2-Connected Graphs

Chih-wen Weng

September 22
2-connected graph

Recall $G$ is 2-connected if $\kappa(G) \geq 2$. Equivalently $G$ is connected and $G - x$ is connected for any vertex $x \in V$.

**Definition 0.1.** Let $u, v$ be two vertices in $V$. Two $u, v$-paths are *internally disjoint* if they have no common internal vertex.
Lemma 0.2. Suppose for any two distinct vertices $u, v$ there exist internally disjoint $u, v$-paths in $G$. Then $G$ is 2-connected.

Proof. Of course $G$ is connected. We shall show $G - x$ is connected for any $x \in V$. Pick two vertices $u, v$ in $G - x$ and two internally disjoint $u, v$-paths. Then at least of the $u, v$-paths is in $G - x$. Hence $G - x$ is connected. \qed

Definition 0.3. For $u, v \in V(G)$, let $\partial(u, v)$ be the length of shortest $u, v$-path. $\partial(u, v)$ is called the distance of $u, v$. 
Whitney Theorem [1932]

Theorem 0.4. Suppose \( G \) is 2-connected with at least three vertices. Then for any two distinct vertices \( u, v \) there exist internally disjoint \( u, v \) paths in \( G \).

Proof. We prove by induction on the distance \( \partial(u, v) \). First suppose \( \partial(u, v) = 1 \). Hence \( uv \) is an edge. We need to find another path in \( G - uv \). Recall that \( \kappa'(G) \geq \kappa(G) \geq 2 \). Hence \( G - uv \) is connected. Next suppose the theorem is true for \( \partial(u, v) \leq k \). \( \square \)
Continue of Proof

Proof. Now assume $\partial(u, v) = k + 1$. Pick a vertex $w$ with $\partial(u, w) = 1$ and $\partial(w, v) = k$. By induction we can find two internally disjoint $w, v$-paths $P, Q$. Since $G - w$ is connected we can find a $u, v$-path $R$ in $G - w$. Let $z$ be the first vertex in $R$ that meets $P$ or $Q$, say $P$. Then the combine of $u, z$-path of $R$ with the $z, v$-path of $P$ is internally disjoint from $uw \cup Q$. \qed
Expanding Lemma

Lemma 0.5. Suppose $G$ is 2-connected and $G'$ is obtained from $G$ by adding a new vertex $y$ with at least 2 neighbors in $G$. Then $G'$ is 2-connected.

Proof. Of course $G'$ is connected. We need to show $G' - x$ is connected for any vertex $x$ in $G'$. Pick two vertices $u, v$ in $G' - x$. If $u, v$ are in $G$ then we can find a $u, v$-path in $G - x$ by the 2-connected property of $G$. So assume one of $u, v$ is $y$, say $v = y$. Choose a neighbor $w$ of $y$ that is not $x$. Pick a $u, w$-path in $G - x$. Combine this $u, w$-path with $wv$ to obtain a $u, v$-path in $G' - x$. □
Characterization

Theorem 0.6. Let $G$ be a graph with at least three vertices. Then the following (i)-(iv) are equivalent.

(i) $G$ is 2-connected.

(ii) For all vertices $u, v \in V$, there are internally disjoint $u, v$-paths.

(iii) For all vertices $u, v \in V$, there is a cycle through $u$ and $v$.

(iv) $\delta(G) \geq 1$ and every pair of edges in $G$ lies on a common cycle.
Proof

Proof. (iv)⇒(iii) Clear.

(iii)⇒(ii) Clear.

(ii)⇒(i) This Lemma 0.2.

(i)⇒(iv) Since $G$ is connected with at least 2 vertices, every vertex has degree at least 1. Let $uv$ and $xy$ be two edges in $G$. Add to $G$ new vertices $w$ with neighborhood $\{u, v\}$ and $z$ with neighborhood $\{x, y\}$. Since $G$ is 2-connected, the Expansion Lemma implies the new graph $G'$ is 2-connected. □
Continue of Proof

Proof. By Theorem 0.4, there are two internally disjoint \( w, z \)-paths \( P, Q \) in \( G' \). Note that \( u, v \) (resp. \( x, y \)) are in different paths since \( w \) (resp. \( z \)) has degree 2. Say \( u, x \in P \) and \( v, y \in Q \). Let \( C \) be the cycle combining the \( w, z \)-path \( P \) and the \( z, w \)-path of reversed \( Q \). Note that \( v, w, u \) and \( x, z, y \) are two paths of length 2 in \( C \). Replacing \( v, w, u \) and \( x, z, y \) by \( v, u \) and \( x, y \) respectively yields the desired cycle through \( vu \) and \( xy \).