3-bounded property in a triangle-free distance-regular graph

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Received 11 May 2006; accepted 15 October 2007

Abstract

Let \( \Gamma \) denote a distance-regular graph with classical parameters \((D, b, \alpha, \beta)\) and \( D \geq 3 \). Assume the intersection numbers \( a_1 = 0 \) and \( a_2 \neq 0 \). We show that \( \Gamma \) is 3-bounded in the sense of the article [C. Weng, \( D \)-bounded distance-regular graphs, European Journal of Combinatorics 18 (1997) 211–229].

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1. Introduction

Let \( \Gamma = (X, R) \) be a distance-regular graph with diameter \( D \geq 3 \) and distance function \( \partial \). Recall that a sequence \( x, y, z \) of vertices of \( \Gamma \) is geodetic whenever

\[ \partial(x, y) + \partial(y, z) = \partial(x, z). \]

A sequence \( x, y, z \) of vertices of \( \Gamma \) is weak-geodetic whenever

\[ \partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1. \]

Definition 1.1. A subset \( \Omega \subseteq X \) is weak-geodetically closed if for any weak-geodetic sequence \( x, y, z \) of \( \Gamma \):

\[ x, z \in \Omega \implies y \in \Omega. \]

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Research partially supported by the NSC grant 95-2115-M-009-002 of Taiwan ROC.

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Weak-geodetically closed subgraphs are called strongly closed subgraphs in [8]. We refer the reader to [7,3,5,9,12,4] for information on weak-geodetically closed subgraphs.

Definition 1.2. $\Gamma$ is said to be i-bounded whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains $x, y$.

The properties of $D$-bounded distance-regular graphs were studied in [13], and these properties were used in the classification of classical distance-regular graphs of negative type [14]. Before stating our main result we give one more definition.

By a parallelogram of length $i$, we mean a 4-tuple $xyzw$ consisting of vertices of $\Gamma$ such that $\partial(x, y) = \partial(z, w) = 0$, $\partial(x, z) = i$, and $\partial(x, w) = \partial(y, w) = \partial(y, z) = i - 1$.

It was proved that if $a_1 = 0$, $a_2 \neq 0$ and $\Gamma$ contains no parallelograms of length 3, then $\Gamma$ is 2-bounded [12, Proposition 6.7], [9, Theorem 1.1]. The following theorem is our main result.

Theorem 1.3. Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\Gamma$ is 3-bounded.

Note that if $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ with $D \geq 3, a_1 = 0$ and $a_2 \neq 0$, then $\Gamma$ contains no parallelograms of any length. See [6, Theorem 1.1] or Theorem 3.3 in this article.

2. Preliminaries

In this section we review some definitions, basic concepts and some previous results concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [10] for more background information.

Let $\Gamma = (X, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set $X$, edge set $R$, distance function $\partial$, and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. By a pentagon, we mean a 5-tuple $x_1x_2x_3x_4x_5$ consisting of vertices in $\Gamma$ such that $\partial(x_i, x_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(x_5, x_1) = 1$.

For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{z \in X \mid \partial(x, z) = i\}$. The valency $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph $\Gamma$ is called regular (with valency $k$) if each vertex in $X$ has valency $k$.

A graph $\Gamma$ is said to be distance-regular whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid z \in \Gamma_i(x) \cap \Gamma_j(y)\}|$$

is independent of $x, y$. The constants $p^h_{ij}$ are known as the intersection numbers of $\Gamma$.

Let $\Gamma = (X, R)$ be a distance-regular graph. For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y),$$
$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y),$$
$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y).$$

Note that

$$|B(x, y)| = p^i_{1,i+1},$$
$$|C(x, y)| = p^i_{1,i-1},$$
$$|A(x, y)| = p^i_{1,i}.$$
are independent of \(x, y\).

For convenience, set \(c_i := p^i_{i-1}\) for \(1 \leq i \leq D\), \(a_i := p^i_{i}\) for \(0 \leq i \leq D\), \(b_i := p^i_{i+1}\) for \(0 \leq i \leq D-1\) and put \(b_D := 0\), \(c_0 := 0\), \(k := b_0\). Note that \(k\) is the valency of \(\Gamma\). It is immediate from the definition of \(p^i_{ij}\) that \(b_i \neq 0\) for \(0 \leq i \leq D-1\) and \(c_i \neq 0\) for \(1 \leq i \leq D\). Moreover

\[
k = a_i + b_i + c_i \quad \text{for } 0 \leq i \leq D.
\]

From now on we assume that \(\Gamma = (X, R)\) is distance-regular with diameter \(D \geq 3\). Recall that a sequence \(x, y, z\) of vertices of \(\Gamma\) is weak-geodetic whenever

\[
\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.
\]

**Definition 2.1.** Let \(\Omega\) be a subset of \(X\), and pick any vertex \(x \in \Omega\). \(\Omega\) is said to be weak-geodetically closed with respect to \(x\) whenever, for all \(z \in \Omega\) and for all \(y \in X\),

\[
x, y, z \text{ are weak-geodetic} \implies y \in \Omega.
\]

Note that \(\Omega\) is weak-geodetically closed with respect to a vertex \(x \in \Omega\) if and only if

\[
C(z, x) \subseteq \Omega \quad \text{and} \quad A(z, x) \subseteq \Omega \quad \text{for all } z \in \Omega
\]

[12, Lemma 2.3]. Also \(\Omega\) is weak-geodetically closed if and only if for any vertex \(x \in \Omega\), \(\Omega\) is weak-geodetically closed with respect to \(x\). We list a few results which will be used later in this paper.

**Theorem 2.2** ([12, Theorem 4.6]). Let \(\Gamma\) be a distance-regular graph with diameter \(D \geq 3\). Let \(\Omega\) be a regular subgraph of \(\Gamma\) with valency \(\gamma\) and set \(d := \min\{i \mid \gamma \leq c_i + a_i\}\). Then the following (i), (ii) are equivalent.

(i) \(\Omega\) is weak-geodetically closed with respect to at least one vertex \(x \in \Omega\).

(ii) \(\Omega\) is weak-geodetically closed with diameter \(d\).

In this case \(\gamma = c_d + ad\).

**Lemma 2.3** ([9, Lemma 2.6]). Let \(\Gamma\) be a distance-regular graph with diameter 2, and let \(x\) be a vertex of \(\Gamma\). Suppose \(\alpha_2 \neq 0\). Then the subgraph induced on \(\Gamma_2(x)\) is connected of diameter at most 3.

**Theorem 2.4** ([12, Proposition 6.7], [9, Theorem 1.1]). Let \(\Gamma\) be a distance-regular graph with diameter \(D \geq 3\). Suppose \(\alpha_1 = 0\), \(\alpha_2 \neq 0\) and \(\Gamma\) contains no parallelograms of length 3. Then \(\Gamma\) is 2-bounded.

**Theorem 2.5** ([12, Lemma 6.9], [9, Lemma 4.1]). Let \(\Gamma\) be a distance-regular graph with diameter \(D \geq 3\). Suppose \(\alpha_1 = 0\), \(\alpha_2 \neq 0\) and \(\Gamma\) contains no parallelograms of any length. Let \(x\) be a vertex of \(\Gamma\), and let \(\Omega\) be a weak-geodetically closed subgraph of \(\Gamma\) with diameter 2. Suppose that there exists an integer \(i\) and a vertex \(u \in \Omega \cap \Gamma_{i-1}(x)\), and suppose \(\Omega \cap \Gamma_{i+1}(x) \neq \emptyset\). Then for all \(t \in \Omega\), we have \(\partial(x, t) = i - 1 + \partial(u, t)\).
3. $Q$-polynomial properties

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Let $\mathbb{R}$ denote the real number field. Let $\text{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over $\mathbb{R}$ with the rows and columns indexed by the elements of $X$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{R})$ defined by the rule

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad \text{for } x, y \in X.$$ 

We call $A_i$ the distance matrices of $\Gamma$. We have

$A_0 = I,$

$A_i^t = A_i$ for $0 \leq i \leq D$ where $A_i^t$ means the transpose of $A_i$,

$A_iA_j = \sum_{h=0}^{D} p_{ij}^h A_h$ for $0 \leq i, j \leq D$.

Let $M$ denote the subspace of $\text{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_D$. Then $M$ is a commutative subalgebra of $\text{Mat}_X(\mathbb{R})$, and is known as the Bose–Mesner algebra of $\Gamma$. By [2, p. 59, 64], $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that

$E_0 = |X|^{-1} J$ where $J$ is all 1’s matrix,

$E_iE_j = \delta_{ij} E_i$ for $0 \leq i, j \leq D$,

$E_0 + E_1 + \cdots + E_D = I$,

$E_i^t = E_i$ for $0 \leq i \leq D$. (3.1)

The $E_0, E_1, \ldots, E_D$ are known as the primitive idempotents of $\Gamma$, and $E_0$ is known as the trivial idempotent. Let $E$ denote any primitive idempotent of $\Gamma$. Then we have

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

(3.2)

for some $\theta_0^*, \theta_1^*, \ldots, \theta_D^* \in \mathbb{R}$, called the dual eigenvalues associated with $E$.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of $V$ as being indexed by $X$. Then the Bose–Mesner algebra $M$ acts on $V$ by left multiplication. We call $V$ the standard module of $\Gamma$. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t,$$

(3.3)

where the 1 is in coordinate $x$. Also, let $\langle , \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \quad \text{for } u, v \in V.$$ (3.4)

Then referring to the primitive idempotent $E$ in (3.2), we compute from (3.1)–(3.4) that for $x, y \in X$,

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^*,$$

(3.5)

where $i = \partial(x, y)$.
Let ∘ denote the entrywise multiplication in Mat$_X$(R). Then

$$A_i \circ A_j = \delta_{ij} A_i \quad \text{for } 0 \leq i, j \leq D,$$

so $M$ is closed under ∘. Thus there exists $q_{ij}^k \in \mathbb{R}$ for $0 \leq i, j, k \leq D$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{D} q_{ij}^k E_k \quad \text{for } 0 \leq i, j \leq D.$$

$\Gamma$ is said to be $Q$-polynomial with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents if for all integers $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two. Let $E$ denote any primitive idempotent of $\Gamma$. Then $\Gamma$ is said to be $Q$-polynomial with respect to $E$ whenever there exists an ordering $E_0, E_1 = E, \ldots, E_D$ of the primitive idempotents of $\Gamma$, with respect to which $\Gamma$ is $Q$-polynomial. If $\Gamma$ is $Q$-polynomial with respect to $E$, then the associated dual eigenvalues are distinct [10, p. 384].

The following theorem about the $Q$-polynomial property will be used in this paper.

**Theorem 3.1** ([11, Theorem 3.3]). Assume $\Gamma$ is $Q$-polynomial with respect to a primitive idempotent $E$, and let $\theta^0_\ast, \ldots, \theta^D_\ast$ denote the corresponding dual eigenvalues. Then for all integers $1 \leq h \leq D$, $0 \leq i, j \leq D$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{\tilde{z} \in \tilde{X}} E\tilde{z} - \sum_{\tilde{z} \in \tilde{X}} E\tilde{z}' = p_{ij}^h \frac{\theta^0_\ast - \theta^i_\ast}{\theta^h_0 - \theta^0_\ast} (E\tilde{x} - E\tilde{y}).$$

(i) $\Gamma$ is said to have classical parameters $(D, b, \alpha, \beta)$ whenever the intersection numbers of $\Gamma$ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left( 1 + \alpha \begin{bmatrix} i - 1 \\ 1 \end{bmatrix} \right) \quad \text{for } 0 \leq i \leq D,$$

$$b_j = \left( \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left( \beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad \text{for } 0 \leq i \leq D,$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}.$$

(ii) $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ for some real constants $\alpha, \beta$.

The following theorem characterizes the distance-regular graphs with classical parameters and \(a_1 = 0, a_2 \neq 0\) in a combinatorial way.

**Theorem 3.3** ([6, Theorem 1.1]). Let \(\Gamma\) denote a distance-regular graph with diameter \(D \geq 3\) and intersection numbers \(a_1 = 0, a_2 \neq 0\). Then the following (i)–(iii) are equivalent.

(i) \(\Gamma\) is \(Q\)-polynomial and contains no parallelograms of length 3.

(ii) \(\Gamma\) is \(Q\)-polynomial and contains no parallelograms of any length \(i \) for \(3 \leq i \leq D\).

(iii) \(\Gamma\) has classical parameters \((D, b, \alpha, \beta)\) for some real constants \(b, \alpha, \beta\) with \(b < -1\).

### 4. Proof of main theorem

Assume \(\Gamma = (X, R)\) is a distance-regular graph with classical parameters \((D, b, \alpha, \beta)\) and \(D \geq 3\). Suppose the intersection numbers \(a_1 = 0\) and \(a_2 \neq 0\). Then \(\Gamma\) contains no parallelograms of any length by Theorem 3.3. We first give a definition.

**Definition 4.1.** For any vertex \(x \in X\) and any subset \(C \subseteq X\), define

\[
[x, C] := \{v \in X \mid \text{there exists } z \in C, \text{ such that } \partial(x, v) + \partial(v, z) = \partial(x, z)\}.
\]

Throughout this section, fix two vertices \(x, y \in X\) with \(\partial(x, y) = 3\). Set

\[
C := \{z \in \Gamma_3(x) \mid B(x, y) = B(x, z)\}
\]

and

\[
\Delta = [x, C].
\]  

(4.1)

We shall prove that \(\Delta\) is a regular weak-geodetically closed subgraph of diameter 3. Note that the diameter of \(\Delta\) is at least 3. If \(D = 3\) then \(C = \Gamma_3(x)\) and \(\Delta = \Gamma\) is clearly a regular weak-geodetically closed graph. Thereafter we assume \(D \geq 4\). By referring to Theorem 2.2, we shall prove that \(\Delta\) is weak-geodetically closed with respect to \(x\), and the subgraph induced on \(\Delta\) is regular with valency \(a_3 + c_3\).

**Lemma 4.2.** For all adjacent vertices \(z, z' \in \Gamma_i(x)\), where \(i \leq D\), we have \(B(x, z) = B(x, z')\).

**Proof.** By symmetry, it suffices to show that \(B(x, z) \subseteq B(x, z')\). Suppose there exists \(w \in B(x, z) \setminus B(x, z')\). Then \(\partial(w, z') \neq i + 1\). Note that \(\partial(w, z') \leq \partial(w, x) + \partial(x, z') = 1 + i\) and \(\partial(w, z') \geq \partial(w, z) = \partial(z, z') = i\). This implies \(\partial(w, z') = i\) and \(wxyz'z\) forms a parallelogram of length \(i + 1\), a contradiction. \(\square\)

We know that \(\Gamma\) is 2-bound by Theorem 2.4. For two vertices \(z, s\) in \(\Gamma\) with \(\partial(z, s) = 2\), let \(\Omega(z, s)\) denote the regular weak-geodetically closed subgraph containing \(z, s\) of diameter 2.

**Lemma 4.3.** Suppose \(stuzw\) is a pentagon in \(\Gamma\), where \(s, u \in \Gamma_3(x)\) and \(z \in \Gamma_2(x)\). Pick \(v \in B(x, u)\). Then \(\partial(v, s) \neq 2\).

**Proof.** Suppose \(\partial(v, s) = 2\). Note \(\partial(z, s) \neq 1\), since \(a_1 = 0\). Note that \(z, w, s, t, u \in \Omega(z, s)\). Then \(s \in \Omega(z, s) \cap \Gamma_2(v)\) and \(u \in \Omega(z, s) \cap \Gamma_4(v) \neq \emptyset\). Hence \(\partial(v, z) = \partial(v, s) + \partial(s, z) = 2 + 2 = 4\) by Theorem 2.5. A contradiction occurs since \(\partial(v, x) = 1\) and \(\partial(x, z) = 2\). \(\square\)

**Lemma 4.4.** Suppose \(stuzw\) is a pentagon in \(\Gamma\), where \(s, u \in \Gamma_3(x)\) and \(z \in \Gamma_2(x)\). Then \(B(x, s) = B(x, u)\).

Proof. Since \(|B(x, s)| = |B(x, u)| = b_3\), it suffices to show \(B(x, u) \subseteq B(x, s)\). By Lemma 4.3, \(B(x, u) \subseteq \Gamma_3(s) \cup \Gamma_4(s)\).

Suppose

\[
|B(x, u) \cap \Gamma_3(s)| = m,

|B(x, u) \cap \Gamma_4(s)| = n.
\]

Then

\[
m + n = b_3. \tag{4.2}
\]

By Theorem 3.1,

\[
\sum_{r \in B(x, u)} Er - \sum_{r' \in B(u, x)} Er' = b_3 \frac{\theta_3^* - \theta_4^*}{\theta_0^* - \theta_3^*} (E \hat{r} - E \hat{u}). \tag{4.3}
\]

Observe \(B(u, x) \subseteq \Gamma_3(s)\); otherwise \(\Omega(u, s) \cap B(u, x) \neq \emptyset\) and this leads \(\partial(x, s) = 4\) by Theorem 2.5, a contradiction. Taking the inner product of \(s\) with both sides of (4.3) and evaluating the result using (3.5), we have

\[
m \theta_3^* + n \theta_4^* - b_3 \theta_3^* = b_3 \frac{\theta_3^* - \theta_4^*}{\theta_0^* - \theta_3^*} (\theta_3^* - \theta_4^*). \tag{4.4}
\]

Solve (4.2) and (4.4) to obtain

\[
n = b_3 \left( \frac{\theta_2^* - \theta_3^*}{\theta_3^* - \theta_4^*} \right) \left( \frac{\theta_0^* - \theta_4^*}{\theta_0^* - \theta_3^*} \right). \tag{4.5}
\]

Simplifying (4.5) using (3.10), we have \(n = b_3\) and then \(m = 0\) by (4.2). This implies \(B(x, u) \subseteq B(x, s)\) and ends the proof. \(\square\)

Lemma 4.5. Let \(z, u \in \Delta\). Suppose \(stuzw\) is a pentagon in \(\Gamma\), where \(z, w \in \Gamma_2(x)\) and \(u \in \Gamma_3(x)\). Then \(w \in \Delta\).

Proof. Observe \(\Omega(z, s) \cap \Gamma_1(x) = \emptyset\) and \(\Omega(z, s) \cap \Gamma_4(x) = \emptyset\) by Theorem 2.5. Hence \(s, t \in \Gamma_2(x) \cup \Gamma_3(x)\). Observe \(s \in \Gamma_3(x)\); otherwise \(w, s \in \Omega(x, z)\), and this implies \(u \in \Omega(x, z)\), a contradiction to the diameter of \(\Omega(x, z)\) being 2. Hence \(B(x, s) = B(x, u)\) by Lemma 4.4. Then \(w \in \Delta\) by construction. \(\square\)

Lemma 4.6. The subgraph \(\Delta\) is weak-geodetically closed with respect to \(x\).

Proof. Clearly \(C(z, x) \subseteq \Delta\) for any \(z \in \Delta\). It suffices to show \(A(z, x) \subseteq \Delta\) for any \(z \in \Delta\). Suppose \(z \in \Delta\). We discuss this case by case in the following. The case \(\partial(x, z) = 1\) is trivial since \(a_1 = 0\). For the case \(\partial(x, z) = 3\), we have \(B(x, y) = B(x, z) = B(x, u)\) for any \(w \in A(z, x)\) by definition of \(\Delta\) and Lemma 4.2. This implies \(A(z, x) \subseteq \Delta\) by the construction of \(\Delta\). For the remaining case \(\partial(x, z) = 2\), fix \(w \in A(z, x)\). There exists \(u \in C\) such that \(z \in C(u, x)\). Observe that \(\partial(w, u) = 2\) since \(a_1 = 0\). Choose \(s \in A(w, u)\) and \(t \in C(u, s)\). Then \(stuzw\) is a pentagon in \(\Gamma\). The result comes immediately by Lemma 4.5. \(\square\)

Proof of Theorem 1.3. By Theorem 2.2 and Lemma 4.6, it suffices to show that \(\Delta\) defined in (4.1) is regular with valency \(a_3 + c_3\). Clearly from the construction and Lemma 4.6, \(|\Gamma_1(z) \cap \Delta| = 2\).
\[ a_3 + c_3 \text{ for any } z \in C. \] First we show that \(|\Gamma_1(x) \cap \Delta| = a_3 + c_3\). Note that \(y \in \Delta \cap \Gamma_3(x)\) by construction of \(\Delta\). For any \(z \in C(x, y) \cup A(x, y)\),

\[ \partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1. \]

This implies \(z \in \Delta\) by Definition 2.1 and Lemma 4.6. Hence \(C(x, y) \cup A(x, y) \subseteq \Delta\). Suppose \(B(x, y) \cap \Delta \neq \emptyset\). Choose \(t \in B(x, y) \cap \Delta\). Then there exists \(y' \in \Gamma_3(x) \cap \Delta\) such that \(t \in C(x, y')\). Note that \(B(x, y) = B(x, y')\). This leads to a contradiction to \(t \in C(x, y')\). Hence \(B(x, y) \cap \Delta = \emptyset\) and \(\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)\). Then we have \(|\Gamma_1(x) \cap \Delta| = a_3 + c_3\).

Since each vertex in \(\Delta\) appears in a sequence of vertices \(x = x_0, x_1, x_2, x_3\) in \(\Delta\), where \(\partial(x, x_j) = j\) and \(\partial(x_{j-1}, x_j) = 1\) for \(1 \leq j \leq 3\), it suffices to show

\[ |\Gamma_1(x_i) \cap \Delta| = a_3 + c_3 \quad (4.6) \]

for \(1 \leq i \leq 2\). For each integer \(0 \leq i \leq 2\), we show

\[ |\Gamma_1(x_i) \setminus \Delta| \leq |\Gamma_1(x_{i+1}) \setminus \Delta| \]

by the 2-way counting of the number of the pairs \((s, z)\) for \(s \in \Gamma_1(x_i) \setminus \Delta, z \in \Gamma_1(x_{i+1}) \setminus \Delta\) and \(\partial(s, z) = 2\). For a fixed \(z \in \Gamma_1(x_{i+1}) \setminus \Delta\), we have \(\partial(x, z) = i + 2\) by Lemma 4.6, so \(\partial(x_i, z) = 2\) and \(s \in A(x_i, z)\). Hence the number of such pairs \((s, z)\) is at most \(|\Gamma_1(x_{i+1}) \setminus \Delta| a_2\).

On the other hand, we show that this number is exactly \(|\Gamma_1(x_i) \setminus \Delta| a_2\). Fix an \(s \in \Gamma_1(x_i) \setminus \Delta\). Observe \(\partial(x, s) = i + 1\) by Lemma 4.6. Observe \(\partial(x_{i+1}, s) = 2\) since \(a_1 = 0\). Pick any \(z \in A(x_{i+1}, s)\). We shall prove \(z \notin \Delta\). Suppose \(z \in \Delta\) in the arguments below and choose any \(w \in C(s, z)\).

Case 1: \(i = 0\).

Observe \(\partial(x, z) = 2, \partial(x, s) = 1\) and \(\partial(x, w) = 2\). This will force \(s \in \Delta\) by Lemma 4.6, a contradiction.

Case 2: \(i = 1\).

Observe \(\partial(x, z) = 3; \) otherwise \(z \in \Omega(x, x_2)\) and this implies \(s \in \Omega(x, x_2) \subseteq \Delta\) by Lemmas 2.3 and 4.6, a contradiction. This also implies \(s \in \Delta\) by Definition 2.1 and Lemma 4.6, a contradiction.

Case 3: \(i = 2\).

Observe \(\partial(x, z) = 2\) or \(3\). Suppose \(\partial(x, z) = 2\). Then \(B(x, x_3) = B(x, s)\) by Lemma 4.4 (with \(x_3 = u, x_2 = t\)). Hence \(s \in \Delta\), a contradiction. So \(z \in \Gamma_3(x)\). Note that \(\partial(x, w) \neq 2, 3\); otherwise \(s \in \Delta\) by Lemmas 4.4 and 4.6 respectively. Hence \(\partial(x, w) = 4\). Then by applying \(\Omega = \Omega(x_2, w)\) in Theorem 2.5 we have \(\partial(x_2, z) = 1\), a contradiction to \(a_1 = 0\).

From the above counting, we have

\[ |\Gamma_1(x_i) \setminus \Delta| a_2 \leq |\Gamma_1(x_{i+1}) \setminus \Delta| a_2 \quad (4.7) \]

for \(0 \leq i \leq 2\). Eliminating \(a_2\) from (4.7), we find

\[ |\Gamma_1(x_i) \setminus \Delta| \leq |\Gamma_1(x_{i+1}) \setminus \Delta|, \quad (4.8) \]

or equivalently

\[ |\Gamma_1(x_i) \cap \Delta| \geq |\Gamma_1(x_{i+1}) \cap \Delta| \quad (4.9) \]

for \(0 \leq i \leq 2\). We already know that \(|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_3) \cap \Delta| = a_3 + c_3\). Hence (4.6) follows from (4.9). \(\square\)
Remark 4.7. The 4-bounded property seems to be much harder to prove. We expect the 3-bounded property to be enough for classifying all the distance-regular graphs with classical parameters, $a_1 = 0$ and $a_2 \neq 0$.

References
