Pooling spaces associated with finite geometry

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Abstract

Motivated by the works of Ngo and Du [H. Ngo, D. Du, A survey on combinatorial group testing algorithms with applications to DNA library screening, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 55 (2000) 171–182], the notion of pooling spaces was introduced [T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Mathematics 282 (2004) 163–169] for a systematic way of constructing pooling designs; note that geometric lattices are among pooling spaces. This paper attempts to draw possible connections from finite geometry and distance regular graphs to pooling spaces: including the projective spaces, the affine spaces, the attenuated spaces, and a few families of geometric lattices associated with the orbits of subspaces under finite classical groups, and associated with $d$-bounded distance-regular graphs.

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1. Introduction

A mechanism has been considered so that each binary vector of length $n$ with weight (at most) $d$ will be associated with a binary vector of length $t$, and the minimum (Hamming) distance among the associated vectors is as large as possible, where $d$ is small comparing with $n$. More precisely, let $\mathcal{P} = \{ x \mid x \in \mathbb{Z}_2^n \text{ with weight at most } d \} \subseteq \mathbb{Z}_2^n$, we are looking for a matrix $M$ of order $t \times n$, such that the minimum distance of the set $\{ Mx \mid x \in \mathcal{P} \} \subseteq \mathbb{Z}_2^t$ is as large as possible.

$$x \in \mathcal{P} \subseteq \mathbb{Z}_2^n \text{ (message)} \rightarrow r(x) = Mx \in \mathbb{Z}_2^t \text{ (encoded message)}$$

$$\rightarrow r(x) + e \in \mathbb{Z}_2^t \text{ (reported message)}$$

$$\rightarrow x \text{ (decoded message).}$$

If the columns of $M$ are identified with the set of $n$ items to be tested in pools (for an unknown positive subset of size at most $d$), and the rows of $M$ are the characteristic vectors of those pools...
(i.e., subsets of items) to be tested the binary vector $x$ corresponds to a (unknown) positive set to be identified, the associated vector $\overline{Mx}$ is called the ideal outcome vector, and indeed the outcome vector $\overline{Mx} + e$ is received with a possible error $e$ occurring during the conducting of experiments. The condition posed over $M$ must guarantee that the correspondence between $x$ and $r(x)$ is one to one for identifying purposes; and moreover the requirement over the minimum distance of the set $\{\overline{Mx} \mid x \in \mathcal{P}\} \subseteq \mathbb{Z}_2^t$ is for error-correcting purposes.

Similar to the situation of classical error-correcting codes, this model provides a mechanism for non-adaptive group testing purpose if $\overline{Mx}$ is specified appropriately, see Section 2, and the matrix $M$ is therefore called a pooling design under this consideration. The notion of traditional group testing can be traced back to around 1941 for blood testing purpose; the items to be tested nowadays have been transformed to DNA segments, refer to [2,15] for an overview of up-to-date results on combinatorial group testing algorithms along with its applications to DNA library screening.

The incidence matrices of the system $\left(\left(\begin{array}{c}n \\ d \end{array}\right), \left(\begin{array}{c}n \\ k \end{array}\right); \subseteq\right)$ and of its $q$-analogue $\left(\left[\begin{array}{c}n \\ d \end{array}\right]_q, \left[\begin{array}{c}n \\ k \end{array}\right]_q; \subseteq\right)$ were studied extensively by Macula [13,14] and by Ngo and Du [16] respectively for the disjunct property, see also [4,5], where $[n] = \{1, 2, \ldots, n\}$, $\left(\begin{array}{c}n \\ i \end{array}\right)$ is the family of all $i$-element subsets of $[n]$, $V = F_q^n$ is a vector space of dimension $n$ over the finite field $F_q$, and $\left[\begin{array}{c}n \\ i \end{array}\right]_q$ is the family of all $i$-dimensional subspaces of $V$. Note that both $\left(\begin{array}{c}n \\ d \end{array}\right)$ and $\left[\begin{array}{c}n \\ d \end{array}\right]_q$ are levels of some well known partially ordered sets and the vertices sets of some distance regular graphs as well. Ngo and Du [15] therefore asked for possible generalizations of the Boolean algebra $B_n$ of power sets of $\{1, 2, \ldots, n\}$; and also for conditions over some two levels of lattices for disjunct purposes; usually some regularity constraints must also be added for avoiding vagueness and for the ease of analysis. It would be better if some information about the capacity of error correct of the matrix being constructed can be derived from the lattices themselves. Inspired by the remarks made by Ngo and Du, a comprehensive treatment of constructions of $d$-disjunct matrices in terms of ranked partially ordered sets was considered by Huang and Weng [8], leading to the introduction of pooling spaces over ranked partially order sets.

It was also pointed out by Ngo and Du [15] that this is a young and interesting field with deep connections to coding theory and design theory; the theory of association schemes, in particular distance regular graphs [1], should play an important role in improving the performance of pooling designs. Based on the notion of pooling spaces, this paper attempts to draw possible connections from finite geometry and association schemes, in particular distance regular graphs to pooling spaces as much as possible. The condition of $d^e$-disjunct and its variations are given in Section 2, followed by a decoding algorithm; the pooling spaces and their capability of error-correcting are given in Section 3. We note that geometric lattices are among pooling spaces, and a few families of geometric lattices associated with the orbits of subspaces under finite classical groups by Wan and Huo [18,19], and associated with $d$-bounded distance-regular graphs of diameter $d$ by Gao et al. [6] respectively, therefore they provide some interesting families of pooling spaces, see Section 4.

2. $d^e$-disjunct matrices and their decodings

For a binary matrix $M$ of order $t \times n$, let $\{C_1, C_2, \ldots, C_n\}$ be the family of subsets of $[t] = \{1, 2, \ldots, t\}$ with the corresponding columns of $M$ as their characteristic vectors; while
\{T_1, T_2, \ldots, T_t\} be the family of subsets of \([n] = \{1, 2, \ldots, n\}\) with the corresponding rows of \(M\) as their characteristic vectors. These two set systems (\(\{T_1\}, \{C_1\} \subseteq [n]\)) and (\(\{T_j\}, \{C_j\} \subseteq [n]\)) are called a dual pair of set systems, or dual pair in short, with respect to the binary matrix \(M\). To identify a positive subset \(P\), unknown at the beginning, of the items \([n]\), the pools \(T_1, T_2, \ldots, T_t\) over the items \([n]\) are arranged in advance; an ideal outcome vector \(z_{P} = (z_1, z_2, \ldots, z_t)^T\) will be reported after these \(t\) tests were performed simultaneously, where \(z_j = 1\) if and only if \(T_j \cap P\) is nonempty. If \(x = (x_1, x_2, \ldots, x_n)^T\), and \(Mx = (y_1, y_2, \ldots, y_t)^T\) is defined as the usual product over the integers, we define \(M^x = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_t)^T\) where \(\tilde{y}_i = 1\) if \(y_i \geq 1\) and \(\tilde{y}_i = 0\) if \(y_i = 0\). Note that \(M^x\) is equivalent to the Boolean sum of those columns corresponding to the nonzero entries of \(x\), and thus \(\overline{M^x}\) corresponds to the union \(\bigcup_{i:X_i=1} C_{i}\) of those columns with \(x_i = 1\). For \(P \subseteq [n]\), \(M(P)\) is defined to be the Boolean sum of the columns vectors corresponding to elements in \(P\).

For the purpose of group testing, the following models have been considered in the literature. A matrix \(M\), and also the corresponding family \(C = \{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}\), is called \(d\)-separable if \(M(P_1) \neq M(P_2)\) for distinct \(P_1, P_2 \subseteq [n]\) with \(|P_1|, |P_2| \leq d\), and \(d\)-disjunct if \(C_I \subseteq M(P)\) whenever \(i \notin P\). The notion of \(d\)-disjunct matrices was first introduced by Kautz and Singleton [12] in 1964. Note that the condition of \(d\)-disjunct is stricter than that of \(d\)-separable, and hence it provides more information for identifying purposes. The one to one correspondence between subsets \(P\) and its outcome vector provides a starting point for the purpose of pooling designs.

(1) If \(\{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}\) is \(d\)-separable, the dual family \(\{T_1, T_2, \ldots, T_t\} \subseteq 2^{[n]}\) satisfies the condition that for each vector \((x_i) \in 2^n\), there exists \(P \subseteq [n]\) with \(|P| \leq d\) such that \(|P \cap T_i| = 0\) if and only if \(x_i = 0\) for \(i \leq t\).

(2) If \(\{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}\) is \(d\)-disjunct and \(P \subseteq [n]\) with \(|P| \leq d\), then the dual family \(\{T_1, T_2, \ldots, T_t\} \subseteq 2^{[n]}\) satisfies the condition that \(\bigcup_{j \notin M(P)} T_j = [n] - P\), and vice versa.

The notion of \(d\)-disjunct matrices has been generalized to \(d^e\)-disjunct matrices [14], (\(s, l\))-superimposed codes and designs [3], (\(s, l\))\(^e\)-generalized cover free families [17] over the past four decades. All these structures can be used in combinatorial group testing algorithms applicable to DNA library screening. The idea used above can be generalized for decoding algorithms for pooling designs based on \(d^e\)-disjunct matrices.

**Definition 2.1.** A binary matrix \(M\) of order \(t \times n\) is called \(d^e\)-disjunct if \(|C_j - \bigcup_{i \in P} C_i| \geq e + 1\) for any \(d\)-element subset \(P\) of \([n]\) and any \(j \in [n] - P\).

A decoding algorithm for pooling designs based on \(d^e\)-disjunct matrices is behind the following theorem, where \(\chi_U\) is the characteristic vector of the set \(U\):

**Theorem 2.1** ([17]). Let \(M\) be a \(d^e\)-disjunct matrix of order \(t \times n\), \(P \subseteq [n]\) with \(|P| \leq d\) and \(U \subseteq [t]\), and let \(T = \{j \mid |C_j - U| \leq \lceil \frac{e}{d} \rceil\}\), then

(a) if \(d_H(M(P), \chi_U) \leq \lceil \frac{e}{d} \rceil\), then \(T = P\).

(b) if \(d_H(M(P), \chi_U) \leq e\) and \(|T| \leq d\), then \(M(P) = \chi_U\) if and only if \(M(T) = \chi_U\).

**Definition 2.2** ([17]). Let \(s, l\) and \(e\) be positive integers, a set system \((X, \mathcal{F})\) with \(\mathcal{F} = \{C_1, C_2, \ldots, C_n\}\) is called an \((s, l; e)\)-cover-free-family provided that \(\bigcap_{j \in S} C_j - \bigcup_{i \in L} C_i \geq e\) for disjoint \(S, L \subseteq [n]\) with \(|S| = s\) and \(|L| = l\).
A geometric lattice is a pooling space. Designs based on such pooling spaces is also included.

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Huang and Weng [9] that the collinearity graphs of the system $\mathcal{C}$ are distance-regular if the parameters $d(i, j) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i - 1\}|$, $a_i(x, y) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i\}|$, $b_i(x, y) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i + 1\}|$ are constants $c_i$, $a_i$, $b_i$ respectively for each $i$ with $0 \leq i \leq d$, independent of the vertices $x$ and $y$ at distance $i$ chosen. For example, the collinearity graphs of the system $\binom{[n]}{d}$ and $\binom{[n]}{d-1}$ and of its $q$-analogue $\binom{V}{d}_q \cdot \binom{V}{d-1}_q$ are the distance-regular Johnson graphs $J(n, d)$ and the Grassmann graphs $J_q(n, d)$ respectively, see [1] for the details. A few families of $d^e$-disjunct matrices defined over the incidence matrices associated with Johnson graphs and Grassmann graphs were considered in [10]. A comprehensive treatment about the containment as the above examples of pooling designs can also be found in [4,5].

3. Pooling spaces and the capacity of error-correcting

Inspired by the remarks made by Ngo and Du [15], a comprehensive treatment of constructions of $d^e$-disjunct matrices in terms of ranked partially ordered sets was given by Huang and Weng [8], leading to the introduction of pooling spaces over ranked partially ordered sets. The notion of pooling spaces together with their capability of error-correcting is included in this section, followed by a few families of examples.

Let $P = (X, \leq)$ be a finite partially ordered set (or poset in short) with the least element $0$. An atom in $P$ is an element in $P$ that covers 0; let $A_P$ be the set of all atoms in $P$. Moreover, let $P_0 = [0]$, $P_1 = A_P$ and let $P_i$ be the set of all rank $i$ elements of $P$ for $i \geq 2$. A ranked poset $P$ is called atomic whenever each element $x \in P$ is the least upper bound of the set $[0, x] \cap P_1$.

A geometric lattice is an upper semi-modular atomic lattice. The projective geometry $PG(n, q)$ and the affine geometry $AG(n, q)$ are typical examples of geometric lattices. Note also that the difference between geometric lattices and combinatorial geometries is similar to that between incidence structures and families of subsets of a set.

Theorem 3.1 ([11]).

(1) A ranked semi-lattice such that each interval is atomic is a pooling space.

(2) A geometric lattice is a pooling space.

Some $d^e$-disjunct matrices can be associated with pooling spaces naturally as shown in the following theorem; moreover, some information of the capacity of error-correcting of the pooling designs based on such pooling spaces is also included.
Theorem 3.2 ([18]). For a pooling space \( P = (X, \leq) \) with rank \( D \geq 1 \), and integers \( 1 \leq d \leq l \leq D \), the binary incidence matrix \( M = M(l, D) \) of the incidence structure \((P_I, P_D, \leq)\) is \( d^e \)-disjunct, where

\[
e + 1 = \min \{ \cup \{ [y, x] \cap P_I \} \},
\]

the minimum is taken over all pairs \((x, T)\) with \( T \subseteq P_D, |T| \leq d \) and \( x \in P_D - T \); the union is taken over all \( y \in [0, x] \cap P_d \) such that \( y \not\leq z \) for all \( z \in T \).

The parameter \( e \) in the above theorem seems complicated; however, the number \( |[y, x] \cap P_I| \) is a constant in the known examples. The truncation of a pooling space is again a pooling space, i.e., if \( P \) is a pooling space with rank \( D \), then so is \( \bigcup_{i=0}^{k-1} P_i \) with rank \( k \) for \( 0 \leq k \leq D \).

Hence we can choose any \( k \) with \( 0 \leq k \leq D \), and replace \( P_D \) by \( P_k \) in the construction of \( M \). The binary incidence matrices of the system \( (\left[ \binom{n}{d} \right], \left[ \binom{n}{k} \right]; \leq) \) and of its \( q \)-analogue

\[
\left( \left[ \begin{array}{c} F_q^n \\ d \end{array} \right], \left[ \begin{array}{c} F_q^n \\ k \end{array} \right]; \subseteq \right)
\]

have been studied extensively [4, 5, 13, 14, 16].

Theorem 3.3. Let \( P = (X, \leq) \) be a pooling space of rank \( D \) such that each interval of rank \( i \) in \( P \) is isomorphic to the projective geometry \( PG(i-1, q) \). For \( 1 \leq d < k \leq D \) with \( k - d \geq 2 \), the incidence matrix \( M(d, k) \) of the incidence structure \((P_d, P_k; \leq)\) is \( s^e \)-disjunct for \( 1 \leq s \leq \frac{q(k-1)^{-1}}{q^{k-d}-1} \), and

\[
e = q^{k-d} \left[ \begin{array}{c} k-1 \\ d-1 \end{array} \right]_q - (s-1)q^{k-d-1} \left[ \begin{array}{c} k-2 \\ d-1 \end{array} \right]_q - 1.
\]

Example 3.1 ([5, 13] The Boolean Algebra). As mentioned before, for \( d < k < n \), the incidence matrix \( J(n, d, k) \) of the incidence structure \( \left( \left[ \binom{n}{d} \right], \left[ \binom{n}{k} \right]; \leq \right) \) is \( d^e \)-disjunct; more precisely,

(1) [5, 13] for \( 1 \leq s \leq d \leq k \leq n \), the matrix \( J(n, d, k) \) is \( s^e \)-disjunct where \( e = \binom{k-s}{k-d} - 1; \)

(2) [14] for \( 1 \leq d \leq k \leq n \) and \( k - d \geq 2 \), let \( K \) be a family of \( k \)-subsets of \( [n] \) such that the Hamming distance between any pair of \( k \)-sets in \( K \) is at least \( 2r \), then the incidence matrix of the system \( \left( \left[ \binom{n}{d} \right], K \right) \) is \( d^e \)-disjunct for \( \alpha_d = \min(r, k-d) \).

Example 3.2 ([5, 8] The Hamming Spaces \( H(n, q) \)). Let \( X = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in F \} \) where \( F = \{ 0, 1, \ldots, q \} \), we define \( x \leq y \) if \( x_i = 0 \) or \( x_i = y_i \) otherwise for \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) in \( X \). The rank of \( x = (x_1, x_2, \ldots, x_n) \) is the weight of \( x \), and \( |P_i| = \binom{n}{i} q^i \).

Let \( 1 \leq s \leq d \leq k \leq n \), then the incidence matrix of the incidence structure \((P_d, P_k; \leq)\) is \( s^e \)-disjunct where \( e = \binom{k-s}{k-d} - 1 \).

Example 3.3 ([4, 5] The Projective Geometry). For \( 1 \leq d < k \leq n \) with \( k - d \geq 2 \), the incidence matrix \( J_q(n, d, k) \) of the incidence structure \( \left( \left[ \binom{n}{d} \right], \left[ \binom{n}{k} \right]; \leq \right) \) is \( s^e \)-disjunct for \( 1 \leq s \leq \frac{q(k-1)^{-1}}{q^{k-d}-1} \), and

\[
e = q^{k-d} \left[ \begin{array}{c} k-1 \\ d-1 \end{array} \right]_q - (s-1)q^{k-d-1} \left[ \begin{array}{c} k-2 \\ d-1 \end{array} \right]_q - 1.
\]
Example 3.4 ([11] The Affine Geometry). Let $V$ be the $n$-dimensional vector space over the finite field $F_q$, and $\mathcal{P}$ be the family of all affine subspaces of $V$ and the empty set $\emptyset$ ordered by inclusion, then $\mathcal{P}$ is a geometric lattice. Let $P_{r+1}$ be the family consisting of all affine $i$-subspaces of $V = F_q^n$. The incidence matrix of the incidence structure $(P_{r+1}, P_{k+1})$ of order $q^{n-r} \binom{n}{r} q$ is $s^e$-disjunct for any $1 \leq s < \frac{q(n-k-1)}{q^{k-r-1}}$ and $e = q^{k-r} \binom{k}{r} q - sq^{k-r-1} \binom{k-1}{r} q - 1$.

We next consider a substructure of $\left( \begin{bmatrix} F_q^m \end{bmatrix}, \begin{bmatrix} F_q^m \end{bmatrix} \right)$ which carries the structure of attenuated spaces and the bilinear forms graphs $H_q(d, n)$ as well.

Example 3.5 ([8] The Attenuated Spaces). Let $V$ be the vector space of dimension $n + d$ over $F_q$ and $W = \langle w_1, w_2, \ldots, w_n \rangle \subseteq V$ a subspace of dimension $n (\geq d)$. Let further $\{w_1, w_2, \ldots, w_n; u_1, u_2, \ldots, u_d\}$ be a basis for $V$, and $U = \langle u_1, u_2, \ldots, u_d \rangle$. Let $P_i = \{A | A \in \begin{bmatrix} V \end{bmatrix}, \dim(A \cap U) = 0\}$, then each $A \in P_d$ corresponds to a unique matrix $M_A$ of order $d \times n$ in the following way: Let $A = \langle u_1 + v_1, u_2 + v_2, \ldots, u_d + v_d \rangle$ for unique choices of $v_1, v_2, \ldots, v_d \in W$, then $M_A = [a_{ij} | d_{\times n}]$ where $v_i = \sum_{j=1}^{n} a_{ij} w_j$ for $1 \leq i \leq d$. If $A, B$ correspond to $M_A, M_B$ respectively as given, then $d - \dim(A \cap B) = \text{rank}(M_A - M_B)$. Moreover, each interval of rank $i$ is isomorphic to the projective geometry $PG(i - 1, q)$. As a consequence of Theorem 3.3, for integers $1 \leq r < k \leq d$ with $k - r \geq 2$, the incidence matrix of the system $(P_r, P_k; \subseteq)$ associated with the attenuated space of rank $d$ is $s^e$-disjunct for $1 \leq s \leq \frac{q(n-k-1)}{q^{k-r-1}}$ and $e = q^{k-r} \binom{k-1}{r-1} q - (s - 1)q^{k-r-1} \binom{k-2}{r-1} q - 1$.

Example 3.6 ([20] The Hermitian Forms Space $\text{Her}(q, d)$). Let $P$ be the set of all weak geodesic subgraphs of the Hermitian forms graph of diameter $d$, ordered by reversed inclusion. Then $P$ is a ranked semi-lattice with atomic intervals, and with $|P_i| = \left[ \frac{d}{i} \right] q^{(2d-i)} (0 \leq i \leq d)$.

4. More examples of pooling spaces

Some families of geometric lattices associated with subspaces of $d$-bounded distance-regular graphs of diameter $d$, and associated with finite geometry and classical groups were given by Gao et al. [6], and by Huo and Wan [18] respectively. All of them turn out to be families of pooling spaces as shown in Section 3, and will be summarized in this section.

4.1. Some $d$-bounded distance-regular graphs

An induced subgraph $\Delta$ of $\Gamma$ of diameter $d$ is called strongly closed if $\{x \mid (d(u, x), d(x, v)) = (1, i - 1) \text{ or } (1, i)\} \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$ at distance $i$ for each $i \leq d$. It is obvious that strongly closed subgraphs are connected and $d_\Delta(x, y) = d_\Delta(x, y)$ for all $x, y \in \Delta$. A subspace of $\Gamma$ is a regular subgraph induced by a strongly closed subset. For subspaces $\Delta_1, \Delta_2$ of $\Gamma$, the join $\Delta_1 + \Delta_2$ of $\Delta_1$ and $\Delta_2$ is the smallest subspace containing $\Delta_1 \cup \Delta_2$. A distance-regular graph $\Gamma$ with diameter $d$ is called $d$-bounded if every strongly closed subgraph of $\Gamma$ is regular, and any two vertices $x, y \in V(\Gamma)$ are contained in a common
strongly closed subgraph of diameter \(d(x, y)\). For a \(d\)-bounded distance-regular graph \(\Gamma\) with diameter \(d \geq 3\), let

(1) \(P(x)\) be the set of strongly closed subgraphs containing the vertex \(x \in V(\Gamma)\),

(2) \(P(x, i) = \{ \Delta \mid \Delta \in P(x) \text{ with diameter } i \}\), and

(3) \(L(x, i)\) be the set of the intersection of elements in \(P(x, i)\), with the convention that \(\Gamma \in L(x, i)\) for \(i \in [d - 1]\). It is called the set generated by the intersection of elements in \(P(x, i)\).

The lattices \((L(x, i), \subseteq)\) and \((L(x, i), \supseteq)\) were studied in [6] for \(d\)-bounded distance-regular graphs with diameter \(d\) at least 3. It was proved that \((L(x, i), \subseteq)\) and \((L(x, i), \supseteq)\) are both finite atomic lattices, and conditions for them being geometric lattices were given.

Example 4.1 ([6]). Let \(\Gamma\) be a \(d\)-bounded distance-regular graph with diameter \(d\) at least 3. For each vertex \(x\) and each \(i \in [d - 1]\),

(1) \((L(x, i), \subseteq)\) is a finite geometric lattice.

(2) \((L(x, i), \supseteq)\) is a finite geometric lattice if and only if \(i = 1\) or \(i = d - 1\) and

\[d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) = d(\Delta_1) + d(\Delta_2), \quad \forall \Delta_1, \Delta_2 \in P(x).\]

4.2. Classical polar spaces

Example 4.2. Let \(V\) be a vector space with a given non-degenerate form over the field \(F_q\), a subspace of \(V\) is called isotropic whenever the form vanishes completely on that subspace. It is known that all the maximal isotropic subspaces have the same dimension, denoted by \(D\). Let \(\mathcal{F}\) be the family of all isotropic subspaces of \(V\), \(A \leq B\) whenever \(A\) is a subspace of \(B\) for \(A, B \in \mathcal{F}\); moreover \(\operatorname{rank}(A) = \dim(A)\).

| Name \(\bigl(_{2}D_{r}^{ D+1}\bigr)\) | \(\dim(V)\) | Form | \(|P_1|\) |
|---|---|---|---|
| \(B_D(q)\) | \(2D + 1\) | Quadratic | \(\left[ \begin{array}{c} D \\ \prod_{k=0}^{i-1} (1 + q^{D-k}) \end{array} \right] \) |
| \(C_D(q)\) | \(2D\) | Alternating | \(\left[ \begin{array}{c} D \\ \prod_{k=0}^{i-1} (1 + q^{D-k}) \end{array} \right] \) |
| \(D_D(q)\) | \(2D\) | Quadratic (with index \(D\)) | \(\left[ \begin{array}{c} D \\ \prod_{k=0}^{i-1} (1 + q^{D-k}) \end{array} \right] \) |
| \(^2D_{D+1}(q)\) | \(2D + 2\) | Quadratic (with index \(D\)) | \(\left[ \begin{array}{c} D \\ \prod_{k=0}^{i-2} (1 + q^{D-k}) \end{array} \right] \) |
| \(^2A_{2D}(q)\) | \(2D + 1\) | Hermitian (\(q = r^2\)) | \(\left[ \begin{array}{c} D \\ \prod_{k=0}^{i-1} (1 + q^{D-k+1}) \end{array} \right] \) |
| \(^2A_{2D-1}(r)\) | \(2D\) | Hermitian (\(q = r^2\)) | \(\left[ \begin{array}{c} D \\ \prod_{k=1}^{i-1} (1 + q^{D-k+1}) \end{array} \right] \) |

Example 4.3. Let \(V\) be a vector space with a given degenerate form over the finite field \(F_q\), a subspace of \(V\) is called isotropic if the form vanishes completely on that space. It is known that all the maximal isotropic subspaces intersecting trivially with \(V^\perp\) have the same dimension, denoted by \(D\). Let \(\dim(V^\perp) = l\).

(1) Let \(\mathcal{F}\) be the family of all isotropic subspaces of \(V\) intersecting trivially with \(V^\perp\), \(A \leq B\) whenever \(A\) is a subspace of \(B\) for \(A, B \in \mathcal{F}\); moreover \(\operatorname{rank}(A) = \dim(A)\).
Let $m$ consisting of all $W$.

Example 4.4

$2A_{2D,l}(r) = 2D + l$ Hermitian $(q = r^2)$

$$q^{2D+l+1-i} \cdot D \left[ q \right] \cdot \prod_{k=0}^{1} (1 + q^{D-k})$$

4.3. Some lattices generated by transitive sets of subspaces

Let $V = F_q^n$ be the $n$-dimensional vector space over the finite field $F_q$ of $q$ elements, and let $G_n$ be one of the classical group of degree $n$ over $F_q$. The family of all subspaces of $V$ is partitioned into orbits under the action of the group $G_n$. Let $M$ be any nontrivial orbit of subspaces under $G_n$, and let $L(M)$ be the set of subspaces generated by $M$, i.e., the family consists of $V$ and of all subspaces of $V$ which are the intersection of those subspaces in $M$. Then both $L(M) \subseteq V$ and $L(M) \supseteq V$ are lattices. The geometry of those lattices were classified by Wan and Huo [18,19] recently.

For the case $G_n = GL_n(F_q)$, the set $L(m, n)$ of subspaces generated by the orbit $M(m, n)$ consisting of all $m$-dimensional subspaces consists of $V$ and all of its subspaces of dimension at most $m$.

Example 4.4 ([18,19]). The lattices $(L(1, n), \supseteq)$, $(L(n - 1, n), \supseteq)$ and $(L(m, n), \subseteq)$ with $1 \leq m \leq n - 1$ are geometric lattices.
For the symplectic case, $G_n$ is the symplectic group $Sp_{2\nu}(F_q)$ of degree $n = 2\nu$ over $F_q$ consisting of all $2\nu \times 2\nu$ matrices $T$ over $F_q$ satisfying $TK'T = K$ where $K = \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}$. An $m$-dimensional subspace $P$ is of type $(m, s)$ if $PK'P$ is of rank $2s$, it is known that subspaces with type $(m, s)$ exist if and only if $2s \leq m \leq v + s$, and that the set $M(m, s; 2\nu)$ of subspaces with the same type $(m, s)$ forms an orbit. The set $L(m, s; 2\nu)$ of intersections of subspaces with type $(m, s)$ consists of $V$ and all subspaces of type $(m_1, s_1)$ with $m - m_1 \geq s - s_1 \geq 0$.

**Example 4.5** ([18,19]). $(L(1, 0; 2\nu), \subseteq), (L(1, 0; 2\nu), \supseteq), (L(2\nu - 1, v - 1; 2\nu), \subseteq), (L(2\nu - 1, v - 1; 2\nu), \supseteq)$ are geometric lattices.

Similar results hold for the unitary case, the orthogonal case and the pseudo-symplectic case; refer to [18,19] for more details.

**References**


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