_s$ orthogonal to $\alpha_s$ relative to $B$ is complementary to line $R_{\alpha_s}$ (Exercise).
- For $s \neq s'$ in $S$, the bilinear form $B$ restricted on $R_{\alpha_s} \oplus R_{\alpha_s'}$ is positive definite if and only if $m(s, s') < \infty$. 
Geometric representation of $W$

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- The subspace $H_s$ orthogonal to $\alpha_s$ (relative to $B$) is complementary to line $\mathbb{R}\alpha_s$ (Exercise).
- For $s \neq s'$ in $S$, the bilinear form $B$ restricted on $\mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_{s'}$ is positive definite if and only if $m(s, s') < \infty$. 
For each \( s \in S \), define a reflection \( \sigma_s : V \to V \) by the rule:

\[
\sigma_s \lambda := \lambda - 2B(\alpha_s, \lambda)\alpha_s.
\]

Basic properties for \( \sigma_s \):

\[\sigma_s \alpha_s = -\alpha_s \text{ and } \sigma_s \text{ fixes } H_s \text{ pointwise.}\]

\[\sigma_s \text{ has order 2 in } GL(V).\]

\[\sigma_s \neq \sigma_{s'} \text{ for } s \neq s' \text{ in } S.\]

\[B(\sigma_s \lambda, \sigma_s \mu) = B(\lambda, \mu) \text{ for all } \lambda, \mu \in V.\]
For each $s \in S$, define a reflection $\sigma_s : V \rightarrow V$ by the rule:

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- $\sigma_s$ has order 2 in $\text{GL}(V)$.
- $\sigma_s \neq \sigma_{s'}$ for $s \neq s'$ in $S$.
- $B(\sigma_s \lambda, \sigma_s \mu) = B(\lambda, \mu)$ for all $\lambda, \mu \in V$. 
Our task is to show that there exists a homomorphism $\sigma : W \rightarrow GL(V)$, sending $s$ to $\sigma_s$. 
It is enough to check that

\[(\sigma_s \sigma_{s'})^{m(s,s')} = 1\]

for \(s \neq s'\). More precisely, we shall show that the order of \(\sigma_s \sigma_{s'}\) is \(m(s, s')\).
Geometric representation of $W$

Consider $\sigma_s \sigma_{s'}$ acts on $\mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_{s'}$. 
Consider \( \sigma_s \sigma_{s'} \) acts on \( R\alpha_s \oplus R\alpha_{s'} \).

**Case I:** \( m := m(s, s') < \infty \).
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**Case I:** $m := m(s, s') < \infty$. Here $B$ is positive definite. Both $\sigma_s$ and $\sigma_{s'}$ acts as reflections in euclidean plane. Since $B(\alpha_s, \alpha_{s'}) = -\cos(\pi/m) = -\cos(\pi - (\pi/m))$, the angle between $\mathbb{R}\alpha_s$ and $\mathbb{R}\alpha_{s'}$ is $\pi - (\pi/m)$, forcing the angle between the two reflecting lines to be $\pi/m$. 
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**Case I:** $m := m(s, s') < \infty$. Here $B$ is positive definite. Both $\sigma_s$ and $\sigma_{s'}$ acts as reflections in euclidean plane. Since $B(\alpha_s, \alpha_{s'}) = -\cos(\pi/m) = -\cos(\pi - (\pi/m))$, the angel between $\mathbb{R}\alpha_s$ and $\mathbb{R}\alpha_{s'}$ is $\pi - (\pi/m)$, forcing the angle between the two reflecting lines to be $\pi/m$. From our previous study, we recognize $\sigma_s \sigma_{s'}$ as a rotation through the angle $2\pi/m$; it therefore has order $m$. 
Geometric representation of $W$

Case II: $m = \infty$. By induction, we can show that $(\sigma_s\sigma_{s'})^k(\alpha_s) = (2k + 1)\alpha_s + 2k\alpha_{s'}$. This implies that $\sigma_s\sigma_{s'}$ has infinite order.
Proposition

There is a unique homomorphism $\sigma : W \to \text{GL}(V)$ sending $s$ to $\sigma_s$, and the group $\sigma(W)$ preserves the form $B$. Moreover, for each pair $s \neq s'$ in $S$, $s \neq s'$ in $W$ and the order of $ss'$ in $W$ is precisely $m(s, s')$. 

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Geometric representation of $W$

Geometric interpretation of the length function
Geometric representation of $W$

**Proposition**

There is a unique homomorphism $\sigma : W \to \text{GL}(V)$ sending $s$ to $\sigma_s$, and the group $\sigma(W)$ preserves the form $B$. Moreover, for each pair $s \neq s'$ in $S$, $s \neq s'$ in $W$ and the order of $ss'$ in $W$ is precisely $m(s, s')$.

We refer to the homomorphism $\sigma$ as the **geometric representation** of $W$. 
Geometric interpretation of the length function
For $\lambda \in V$, we write $\omega(\lambda)$ in place of $\sigma(\omega)(\lambda)$. 

\[ \Phi := \{ \omega(\alpha_S) \mid \omega \in W, s \in S \}. \]

$\Phi$ is called the root system of $W$. 

$\Phi = -\Phi$, since $s(\alpha_S) = -\alpha_S$. 

$\omega(\alpha_S)$ is a unit vector, i.e. $B(\omega(\alpha_S), \omega(\alpha_S)) = 1$. 

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Let $\Phi := \{\omega(\alpha_s) \mid \omega \in W, s \in S\}$. $\Phi$ is called the **root system** of $W$.

- $\Phi = -\Phi$, since $s(\alpha_s) = -\alpha_s$.
- $\omega(\alpha_s)$ is a unit vector, i.e. $B(\omega(\alpha_s), \omega(\alpha_s)) = 1$. 


If $\alpha \in \Phi$, we can write it uniquely in the form

$$\alpha = \sum_{s \in S} c_s \alpha_s \ (c_s \in R).$$

Call $\alpha \in \Phi$ **positive** (resp. **negative**) and write $\alpha > 0$ (resp. $\alpha < 0$) if all $c_s \geq 0$ (resp. all $c_s \leq 0$). For example, $\alpha_s > 0$ for $s \in S$. 

**Geometric interpretation of the length function**
Geometric interpretation of the length function

If $\alpha \in \Phi$, we can write it uniquely in the form

$$\alpha = \sum_{s \in S} c_s \alpha_s \ (c_s \in \mathbb{R}).$$

Call $\alpha \in \Phi$ **positive** (resp. **negative**) and write $\alpha > 0$ (resp. $\alpha < 0$) if all $c_s \geq 0$ (resp. all $c_s \leq 0$). For example, $\alpha_s > 0$ for $s \in S$.

Let $\Phi^+ := \{ \alpha \in \Phi \mid \alpha > 0 \}$ and $\Phi^- := \{ \alpha \in \Phi \mid \alpha < 0 \}$. 
Geometric interpretation of the length function

**Theorem**

Let $\omega \in W$ and $s \in S$. If $\ell(\omega s) > \ell(\omega)$, then $\omega(\alpha_s) > 0$. If $\ell(\omega s) < \ell(\omega)$, then $\omega(\alpha_s) < 0$. 

Proof. Omitted.
Theorem

Let $\omega \in W$ and $s \in S$. If $\ell(\omega s) > \ell(\omega)$, then $\omega(\alpha_s) > 0$. If $\ell(\omega s) < \ell(\omega)$, then $\omega(\alpha_s) < 0$.

Proof. Omitted.
Geometric interpretation of the length function

Corollary

\[ \Phi = \Phi^+ \cup \Phi^- . \]
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Corollary

The representation \( \sigma : W \rightarrow \text{GL}(V) \) is faithful.

Proof.
Geometric interpretation of the length function

Corollary

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Corollary

*The representation* \( \sigma : W \rightarrow \text{GL}(V) \) *is faithful.*

*Proof. Let* \( \omega \in \text{Ker} \sigma \). If \( \omega \neq 1 \), there exists \( s \in S \) for which \( \ell(\omega s) < \ell(\omega) \).
Geometric interpretation of the length function

Corollary

\[ \Phi = \Phi^+ \cup \Phi^- . \]

Proof. Let \( \omega \in \text{Ker} \sigma \). If \( \omega \neq 1 \), there exists \( s \in S \) for which \( \ell(\omega s) < \ell(\omega) \). The above theorem says that \( \omega(\alpha_s) < 0 \). But \( \omega(\alpha_s) > 0 \), which is a contradiction.
The following is a precise description of the way in which $W$ permutes $\Phi$.

**Proposition**

*For any $\omega \in W$, $\ell(\omega)$ equals the number of positive roots sent by $\omega$ to negative roots.*

*Sketch of Proof.*
The following is a precise description of the way in which $W$ permutes $\Phi$.

**Proposition**

*For any $\omega \in W$, $\ell(\omega)$ equals the number of positive roots sent by $\omega$ to negative roots.*

*Sketch of Proof.* Let $n(\omega) := \text{Card}(\Phi^+ \cap \omega^{-1}(\Phi^-))$. Evidently, $n(1) = 0$. 
The following is a precise description of the way in which $W$ permutes $\Phi$.

**Proposition**

For any $\omega \in W$, $\ell(\omega)$ equals the number of positive roots sent by $\omega$ to negative roots.

**Sketch of Proof.** Let $n(\omega) := \text{Card}(\Phi^+ \cap \omega^{-1}(\Phi^-))$. Evidently, $n(1) = 0$. From the above theorem, we see that it suffices to show that if $\omega(\alpha_s) > 0$ then $n(\omega s) = n(\omega) + 1$, and if $\omega(\alpha_s) < 0$ then $n(\omega s) = n(\omega) - 1$. $\square$
Geometric interpretation of the length function

The geometric characterization for the length function is useful.
Geometric interpretation of the length function

Exercise

If $W$ is infinite, prove that the length function takes arbitrarily large values, hence that $\Phi$ is infinite. (Therefore the scalar $-1 \in \text{GL}(V)$ does not lie in $\sigma(W)$.) If $W$ is finite, prove that there is one and only one element $\omega_0 \in W$ of maximum length, and that $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. 

Proof.
Exercise

If $W$ is infinite, prove that the length function takes arbitrarily large values, hence that $\Phi$ is infinite. (Therefore the scalar $-1 \in \text{GL}(V)$ does not lie in $\sigma(W)$.) If $W$ is finite, prove that there is one and only one element $\omega_0 \in W$ of maximum length, and that $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. 

Proof. If the length function $\ell$ has a large value $k$, then $|W| \leq \sum_{0 \leq i \leq k} |S|^i$ is finite. This shows the first assertion.
Exercise

If $W$ is infinite, prove that the length function takes arbitrarily large values, hence that $\Phi$ is infinite. (Therefore the scalar $-1 \in \text{GL}(V)$ does not lie in $\sigma(W)$.) If $W$ is finite, prove that there is one and only one element $\omega_0 \in W$ of maximum length, and that $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. 

**Proof.** If the length function $\ell$ has a large value $k$, then $|W| \leq \sum_{0 \leq i \leq k} |S|^i$ is finite. This shows the first assertion. Suppose $W$ is finite. Let $\omega_0 \in W$ be an element of maximum length. Evidently, $\ell(\omega_0 s) < \ell(\omega_0)$. By the above proposition, $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. 
Exercise

If $W$ is infinite, prove that the length function takes arbitrarily large values, hence that $\Phi$ is infinite. (Therefore the scalar $-1 \in \text{GL}(V)$ does not lie in $\sigma(W)$.) If $W$ is finite, prove that there is one and only one element $\omega_0 \in W$ of maximum length, and that $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. 

Proof. If the length function $\ell$ has a large value $k$, then $|W| \leq \sum_{0 \leq i \leq k} |S|^i$ is finite. This shows the first assertion. Suppose $W$ is finite. Let $\omega_0 \in W$ be an element of maximum length. Evidently, $\ell(\omega_0 s) < \ell(\omega_0)$. By the above proposition, $\omega_0$ maps $\Phi^+$ onto $\Phi^-$. Let $u \in W$ be another element of maximum length. Then $\omega_0 u^{-1}$ maps $\Phi^+$ onto $\Phi^+$. This implies that $\ell(\omega_0 u^{-1}) = 0$, i.e. $\omega_0 = u$. This shows the second assertion. □
Question: Suppose $W$ is finite. How to find the unique longest element $\omega_0$ in $W$.

Solution.
Question

Suppose $W$ is finite. How to find the unique longest element $\omega_0$ in $W$.

Solution. By the geometric characterization for the length function $\ell$, we see $\omega_0$ as the unique element in $W$ satisfying $\ell(\omega s) < \ell(\omega)$ for all $s \in S$. 
Geometric interpretation of the length function

Question
Suppose $W$ is finite. How to find the unique longest element $\omega_0$ in $W$.

Solution. By the geometric characterization for the length function $\ell$, we see $\omega_0$ as the unique element in $W$ satisfying $\ell(\omega s) < \ell(\omega)$ for all $s \in S$. Start at $\omega = s_1$ where $s_1 \in S$. We can successively multiply $\omega$ on the right by $s \in S$ (increasing the length by 1) until this is no longer possible and $\omega_0$ is obtained.
Thanks for your attention