Partially Distance-regular Graphs and Partially Walk-regular Graphs

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Abstract

We study partially distance-regular graphs and partially walk-regular graphs as generalizations of distance-regular graphs and walk-regular graphs respectively. We conclude that the partially distance-regular graphs can be viewed as some extremal graphs of partially walk-regular graphs. In the special case that the graph is assumed to be regular with four distinct eigenvalues, a well known class of walk-regular graphs, we show that there exists a rational function $f$ in the expression of the order and the four eigenvalues of the graph such that $k_2(x)$, the number of vertices with distance 2 to a vertex $x$, satisfies $k_2(x) \geq f$; furthermore we show the equality holds for each vertex $x$ if and only if the graph is distance-regular with diameter 3.

Keywords: Partially distance-regular graphs; partially walk-regular graphs, eigenvalues.

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1 Introduction

Partially distance-regular graphs and partially walk-regular graphs (formal definition in Section 2 and Section 3) are generalizations of distance-regular graphs and walk-regular graphs respectively. They were introduced by M. A. Fiol and E. Garriga when they studied the range of the spectrum of a graph [6].

We study these two classes of graphs and find their properties by following the classic way in the study of distance-regular graphs and walk-regular graphs respectively. We study the link between these two classes of graphs, and conclude that the partially distance-regular graphs can be viewed as some extremal graphs of partially walk-regular graphs. See Theorem 3.3 for detailed description.

We apply Theorem 3.3 to the case when the graph $\Gamma$ is assumed to be regular and has exactly four distinct eigenvalues. It is well known that $\Gamma$ is walk-regular. See Lemma 5.1 for the generalization of this result. We show that there exists a rational function $f$ in the expression of the order and the four eigenvalues of $\Gamma$ such that $k_2(x)$, the number of vertices with distance 2 to a vertex $x$, satisfies $k_2(x) \geq f$. Furthermore we show the equality holds for each vertex $x$ of $\Gamma$ if and only if $\Gamma$ is distance-regular with diameter 3. See Theorem 6.1 for detail. Theorem 6.1 answers a conjecture in [4].

It is well known that the Godsil switching, a way of switching edges of a graph (described in Section 4), will destroy the distance-regularity of a graph while preserving its spectrum. We show in Corollary 4.3 that the walk-regularity of a graph is preserved by Godsil switching provided the switching exists.

2 Partially Distance-regular Graphs

Throughout the paper, let $\Gamma = (X, R)$ denote a simple connected graph with diameter $d$, order $n$, and length distance function $\partial$. For an integer $i$ and $x \in X$, set

$$\Gamma_i(x) := \{y \mid y \in X, \ \partial(y, x) = i\}.$$
Γ is \( k \)-regular or regular for short if \( |\Gamma_1(x)| = k \) for all \( x \in X \). The \( i \)-th distance matrix \( A_i \) is an \( n \times n \) matrix with rows and columns indexed by \( X \) such that
\[
(A_i)_{xy} = \begin{cases} 
1, & \text{if } \partial(x, y) = i; \\
0, & \text{else},
\end{cases} \quad (x, y \in X).
\]
Hence \( A_i = 0 \) for \( i < 0 \) or \( i > d \). \( A = A_1 \) is called the adjacency matrix of \( \Gamma \).

**Definition 2.1.** We say that \( \Gamma \) is \( t \)-partially distance-regular whenever for each integer \( 0 \leq i \leq t \), there exists a polynomial \( v_i(\lambda) \in \mathbb{R}[\lambda] \) of degree \( i \) such that \( A_i = v_i(A) \). \( \Gamma \) is distance-regular if \( \Gamma \) is \( d \)-partially distance-regular.

Hence any graph is 1-partially distance-regular from the above definition.

**Definition 2.2.** Fix integers \( 0 \leq i, j, h \leq d \). We say \( P_{ij}^h \) is well-defined in \( \Gamma \) whenever for any two vertices \( x, y \in X \) with \( \partial(x, y) = h \), the number
\[
p_{ij}^h(x, y) := |\Gamma_i(x) \cap \Gamma_j(y)|
\]
is independent of the choice of \( x, y \). In this case we write \( p_{ij}^h \) for \( p_{ij}^h(x, y) \), and call \( p_{ij}^h \) the intersection number of \( \Gamma \). For convenience we will write \( c_h, a_h, b_h \), and \( k_h \) for the symbols \( p_{1-h}^1, p_1^h, p_1^{h+1} \) and \( p_{hh}^0 \) respectively.

Note that \( k_1(x) \) is the valency of \( x \in X \). Observe that for \( h \geq 1 \), \( a_h(x, y) + b_h(x, y) + c_h(x, y) = k_1(x) \) for \( x, y \in X \) with \( \partial(x, y) = h \). The following proposition is similar to a well-known property in the study of distance-regular graphs [1, Proposition 1.1].

**Proposition 2.3.** Fix an integer \( 1 \leq t \leq d \). Then the following are equivalent.

(i) \( \Gamma \) is \( t \)-partially distance-regular.
(ii) \( p_{ij}^h \) are well-defined for \( 0 \leq i + j, k \leq t \).
(iii) \( c_i, a_{i-1}, b_{i-2} \) are well-defined for \( 1 \leq i \leq t \).

**Proof.** ( (i) \( \implies \) (ii) ) Fix two integers \( i, j \) with \( 0 \leq i+j \leq t \). By the assumption (i) we know \( \{A_0, A_1, \ldots, A_t\} \) is a basis of the vector space \( \{ f(A) \mid f(\lambda) \in \mathbb{R}[\lambda] \text{ has degree at most } t \} \) over \( \mathbb{R} \), and \( A_iA_j \) is in the vector space. Hence
\[
A_iA_j = \sum_{k=0}^{t} c_{ij}^k A_k \tag{2.1}
\]
for some constants $c^k_{ij} \in \mathbb{R}$. For any two vertices $x, y \in X$ with $\partial(x, y) = k \leq t$, comparing the $xy$ entry on both sides of (2.1), we find that $p^k_{ij}(x, y) = c^k_{ij}$ is independent of $x, y$.

((ii)$\implies$(iii)) For $1 \leq i \leq t$, $c_i = p^i_{i-1}$, $a_{i-1} = p^{i-1}_{i-1}$, $b_{i-2} = p^{i-2}_{i-1}$ are well-defined, since the sum is $i \leq t$ in each of the subscripts of intersection numbers.

((iii)$\implies$(i)) Note that

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i$$

by comparing entries on both sides, or equivalently

$$A_i = c_i^{-1}((A - a_{i-1}I)A_{i-1} - b_{i-2}A_{i-2})$$

for $1 \leq i \leq t - 1$. Hence $A_i = v_i(A)$ where $v_i(\lambda) \in \mathbb{R}[\lambda]$ has degree $i$ and is defined recursively by $v_0(\lambda) = 1$, $v_1(\lambda) = \lambda$, and

$$v_i(\lambda) = c_i^{-1}((\lambda - a_{i-1})v_{i-1}(\lambda) - b_{i-2}v_{i-2}(\lambda))$$

for $2 \leq i \leq t$. \hfill $\square$

It is not clear at this moment that $k_t = p^0_0$ is well-defined from Proposition 2.3(ii). The following lemma explains this.

**Lemma 2.4.** Suppose $\Gamma$ is $t$-partially distance-regular, where $t \geq 2$. Then $b_{t-1}$ and $k_i$ are well-defined for $0 \leq i \leq t$.

**Proof.** We apply Proposition 2.3. Note $b_0$ is well-defined since $t \geq 2$. Since $b_{t-1} = b_0 - a_{t-1} - c_{t-1}$, we find $b_{t-1}$ is well-defined. As in [2, Chapter 5], we have $k_i = b_0b_1 \cdots b_{i-1}/(c_1c_2\cdots c_i)$, and hence $k_i$ is well-defined for $0 \leq i \leq t$. \hfill $\square$

### 3 Partially Walk-regular Graphs

We now give the definition of the second class of graphs in the title of the paper.

**Definition 3.1.** We say that $\Gamma$ is $t$-partially walk-regular whenever for each integer $1 \leq i \leq t$, $(A^i)_{xx}$ is a constant depending on $i$, but not on $x \in X$. $\Gamma$ is walk-regular if $\Gamma$ is $t$-partially walk-regular for any integer $t$. 

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Hence in a $t$-partially walk-regular graph, the number of closed walks of length $i$ from a vertex $x$ to itself is a constant, depending on $i \leq t$ not on $x \in X$. In particular, a 2-partially walk-regular graph is regular with valency $(A^2)_{xx}$ for any $x \in X$.

**Lemma 3.2.** Suppose $\Gamma$ is $t$-partially distance-regular, where $t \geq 2$. Then $\Gamma$ is $2t$-partially walk-regular.

**Proof.** Fix a positive integer $u \leq t$. Suppose

$$A^{u-1} = \sum_{i=0}^{u-1} t_i A_i, \quad A^u = \sum_{i=0}^{u} s_i A_i$$

for some $t_i, s_i \in \mathbb{R}$. Note that $k_i$ is well-defined for $0 \leq i \leq t$ by Lemma 2.4. Then

$$\begin{align*}
(A^{2u-1})_{xx} &= \sum_{y \in X} (A^{u-1})_{xy}(A^u)_{yx} \\
&= \sum_{y \in X} (\sum_{i=0}^{u-1} t_i(A_i)_{xy})(\sum_{i=0}^{u} s_i(A_i)_{xy}) \\
&= \sum_{i=0}^{u-1} k_it_is_i,
\end{align*}$$

and

$$\begin{align*}
(A^{2u})_{xx} &= \sum_{y \in X} (\sum_{i=0}^{u} s_i(A_i)_{xy})^2 \\
&= \sum_{i=0}^{u} k_is_i^2
\end{align*}$$

are independent of the choice of $x \in X$. \hfill \Box

The converse of the above lemma is false. $\overline{C_6}$, the complement of a cycle of length 6, is a graph of diameter 2, which is walk-regular, but not distance-regular.

**Theorem 3.3.** Suppose $\Gamma$ is regular and $t$-partially distance-regular. Then for $x \in X$, we have $|\Gamma_{t+1}(x)| \geq f$, where $f$ is a function of intersection.
numbers and \((A^{2t+1})_{xx}, (A^{2t+2})_{xx}\). Furthermore suppose \(\Gamma\) is \((2t+2)\)-partially walk-regular. Then the above equality holds for each \(x \in X\) if and only if \(\Gamma\) is \((t+1)\)-partially distance-regular.

Proof. If \(\Gamma_{t+1}(x) = \emptyset\), we choose \(f = 0\) and the first part of the theorem holds clearly. We assume \(\Gamma_{t+1}(x) \neq \emptyset\). By the assumption we can write

\[ A^t = \sum_{i=0}^t s_i A_i \]

for some \(s_i \in \mathbb{R}\) with \(s_t \neq 0\). Also \(c_i, b_{i-1}, a_{i-1}\) and \(k_i\) are well-defined in \(\Gamma\) for \(1 \leq i \leq t\) by Proposition 2.3 and Lemma 2.4. Then

\[
(A^{2t+1})_{xx} = \sum_{y \in X} \left( \sum_{z \in X} \sum_{i=0}^t s_i (A_i)_{xz} A_{zy} (\sum_{i=0}^t s_i (A_i)_{yx}) \right) \\
= \sum_{i=0}^{t-1} k_i (c_i s_{i-1} + a_i s_i + b_i s_{i+1}) s_i \\
+ (k_t c_t s_{t-1} + \sum_{y \in \Gamma_t(x)} a_t (y, x) s_t) s_t
\]

by summing \(y\) according to its distance \(i\) to \(x\). From (3.1) we find

\[
\sum_{y \in \Gamma_t(x)} a_t (y, x)
\]

can be determined from the well-defined intersection numbers of \(\Gamma\) and an additional constant \((A^{2t+1})_{xx}\). Similarly,

\[
(A^{2t+2})_{xx} = \sum_{y \in X} (A^{t+1})_{xy}^2 \\
= \sum_{y \in X} \left( \sum_{z \in X} \sum_{i=0}^t s_i (A_i)_{xz} (A_i)_{zy} \right)^2 \\
= \sum_{i=0}^{t-1} k_i (c_i s_{i-1} + a_i s_i + b_i s_{i+1})^2 \\
+ \sum_{y \in \Gamma_t(x)} (c_t s_{t-1} + a_t (y, x) s_t)^2 \\
+ \sum_{y \in \Gamma_{t+1}(x)} (c_{t+1} (y, x) s_t)^2.
\]
By applying Cauchy’s inequality in (3.3),
\[
\sum_{y \in \Gamma_t(x)} (c_{t-1} + a_t(y, x)s_t)^2 \geq \frac{1}{k_t} \left( \sum_{y \in \Gamma_t(x)} (c_{t-1} + a_t(y, x)s_t) \right)^2.
\tag{3.5}
\]
and equality holds in (3.5) iff \(a_t(y, x)\) is independent of the choice of \(y \in \Gamma_t(x)\). Similarly for (3.4) we have
\[
\sum_{y \in \Gamma_{t+1}(x)} (c_{t+1}(y, x)s_t)^2 \geq \frac{1}{k_{t+1}(x)} \left( \sum_{y \in \Gamma_{t+1}(x)} c_{t+1}(y, x)s_t \right)^2 = \frac{1}{k_{t+1}(x)} \left( \sum_{y \in \Gamma_t(x)} (b_t - c_t - a_t(y, x))s_t \right)^2,
\tag{3.6}
\]
and equality holds iff \(c_{t+1}(y, x)\) is independent of the choice \(y \in \Gamma_{t+1}(x)\).

Set \(T_1, T_2, (k_{t+1}(x))^{-1}T_3\) to be the expressions in (3.2), (3.5) and (3.6) respectively, and note that \(T_1, T_2, T_3\) can be computed from the intersection numbers of \(\Gamma\) and the additional constant \((A^{2t+2})_{xx}\). Now we have
\[
(A^{2t+2})_{xx} \geq T_1 + T_2 + (k_{t+1}(x))^{-1}T_3.
\]
Note that \(T_3 > 0\). Then \((A^{2t+2})_{xx} - T_1 - T_2 > 0\). So we can rewrite the above inequality as
\[
k_{t+1}(x) \geq \frac{T_3}{(A^{2t+2})_{xx} - T_1 - T_2}.
\tag{3.7}
\]
The first part of the theorem is obtained by setting \(f\) to be the right hand side of (3.7).

Suppose \(\Gamma\) is \((2t+2)\)-partially walk-regular. Then the \(f\) is a constant, not depending on \(x \in X\). We consider two cases according to \(f = 0\) or not. Note that \(f = |\Gamma_{t+1}(x)| = 0\) for all \(x \in X\) iff \(d \leq t\), and this is equivalent to that \(\Gamma\) is distance-regular. Suppose \(|\Gamma_{t+1}(x)| \neq 0\) for some \(x \in X\). Then the equality hold in (3.7) for each \(x \in X\) iff \(c_{t+1}, a_t, b_t = b_0 - c_t - a_t\) are well-defined, and this is equivalent to that \(\Gamma\) is \((t + 1)\)-partially distance-regular.

Remark 3.4. The inequality in Theorem 3.3 essentially comes from Cauchy’s inequality. A similar argument also appears in [5].
4 Godsil Switching

We shall prove the walk-regularity are preserved by two operations on graphs in this section.

The complement $\overline{\Gamma}$ of $\Gamma = (X, R)$ is a graph with vertex set $X$ and adjacency matrix $A = J - I - A$, where $A$ is the adjacency matrix of $\Gamma$, $I$ is the identity matrix and $J$ is the all 1’s matrix.

Lemma 4.1. Suppose $\Gamma = (X, R)$ is $t$-partially walk-regular. Then the complement $\overline{\Gamma}$ of $\Gamma$ is $t$-partially walk regular.

Proof. This is clear if $t \leq 1$ since every graph is 1-partially walk-regular. Assume $t \geq 2$. In particular $\Gamma$ is $k$-regular for some nonnegative integer $k$. Since $JA = AJ = kJ$ and $JJ = nJ$, we find $\overline{A}^i = (J - I - A)^i$ is a linear combination of $J, I, A$; in particular $\overline{A}^i$ has identical diagonal entries for $0 \leq i \leq t$.

Suppose $\pi = (C_1, C_2, \ldots, C_k, C_{k+1})$ is a partition of the vertex set $X$ such that the following (i)-(ii) hold.

(i) For $1 \leq i, j \leq k$, there exists a constant $t_{ij}$ such that

$$|\Gamma_1(x) \cap C_j| = t_{ij}$$ (4.1)

for all $x \in C_i$.

(ii) For $x \in C_{k+1}$ and $1 \leq i \leq k$,

$$|\Gamma_1(x) \cap C_i| = 0, |C_i|/2, \text{ or } |C_i|.$$

Suppose the above partition $\pi$ exists in $\Gamma = (X, R)$. Let $\Gamma^{(\pi)}$ denote the graph with same vertex set $X$ and the same edges of $\Gamma$ except that for each $x \in C_{k+1}$ and each $1 \leq i \leq k$ with $|\Gamma_1(x) \cap C_i| = |C_i|/2$, the edges between $x$ and $C_i$ are deleted and all the edges from $x$ to vertices in $C_i - \Gamma_1(x)$ are added. We say the graph $\Gamma^{(\pi)}$ is obtained from $\Gamma$ by the Godsil switching with respect to $\pi$. To describe the adjacency matrix $A^{(\pi)}$ of $\Gamma^{(\pi)}$ as shown in [7], we need the following setting. For positive integers $m, t$, let $J_m$ (resp. $j_m$) denote the $m \times m$ (resp. $m \times 1$) all 1’s matrix, $I_m$ denote the $m \times m$ identity matrix and $Q_m = (2/m)J_m - I_m$. The following (a)-(d) are easily verified.
(a) $Q_m^2 = I_m$.

(b) If $X$ is an $m \times t$ matrix with a constant row sum and a constant column sum, then $Q_m X Q_t = X$.

(c) If $X$ is an $m \times 1(1 \times m)$ matrix with column sum (row sum) $m/2$, then $Q_m X = j_m - X ((j_m)^T - X = X Q_m)$.

(d) $Q_m j_m = j_m$.

We may assume that the vertices of $\Gamma$ are ordered so that $A$ can be written as

\[
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1k+1} \\
B_{21} & B_{22} & \cdots & B_{2k+1} \\
\vdots & \vdots & \ddots & \vdots \\
B_{k+1,1} & B_{k+1,2} & \cdots & B_{k+1,k+1}
\end{pmatrix},
\]

where $B_{ii}$ is the adjacency matrix of the graph induced by $C_i$. Note that $B_{ij}$ has a constant row sum and a constant column sum for each pair $1 \leq i, j \leq k$. The partition $\pi = (C_1, C_2, \ldots, C_k, C_{k+1})$ of $X$ is equitable if (4.1) holds for $1 \leq i, j \leq k + 1$; in this case $B_{ij}$ has a constant row sum and a constant column sum for each pair $1 \leq i, j \leq k + 1$. Let $Q$ be the block diagonal matrix with $k + 1$ blocks, where the $i$-th diagonal block is $Q_{m_i}$ if $i \leq k$ and the $(k+1)$-th block is the identity matrix $I_{m_{k+1}}$, with $m_i = |C_i|$. From (a)-(d) above and the constriction, we have $Q^2 = I$ and

\[
A^{(\pi)} = QAQ = \begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1k} & Q_{m_1} B_{1k+1} \\
B_{21} & B_{22} & \cdots & B_{2k} & Q_{m_2} B_{2k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{k+1,1} Q_{m_1} & B_{k+1,2} Q_{m_2} & \cdots & B_{k+1,k} Q_{m_k} & B_{k+1,k+1}
\end{pmatrix}.
\]

Suppose $A^s$ is written in the block matrix form as

\[
A^s = \begin{pmatrix}
B_{11}^{(s)} & B_{12}^{(s)} & \cdots & B_{1k+1}^{(s)} \\
B_{21}^{(s)} & B_{22}^{(s)} & \cdots & B_{2k+1}^{(s)} \\
\vdots & \vdots & \ddots & \vdots \\
B_{k+1,1}^{(s)} & B_{k+1,2}^{(s)} & \cdots & B_{k+1,k+1}^{(s)}
\end{pmatrix}
\]

for any nonnegative integer $s$, where $B_{ij}^{(1)} = B_{ij}$ for $1 \leq i, j \leq k + 1$.  

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Proposition 4.2. Let $\Gamma = (X, R)$ be a regular graph, and let $\pi$ be an equitable partition of $X$ satisfying (ii) above. Fix a nonnegative integer $s$ and suppose $A^s$ is as in (4.2). Then

$$(A^{(\pi)})^s = \begin{pmatrix}
B_{11}^{(s)} & B_{12}^{(s)} & \cdots & B_{1k}^{(s)} & Q_{m_1}B_{k+1}^{(s)} \\
B_{21}^{(s)} & B_{22}^{(s)} & \cdots & B_{2k}^{(s)} & Q_{m_2}B_{k+1}^{(s)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{k1}^{(s)} & B_{k2}^{(s)} & \cdots & B_{kk}^{(s)} & Q_{m_k}B_{k+1}^{(s)} \\
B_{k+1}^{(s)}Q_{m_1} & B_{k+1}^{(s)}Q_{m_2} & \cdots & B_{k+1}^{(s)}Q_{m_k} & B_{k+1}^{(s)}
\end{pmatrix}.$$ 

Proof. The $B_{ij}$ described above has a constant row sum and a constant column sum for each pair $1 \leq i, j \leq k + 1$, since $\pi$ is equitable. This implies that $B_{ij}^{(s)}$ has a constant row sum and a constant column sum. Applying the above (b) to $(A^{(\pi)})^s = QA^sQ$ and simplifying, we have the proposition.

Corollary 4.3. Let $\Gamma = (X, R)$ denote a $t$-partially walk-regular graph, and let $\pi$ be an equitable partition of $X$ satisfying (ii) above. Then $\Gamma^{(\pi)}$ is $t$-partially walk-regular.

Proof. The corollary follows immediately from Proposition 4.2 since $A^s$ and $(A^{(\pi)})^s$ have the same diagonal blocks for any nonnegative integer $s$.

Remark 4.4. ([8]) The Gosset graph $\Gamma = (X, R)$ is the unique distance-regular graph on 56 vertices of diameter 3 with $b_0 = 27$, $b_1 = 10$, $b_2 = 1$, $c_2 = 10$, and $c_3 = 27$. There exists an equitable partition $\pi$ of $X$ that satisfies (ii) above. The graph $\Gamma^{(\pi)}$ obtained from $\Gamma$ by Godsil switching with respect to $\pi$ is not distance-regular.

5 Graphs with $s$ Distinct Eigenvalues

In this section we assume $\Gamma = (X, R)$ is a simple connected $k$-regular graph with diameter $d$, $s$ distinct eigenvalues

$$k > \lambda_1 > \lambda_2 > \ldots > \lambda_{s-1},$$
and order $n$. It is well-known that $s \geq d + 1$ [7, Lemma 5.2], and
\[
J = \frac{n}{q(k)} q(A),
\] (5.1)
where $q(\lambda) := \prod_{i=1}^{s-1} (\lambda - \lambda_i) \in \mathbb{R}[\lambda]$ [3, Corollary 3.3].

**Lemma 5.1.** Suppose that $\Gamma$ is $(s-2)$-partially walk-regular, where $s \geq 4$. Then $\Gamma$ is walk-regular. In particular, for any nonnegative integers $t$, $(A^t)_{xx}$ is determined by $n$ and the eigenvalues of $\Gamma$ for all $x \in X$.

*Proof.* For $s \geq 4$, $\Gamma$ is regular. Note that the $q(\lambda)$ of $\Gamma$ has degree $s-1$ and $k^t J = \frac{n}{q(k)} q(A) A^t$

for all nonnegative integers $t$, where $k$ is the valency of $\Gamma$. Hence $(A^{s-1+t})_{xx}$ is a function of $k = (A^2)_{xx}, (A^3)_{xx}, \ldots, (A^{s-2})_{xx}$ for all nonnegative integers $t$. $\square$

**Lemma 5.2.** Suppose that $\Gamma$ is $(d-1)$-partially distance-regular and has $d + 1$ distinct eigenvalues, where $d \geq 3$. Then $\Gamma$ is distance-regular.

*Proof.* $\Gamma$ is regular since $d \geq 3$. The $q(\lambda)$ of $\Gamma$ has degree $d$ since $\Gamma$ has $d + 1$ eigenvalues. Hence by referring to (5.1), for any two vertices $x, y \in X$ with $\partial(x, y) = d$,
\[
(A^d)_{xy} = \frac{q(k)}{n}
\]
is independent of the choice of $x, y$, and note that
\[
(A^d)_{xy} = c_d(y, x) c_{d-1} c_{d-2} \cdots c_1.
\]
Hence $c_d$ is well-defined. Similarly, for any two vertices $x, y \in X$ with $\partial(x, y) = d - 1$,
\[
(A^d)_{xy} = \frac{q(k)}{n} - c_{d-1} c_{d-2} \cdots c_1 \times \text{the coefficient of } \lambda^{d-1} \text{ in } q(\lambda)
\]
is independent of the choice of $x, y$, and note that
\[
(A^d)_{xy} = (a_{d-1}(y, x) + a_{d-2} + \ldots + a_1)(c_{d-1} c_{d-2} \cdots c_1).
\]
Hence $a_{d-1}$ is well-defined. Since $\Gamma$ is regular with diameter $d$, we find $c_d, a_{d-1}, b_{d-2}$ are well-defined. $\square$
6 Regular Graphs with Four Eigenvalues

We apply the previous results to the connected regular graphs with four distinct eigenvalues in this section.

**Theorem 6.1.** Let $\Gamma = (X, R)$ denote a connected $k$-regular graph with $n$ vertices and four distinct eigenvalues $k > \lambda_1 > \lambda_2 > \lambda_3$. Then $\Gamma$ is walk-regular with diameter 2 or 3. Moreover there exists a rational function $f(n, k, \lambda_1, \lambda_2, \lambda_3)$ in the variables $n, k, \lambda_1, \lambda_2, \lambda_3$ such that for any $x \in X$

$$k_2(x) \geq f(n, k, \lambda_1, \lambda_2, \lambda_3).$$

(6.1)

Furthermore the equality holds for each $x \in X$ if and only if $\Gamma$ is distance-regular with diameter 3.

**Proof.** $\Gamma$ is walk-regular by Lemma 5.1 and clear to have diameter 2 or 3. It is well known that if $\Gamma$ has diameter 2 then it is not distance-regular[7, Lemma 4.1]. Now the theorem follows from Theorem 3.3 with the case $t = 1$.

**Remark 6.2.** The second part of Theorem 6.1 is essentially a result of E. R. van Dam and W. H. Haemers [4] with slightly different variables in the expression of $f$ [4]. The inequality (6.1) is also obtained there with other additional assumptions. They conjectured these additional assumptions can be removed. Theorem 6.1 fulfills their conjecture.

**References**


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