Vertex-Transitive and Cayley Graphs

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- Cayley graphs: Let $G$ be a finite group and $S$ a subset of $G$ not containing the identity element 1. Assume $S^{-1} = S$. We define the Cayley graph $\Gamma = \text{Cay}(G, S)$ on $G$ with respect to $S$ by

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V(\Gamma) &= G, \\
E(\Gamma) &= \{(g, sg) \mid g \in G, s \in S\}.
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Cayley graphs

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on $G$ with respect to $S$.

1. $\text{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of $G$, so $\Gamma$ is vertex-transitive.

2. $\Gamma$ is connected if and only if $G = \langle S \rangle$.

A graph $\Gamma = (V, E)$ is a Cayley graph of a group $G$ if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup isomorphic to $G$.

Above Proposition $\Rightarrow$ Cayley graphs are just those vertex-transitive graphs whose full automorphism groups have a regular subgroup.
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The Petersen graph is not Cayley.

Outline of a proof:
If it is Cayley, then the group has order 10.
There are two non-isomorphic groups: cyclic and dihedral.
The girth of Cayley graphs on abelian groups are 3 or 4. So $G = D_{10}$.

$G = \langle a, b | a^5 = b^2 = 1, bab \rangle$.

$S = \{a, a^{-1}, b\}$ or three involutions.
For the former, since $abab$ is a cyclic of size 4, this is not the case.
For the latter, the product of any two involutions is 1 or of order 5, a contradiction.

Note: For a positive integer $n$, if every transitive group has a regular subgroup then every vertex-transitive graph is Cayley.
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  (Example: $n = p$, a prime.)
A Question

Let \( NR = \{ n \in \mathbb{N} \mid \text{there is a transitive group of degree } n \text{ without a regular subgroup} \} \)

\( NC = \{ n \in \mathbb{N} \mid \text{there is a vertex-transitive graph of order } n \text{ which is non-Cayley} \} \)

Then \( NR \subseteq NC \).

Question: \( NR = NC \)?

Answer: \( NR \nsubseteq NC \). For example, 12 \( \not\in NC \), but 12 \( \in NR \) since \( M_{11} \), acting on 12 points, has no regular subgroup.

Exercise: 6 is the smallest number in \( NR \setminus NC \) since \( A_6 \) has no regular subgroups.
A Question

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\[ \mathcal{NR} = \{ n \in \mathbb{N} \mid \text{there is a transitive group} \]
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\[ \mathcal{NC} = \{ n \in \mathbb{N} \mid \text{there is a vertex-transitive graph} \]
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Then \( \mathcal{NR} \supseteq \mathcal{NC} \).

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- **Answer:** \( \mathcal{NR} \supsetneq \mathcal{NC} \). For example, \( 12 \notin \mathcal{NC} \), but \( 12 \in \mathcal{NR} \) since \( M_{11} \), acting on 12 points, has no regular subgroup.
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- **Question:** $$\mathcal{NR} = \mathcal{NC}$$?
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- **Exercise:** 6 is the smallest number in $$\mathcal{NR} \setminus \mathcal{NC}$$ since $$A_6$$ has no regular subgroups.
Any transitive group $G$ of degree $p^2$ on $\Omega$ has a regular subgroup, i.e., $p^2 \in \mathbb{N}$. Outline of a proof: Take a minimal transitive subgroup $P$ of $G$. Then $P$ is a $p$-group and every maximal subgroup $M$ of $P$ is intranisitive. For any $\alpha \in \Omega$, we have $|P_\alpha| = |P|/p^2$ and $|M_\alpha| > |M|/p^2$, so $M_\alpha = P_\alpha$. It follows that $P_\alpha \leq M$ and hence $P_\alpha \leq \Phi(P)$. If $|P:\Phi(P)| = p$, then $P$ is cyclic and is regular. If $|P:\Phi(P)| = p^2$, then $P_\alpha = \Phi(P)$. Since $\Phi(P)$ is normal in $P$ and $P_\alpha$ is core-free, we have $P_\alpha = 1$ and hence $P \sim Z_2p$ is regular.
$p^2 \notin NC$

- (Marušičič) Any transitive group $G$ of degree $p^2$ on $\Omega$ has a regular subgroup, i.e., $p^2 \notin NR$. 

Outline of a proof: Take a minimal transitive subgroup $P$ of $G$. Then $P$ is a $p$-group and every maximal subgroup $M$ of $P$ is intranisitive. For any $\alpha \in \Omega$, we have $|P_\alpha| = |P|/p^2$ and $|M_\alpha| > |M|/p^2$, so $M_\alpha = P_\alpha$. It follows that $P_\alpha \leq M$ and hence $P_\alpha \leq \Phi(P)$. If $|P:\Phi(P)| = p$, then $P$ is cyclic and is regular. If $|P:\Phi(P)| = p^2$, then $P_\alpha = \Phi(P)$. Since $\Phi(P)$ is normal in $P$ and $P_\alpha$ is core-free, we have $P_\alpha = 1$ and hence $P \sim Z_{p^2}$ is regular.
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A Question

Let $G$ be the following group of order $p^4$:

$$G = \langle a, b \mid a^{p^2} = b^{p^3} = c^{p^4} = 1, [a, b] = c, [c, a] = a^{p^{2a}}, [c, b] = 1 \rangle.$$ 

Let $H = \langle c \rangle$. Consider the transitive permutation representation $\varphi$ of $G$ acting on the coset space $G : H$.

Then $\varphi(G)$ is a transitive group of degree $p^3$, and $\varphi(G)$ has no regular subgroups.
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- (Marušič ič) $p^3 \notin \mathcal{NC}$.
- $p^3 \in \mathcal{NR}$ (For $p > 2$).

Let $G$ be the following group of order $p^4$

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Let \( G \) be the following group of order \( p^4 \)

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Let \( H = \langle c \rangle \). Consider the transitive permutation representation \( \varphi \) of \( G \) acting on the coset space \([G : H] \).

Then \( \varphi(G) \) is a transitive group of degree \( p^3 \), and \( \varphi(G) \) has no regular subgroups.
A Question

- \( p^3 \in \mathcal{N}\mathcal{R} \) (For \( p = 2 \)).

Let

\[
G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^4 = 1, \\
[a, b] = [b, c] = [c, a] = 1, a^d = ab, b^d = bc, c^d = c \rangle.
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Then \( G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4 \) has order \( 2^5 \). Let \( H = \langle b, d^2 \rangle \) and \( \varphi \) be the transitive permutation representation of \( G \) acting on the coset space \( [G : H] \).

Then \( \varphi(G) \) is a transitive group of degree \( 2^3 \) and has no regular subgroup.
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  Then $\varphi(G)$ is a transitive group of degree $2^3$ and has no regular subgroup.
Determine the set $\mathcal{N}$.
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**Theorem**

Let $n$ be a positive integer greater than 1. Then $n \in \mathcal{N}\mathcal{R}$ unless $n = p$ or $p^2$ for a prime $p$. 
Let $p < q$ be two primes. Then $pq \in \mathcal{N}\mathcal{R}$.

**Theorem**

*Let $n$ be a positive integer greater than 1. Then $n \in \mathcal{N}\mathcal{R}$ unless $n = p$ or $p^2$ for a prime $p$.***
Further Questions
Let \( \mathbb{N}_2 \mathbb{R} = \{ n \in \mathbb{N} \mid \text{there is a 2-closed transitive group of degree } n \text{ without a regular subgroup} \} \)

\( \mathbb{N}_D = \{ n \in \mathbb{N} \mid \text{there is a vertex-transitive digraph of order } n \text{ which is non-Cayley} \} \)

Question 1: Is \( \mathbb{N}_2 \mathbb{R} = \mathbb{N}_C \)?

Question 2: Is \( \mathbb{N}_D = \mathbb{N}_C \)?
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Let \( PNR = \{ n \in \mathbb{N} \mid \text{there is a primitive group of degree} \ n \ \text{without a regular subgroup} \} \).

Determine the set \( PNR \).

Note: Different from the set \( NR \), we know that \( p_n / \in PNR \) for any prime \( p \) and any positive integer \( n \). Thus, determining the set \( PNR \) should be much harder than \( NR \).
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