Spectral Radius and Average 2-Degree Sequence of a Graph

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Abstract

Let $G$ be a simple connected graph of order $n$ with average 2-degree sequence $M_1 \geq M_2 \geq \cdots \geq M_n$. Let $\rho(G)$ denote the spectral radius of the adjacency matrix of $G$. We show that for each $1 \leq \ell \leq n$ and for any $b \geq \max \{d_i/d_j \mid i \sim j\}$,

$$\rho(G) \leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2}$$

with equality if and only if $M_1 = M_2 = \cdots = M_n$.

Keywords: Graph, adjacency matrix, spectral radius, average 2-degree.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$, edge set $E(G)$ and $n = |V(G)|$. For any vertex $v \in V(G)$, let $d_v$ denote the degree of $v$, define the average 2-degree $M_v$ of $v$ to be the average degree of the neighbors of $v$. In other words, $M_v = \sum_{u \sim v} d_u/d_v$, where $u \sim v$ means vertices $u$ and $v$ are adjacent. Throughout the article, label the vertices of $G$ by $1, 2, \cdots, n$

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such that $M_1 \geq M_2 \geq \cdots \geq M_n$. Let $A = (a_{ij})$ be the adjacency matrix of $G$, a binary matrix of order $n$ such that for any pair $i, j \in V(G)$, $a_{ij} = 1$ iff $i, j$ are adjacent in $G$. The spectral radius $\rho(G)$ of $G$ is the largest eigenvalue of its adjacency matrix, this parameter has been studied by many authors \cite{1, 2, 5, 6, 7, 8, 9, 11, 12, 13} and can be used to induce some other bounds such as the upper bounds of signless Laplacian eigenvalues\cite{3, 4}.

The following theorem is well-known and referred as Perron-Frobenius Theorem \cite[Chapter 2]{10}.

**Theorem 1.1.** If $B$ is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(B)$ and row-sums $r_1, r_2, \ldots, r_n$, then

$$\rho(B) \leq \max_{1 \leq i \leq n} r_i$$

with equality if and only if the row-sums of $B$ are all equal. \hfill \Box

In this article, we pay attention to the upper bounds of spectral radius of graphs in terms of the average 2-degree sequence. By setting $B = U^{-1}AU$, where $U = \text{diag}\,(d_1, d_2, \cdots, d_n)$, the following fact is easily seen from Theorem 1.1.

**Theorem 1.2.**

$$\rho(G) \leq M_1$$

with equality if and only if $M_1 = M_2 = \cdots = M_n$. \hfill \Box

The graph with the equality holds in Theorem 1.2 is called pseudo-regular in \cite{13}.

In 2011 \cite[Theorem 2.1]{3}, Chen, Pan and Zhang gave the following bound.

**Theorem 1.3.** Let $a := \max\\{d_i/d_j \mid 1 \leq i, j \leq n\}$. Then

$$\rho(G) \leq \frac{M_2 - a + \sqrt{(M_2 + a)^2 + 4a(M_1 - M_2)}}{2},$$

with equality if and only if $G$ is pseudo-regular. \hfill \Box

We will show in Corollary 3.3 that Theorem 1.3 is indeed a generalization of Theorem 1.2. Moreover, we give the following Theorem to generalize Theorem 1.3.
Theorem 1.4. For any $b \geq \max \{d_i/d_j \mid i \sim j\}$ and $1 \leq \ell \leq n$,
\[
\rho(G) \leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2},
\]
with equality if and only if $G$ is pseudo-regular.

Note that Theorem 1.3 is a special case of Theorem 1.4 by taking $b = a$ and $\ell = 2$. Our proof of Theorem 1.4 is a subtle application of Perron-Frobenius Theorem. This idea is previously employed in [11, 9].

We provide some examples of pseudo-regular graphs that are not regular in Example 2.1. The lowest upper bound among the choices of $b$ and $\ell$ is investigated in Section 3.

2 Proof of Theorem 1.4

Proof. For each $1 \leq i \leq \ell - 1$, let $x_i \geq 1$ be a variable to be determined later. Let $U = \text{diag}(d_1x_1, \ldots, d_{\ell-1}x_{\ell-1}, d_\ell, \ldots, d_n)$ be a diagonal matrix of size $n \times n$. Consider the matrix $B = U^{-1}AU$. Note that $A$ and $B$ have the same eigenvalues. Let $r_1, r_2, \ldots, r_n$ be the row-sums of $B$. Then for $1 \leq i \leq \ell - 1$ we have

\[
\begin{align*}
\rho(G) &\leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2},
\end{align*}
\]

\[
\begin{align*}
\frac{1}{x_i} \sum_{k=1}^{\ell-1} (x_k - 1)a_{ik} \frac{d_k}{d_i} + \frac{1}{x_i} \sum_{k=1}^{n} a_{ik} \frac{d_k}{d_i}
\end{align*}
\]

\[
\leq \frac{b}{x_i} \left( \sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i,
\]

(2.1)
since \(a_{ik}d_k/d_i \leq b\). Similarly for \(\ell \leq j \leq n\) we have

\[
r_j = \sum_{k=1}^{\ell-1} x_k a_{jk} \frac{d_k}{d_j} + \sum_{k=\ell}^{n} a_{jk} \frac{d_k}{d_j}
\]

\[
= \sum_{k=1}^{\ell-1} (x_k - 1)a_{jk} \frac{d_k}{d_j} + \sum_{k=1}^{n} a_{jk} \frac{d_k}{d_j}
\]

\[
\leq b \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) + M_{\ell}. \quad (2.2)
\]

Let

\[
\phi_{\ell} = \frac{M_{\ell} - b + \sqrt{(M_{\ell} + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_{\ell})}}{2}.
\]

For \(1 \leq i \leq \ell - 1\) let

\[
x_i = 1 + \frac{M_i - M_{\ell}}{\phi_{\ell} + b} \geq 1. \quad (2.3)
\]

Then for \(1 \leq i \leq \ell - 1\) we have

\[
r_i \leq \frac{b}{x_i} \left( \sum_{k=1,k \neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i
\]

\[
= \frac{b \sum_{i=1}^{\ell-1}(M_i - M_{\ell}) + \phi_{\ell} M_i + bM_{\ell}}{\phi_{\ell} + b + M_i - M_{\ell}}
\]

\[
= \frac{\frac{1}{2}((M_{\ell} + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_{\ell}) - M_{\ell}^2 - b^2 + 2bM_{\ell}) + \phi_{\ell} M_i}{\phi_{\ell} + b + M_i - M_{\ell}}
\]

\[
= \frac{\phi_{\ell}^2 + \phi_{\ell} b + \phi_{\ell} M_i - \phi_{\ell} M_{\ell}}{\phi_{\ell} + b + M_i - M_{\ell}}
\]

\[
= \phi_{\ell}.
\]

Similarly for \(\ell \leq j \leq n\) we have

\[
r_j \leq b \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) + M_{\ell} = \phi_{\ell}.
\]

Hence by Theorem 1.1,

\[
\rho(G) = \rho(B) \leq \max_{1 \leq i \leq n} \{ r_i \} \leq \phi_{\ell}.
\]
The first part of Theorem 1.4 follows.

Suppose \( M_1 = M_2 = \cdots = M_n \). Then \( \rho(G) = M_1 = \phi_\ell \) by Theorem 1.2. Hence the equality in Theorem 1.4 follows.

To prove the necessary condition, suppose \( \rho(G) = \phi_\ell \). Applying Theorem 1.1 and the inequalities in (2.1) and (2.2), \( \phi_\ell = \rho(G) \leq \max_{1 \leq i \leq n} r_i \leq \phi_\ell \). Hence \( r_1 = r_2 = \cdots = r_n = \phi_\ell \), and the equalities in (2.1) and (2.2) hold. In particular,

\[
b = a_{ik} \frac{d_k}{d_i}
\]

(2.4)

for any \( 1 \leq i \leq n \) and \( 1 \leq k \leq \ell - 1 \) with \( x_k - 1 > 0 \), and \( M_\ell = M_n \). We consider three cases:

(i) Suppose \( M_1 = M_\ell \) : Clearly \( M_1 = M_n \).

(ii) \( M_{t-1} > M_\ell = M_t \) for some \( 3 \leq t \leq \ell \) : Then \( x_k > 1 \) for \( 1 \leq k \leq \ell - 1 \) by (2.3). Hence by (2.4)

\[
b = a_{i2} \frac{d_2}{d_1} = a_{21} \frac{d_1}{d_2} = 1,
\]

and \( d_i = n - 1 \) for all \( i = 1, 2, \cdots, n \). This implies \( G \) is regular, a contradiction.

(iii) \( M_1 > M_2 = M_\ell \) : Then \( x_1 > 1 \) by (2.3). Hence by (2.4), \( b = a_{i1} d_1 / d_i \) for \( 2 \leq i \leq n \). Hence \( d_1 = n - 1 \) and \( d_2 = d_3 = \cdots = d_n = (n - 1)/b \). Then \( (n - 1)/b = M_1 > M_2 = M_n = (n - 1)/b - 1 + b \). This implies \( b < 1 \), a contradiction. This completes the proof of the theorem.

\( \square \)

**Example 2.1.** The following graphs are pseudo-regular graphs but not regular.
A graph in Figure 3 has a cycle $C_k$ of $k$ vertices, and shares each vertex of $C_k$ with a triangle $K_3$.

An interesting problem could be characterizing all the nonregular pseudo-regular graphs.
3 The shape of the sequence $\phi_1, \phi_2, \ldots, \phi_n$

Given a decreasing sequence $M_1 \geq M_2 \geq \cdots \geq M_n$ of positive integers, consider the functions

$$\phi_\ell(x) = \frac{M_\ell - x + \sqrt{(M_\ell + x)^2 + 4x \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2}$$

for $x \in [1, \infty)$. Note that $\phi_\ell(b)$ is the upper bound of $\rho(G)$ in Theorem 1.4.

The following proposition shows that the smaller the $b$ in Theorem 1.4 is, the lower the upper bound of $\rho(G)$ reaches.

**Proposition 3.1.** For any $1 \leq \ell \leq n$, $\phi_\ell(x)$ is increasing on $[1, \infty)$.

*Proof.* For convenience, let

$$S = \sum_{i=1}^{\ell-1} (M_i - M_\ell).$$

To show that $\phi_\ell(x)$ is increasing on $[1, \infty)$, it is sufficient to show that the derivative of $\phi_\ell(x)$ is nonnegative. This follows from the following equivalent steps.

$$\phi'_\ell(x) \geq 0 \iff -1 + \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} \geq 0$$

$$\iff \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} \geq 1$$

$$\iff (M_\ell + x + 2S)^2 \geq (M_\ell + x)^2 + 4Sx$$

$$\iff 4SM_\ell + 4S^2 \geq 0.$$  

$\square$

Notice that for $1 \leq s \leq n - 1$, $M_s = M_{s+1}$ implies $\phi_s(x) = \phi_{s+1}(x)$. The following proposition describes when the bound gets improved throughout the sequence $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$. 

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Proposition 3.2. Suppose $M_s > M_{s+1}$ for some $1 \leq s \leq n-1$, and let the symbol $\succeq$ denote $>$ or $\geq$. Then

$$\phi_s(x) \succeq \phi_{s+1}(x) \iff \sum_{i=1}^{s} M_i \geq xs(s-1).$$

Proof. Consider the following equivalent relations step by step.

$$\phi_s(x) > \phi_{s+1}(x) \iff M_s - M_{s+1} + \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} > \sqrt{(M_{s+1} + x)^2 + 4x \sum_{i=1}^{s} (M_i - M_{s+1})}$$

$$\iff \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} > 2xs - (M_s + x)$$

$$\iff (M_s + x)^2 + 4x \sum_{i=1}^{s} (M_i - M_s) > 4x^2 s^2 - 4xs(M_s + x) + (M_s + x)^2$$

$$\iff \sum_{i=1}^{s} M_i > xs(s-1),$$

where the third relation is obtained from the second by taking square on both sides, simplifying it, and deleting the common term $M_s - M_{s+1}$. Similarly, note that if $\sum_{i=1}^{s} M_i = xs(s-1)$, $M_s \leq xs$ and then $2xs - (M_s + x) \geq 0$. Hence
\[
\phi_s(x) = \phi_{s+1}(x) \quad (3.1)
\]
\[
\Leftrightarrow \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} = 2xs - (M_s + x)
\]
\[
\Leftrightarrow (M_s + x)^2 + 4x \sum_{i=1}^{s} (M_i - M_s) = 4x^2s^2 - 4xs(M_s + x) + (M_s + x)^2
\]
\[
\Leftrightarrow \sum_{i=1}^{s} M_i = xs(s - 1),
\]

The following corollary shows that Theorem 1.3 is an improvement of Theorem 1.2.

**Corollary 3.3.** For any \(x \in [1, \infty)\), \(\phi_2(x) \leq M_1\) with equality iff \(M_2 = M_1\).

**Proof.** If \(M_2 = M_1\) then \(\phi_2(x) = M_2 \leq M_1\). Suppose \(M_2 < M_1\). Choose \(s = 1\) and the symbol \(\geq\) to be \(>\) in Proposition 3.2,

\[M_1 = \phi_1(x) > \phi_2(x).\]

Choosing \(b = \max \{d_i/d_j \mid i \sim j\}\), by Proposition 3.2 with \(s = 2\) and \(x = b\), if \(M_2 > M_3\) and \(M_1 + M_2 > 2b\), then \(\phi_2(b) > \phi_3(b)\). This is a case when Theorem 1.4 is truly an improvement of Theorem 1.3.

**Example 3.4.** In the following graph, \(M_1 = M_2 = 4\), \(M_3 = 7/2\), \(b = 4/3\), \(\phi_1(b) = \phi_2(b) = 4\), \(\phi_3(b) = 3.762\) and \(\rho(G) = 1 + \sqrt{7} = 3.646\).
Notice that $\phi_1(x) = M_1 \geq \phi_2(x)$ by Corollary 3.3, and for $2 \leq t \leq n - 1$, 
$\sum_{i=1}^{t} M_i < xt(t-1)$ implies $M_t < x(t-1)$, and hence $\sum_{i=1}^{t+1} M_i < xt(t-1) + x(t-1) < xt(t+1)$. This implies that the sequence $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ is composed by two parts. The first part is decreasing and the second part is increasing. In particular, if we choose $x = M_1$, $M_2 > M_3$, $s = 2$ and $\geq$ to be $>$ in Proposition 3.2, then $M_1 + M_2 \geq 2M_1 = xs(s-1)$, so $\phi_2(M_1) \leq \phi_3(M_1)$. Hence $\phi_2(M_1)$ is smallest among $\phi_1(M_1), \phi_2(M_1), \ldots, \phi_n(M_1)$.

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