Abstract

Let $\Gamma$ denote a distance-regular graph with diameter $D$ and intersection numbers $a_2 > a_1 = 0$. We show that for each $1 \leq d \leq D-1$, if $\Gamma$ contains no parallelograms of lengths up to $d+1$ then $\Gamma$ is $d$-bounded in the sense of the article [Distance-regular graphs, European Journal of Combinatorics(1997)18, 211-229]. By applying this result we show the nonexistence of distance-regular graphs with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$ for any $D \geq 4$. In the end, we survey the progress on the classification of distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $b < -1$.

Keywords: Distance-regular graph, classical parameters, parallelogram, weak-geodetically closed subgraph, $D$-bounded.

1 Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. Recall that a sequence $x, z, y$ of vertices of $\Gamma$ is geodetic whenever

$$\partial(x, z) + \partial(z, y) = \partial(x, y),$$

where $\partial$ denotes the distance function.

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where $\partial$ is the distance function of $\Gamma$. A sequence $x, z, y$ of vertices of $\Gamma$ is \textit{weak-geodetic} whenever

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1.$$  

\textbf{Definition 1.1.} A subset $\Delta \subseteq X$ is \textit{weak-geodetically closed} if for any weak-geodetic sequence $x, z, y$ of $\Gamma$,

$$x, y \in \Delta \implies z \in \Delta.$$  

Weak-geodetically closed subgraphs are called \textit{strongly closed subgraphs} in [12]. If a weak-geodetically closed subgraph $\Delta$ of diameter $d$ is regular then it has valency $a_d + c_d = b_0 - b_d$, where $a_d, c_d, b_0, b_d$ are intersection numbers of $\Gamma$. Furthermore $\Delta$ is distance-regular with intersection numbers $a_i(\Delta) = a_i(\Gamma)$ and $c_i(\Delta) = c_i(\Gamma)$ for $1 \leq i \leq d$ [17, Theorem 4.5].

\textbf{Definition 1.2.} $\Gamma$ is said to be \textit{i-bounded} whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains $x$ and $y$.

Note that a $(D - 1)$-bounded distance-regular graph is clear to be $D$-bounded. The properties of $D$-bounded distance-regular graphs were studied in [18], and these properties were used in the classification of classical distance-regular graphs of negative type [19]. Before stating our main result we make one more definition.

By a \textit{parallelogram of length} $i$, we mean a 4-tuple $xyzw$ consisting of vertices of $\Gamma$ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, w) = i$, and $\partial(x, z) = \partial(y, w) = \partial(y, z) = i - 1$. The following theorem is our main result.

\textbf{Theorem 1.3.} Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Fix an integer $1 \leq d \leq D - 1$ and suppose that $\Gamma$ contains no parallelograms of any length up to $d + 1$. Then $\Gamma$ is $d$-bounded.

Theorem 1.3 answers the problem proposed in [17, p. 299]. Many previous results deal with its complement case $a_1 \neq 0$, for examples under an additional assumption $c_2 > 1$ [17] and under the assumptions $a_2 > a_1 > c_2 = 1$ [13]. More precisely, for the case under the assumptions $a_2 > a_1$ and $c_2 = 1$, H. Suzuki proves the case $d = 2$ in Theorem 1.3 [13]; in particular $\Gamma$ contains a
regular weak-geodetically closed subgraph Ω of diameter 2. Since the Friendship Theorem [20, Theorem 8.6.39] asserts no such Ω in the case \(a_1 = c_2 = 1\), there must be no such distance-regular graph \(Γ\) with \(a_2 > a_1 = c_2 = 1\) and \(Γ\) contains no parallelograms of length 3. Note that the assumption \(a_1 \neq 0\) implies \(a_2 \neq 0\) [2, Proposition 5.5.1(i)]. Hence Theorem 1.3 is also true under the weaker assumptions \(b_1 > b_2\) and \(a_2 \neq 0\) (without assuming \(a_1 = 0\)). Our method in proving Theorem 1.3 also works for the case \(b_1 > b_2\) and \(a_2 \neq 0\) after a slight modification, but we decide not to duplicate the previous works.

On the other hand we suppose that \(Γ\) is \(d\)-bounded for \(d \geq 2\). Let \(Ω \subseteq Δ\) be two regular weak-geodetically closed subgraphs of diameters 1, 2 respectively. Since Ω and Δ have different valency \(b_0 - b_1\) and \(b_0 - b_2\) respectively, we have \(b_1 > b_2\). It is also easy to see that \(Γ\) contains no parallelograms of any length up to \(d + 1\) [17, Lemma 6.5]. With these comments, Theorem 1.3 is the final step in the following characterization of \(d\)-bounded distance-regular graphs in terms of forbidden parallelograms.

**Theorem 1.4.** Let \(Γ\) denote a distance-regular graph with diameter \(D \geq 3\). Suppose the intersection number \(a_2 \neq 0\). Fix an integer \(2 \leq d \leq D - 1\). Then the following two conditions (i), (ii) are equivalent:

(i) \(Γ\) is \(d\)-bounded.

(ii) \(Γ\) contains no parallelograms of any length up to \(d + 1\) and \(b_1 > b_2\). \(\square\)

Theorem 1.3 is a generalization of [2, Lemma 4.3.13], [10], and is also proved under an additional assumption \(c_2 > 1\) by A. Hiraki [4]. To prove Theorem 1.3, we need many previous results of [4]. These will be stated independently in Section 3. Some applications of Theorem 1.3 were previously given in [4], [11]. The following is a new application of Theorem 1.3.

**Theorem 1.5.** There is no distance-regular graph with classical parameters \((D, b, α, β) = (D, -2, -2, ((-2)^{D+1} - 1)/3), where D \geq 4.\)

A consequence of Theorem 1.5 is the following.

**Corollary 1.6.** Let \(Γ\) denote a distance-regular graph with classical parameters \((D, b, α, β), D \geq 4\) and \(c_2 = 1\). Then \(a_2 = a_1\) and \(a_1 \neq 0\).

We prove Theorem 1.3 in Section 4, and prove Theorem 1.5, Corollary 1.6 in Section 5. We survey the progress on the classification of distance-regular graph with classical parameters \((D, b, α, β)\) and \(b < -1\) in Section 6.
2 Preliminaries

In this section we review some definitions, basic concepts and some previous results concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [14] for more background information.

Let $\Gamma = (X, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set $X$, edge set $R$, distance function $\partial$, and diameter $D := \max \{ \partial(x, y) \mid x, y \in X \}$. By a pentagon, we mean a 5-tuple $u_1 u_2 u_3 u_4 u_5$ consisting of distinct vertices in $\Gamma$ such that $\partial(u_i, u_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(u_5, u_1) = 1$.

For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{ z \in X \mid \partial(x, z) = i \}$. The valency $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph $\Gamma$ is called regular (with valency $k$) if each vertex in $X$ has valency $k$. A graph $\Gamma$ is said to be distance-regular whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of $x, y$. The constants $p^h_{ij}$ are known as the intersection numbers of $\Gamma$.

From now on let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y),$$
$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y),$$
$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y).$$

Note that

$$|B(x, y)| = p^i_{i+1},$$
$$|C(x, y)| = p^i_{i-1},$$
$$|A(x, y)| = p^i_i$$

are independent of $x, y$. For convenience, set $c_i := p^i_{i-1}$ for $1 \leq i \leq D$, $a_i := p^i_i$ for $0 \leq i \leq D$, $b_i := p^i_{i+1}$ for $0 \leq i \leq D - 1$ and put $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that $k$ is the valency of $\Gamma$. It is immediate from the
definition of $p_{ij}^h$ that $b_i \neq 0$ for $0 \leq i \leq D - 1$ and $c_i \neq 0$ for $1 \leq i \leq D$. Moreover

$$k = a_i + b_i + c_i \quad \text{for} \quad 0 \leq i \leq D.$$  

(2.1)

A subset $\Omega$ of $X$ is weak-geodetically closed with respect to a vertex $x \in \Omega$ if

$$C(y, x) \subseteq \Omega \quad \text{and} \quad A(y, x) \subseteq \Omega \quad \text{for all} \quad y \in \Omega.$$  

(2.2)

Note that $\Omega$ is weak-geodetically closed if and only if for any vertex $x \in \Omega$, $\Omega$ is weak-geodetically closed with respect to $x$ [17, Lemma 2.3]. We list a few results which will be used later in this paper.

**Theorem 2.1.** ([17, Theorem 4.6]) Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$. Let $\Omega$ be a regular subgraph of $\Gamma$ with valency $\gamma$ and set $d := \min\{i \mid \gamma \leq c_i + a_i\}$. Then the following (i),(ii) are equivalent.

(i) $\Omega$ is weak-geodetically closed with respect to at least one vertex $x \in \Omega$.

(ii) $\Omega$ is weak-geodetically closed with diameter $d$.

In this case $\gamma = c_d + a_d$. □

**Definition 2.2.** Fix a vertex $x \in X$. A pentagon $u_1u_2u_3u_4u_5$ has shape $i_1, i_2, i_3, i_4, i_5$ with respect to $x$ if $i_j = \partial(x, u_j)$ for $1 \leq j \leq 5$.

**Theorem 2.3.** ([17, Lemma 6.9],[13, Lemma 4.1]) Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$. Suppose $a_1 = 0$, $a_2 \neq 0$ and $\Gamma$ contains no parallelograms of length up to $d + 1$ for some integer $d \geq 2$. Let $x$ be a vertex of $\Gamma$, and let $u_1u_2u_3u_4u_5$ be a pentagon of $\Gamma$ such that $\partial(x, u_1) = i - 1$ and $\partial(x, u_3) = i + 1$ for $1 \leq i \leq d$. Then the pentagon $u_1u_2u_3u_4u_5$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$. □

### 3 A few lemmas

Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Such graphs are also studied in [4, 8, 9, 10, 11]. Note that any two vertices at distance 2 are always contained in a pentagon since $a_2 \neq 0$, and two nonconsecutive
vertices in a pentagon of $\Gamma$ have distance 2 since $a_1 = 0$. In this section we give a few lemmas which will be used in the next section. These results were formulated by A. Hiraki in [4] under an additional assumption $c_2 > 1$, but this assumption is essentially not used in his proofs. For the completeness, we still provide the proofs.

**Lemma 3.1.** Fix an integer $1 \leq d \leq D - 1$, and suppose $\Gamma$ does not contain parallelograms of length up to $d + 1$. Then for any two vertices $z, z' \in X$ such that $\partial(x, z) \leq d$ and $z' \in A(z, x)$, we have $B(x, z) = B(x, z')$.

**Proof.** By symmetry, it suffices to show $B(x, z) \subseteq B(x, z')$. Suppose there exists $w \in B(x, z) \setminus B(x, z')$. Then $\partial(w, z') \neq \partial(x, z) + 1$. Note that $\partial(w, z') \leq \partial(w, x) + \partial(x, z) = 1 + \partial(x, z)$ and $\partial(w, z') \geq \partial(w, z) - \partial(z, z') = \partial(x, z)$. This implies $\partial(w, z') = \partial(x, z)$ and $wxz'z$ forms a parallelogram of length $\partial(x, z) + 1$, a contradiction. □

**Lemma 3.2.** Fix integers $1 \leq i \leq d \leq D - 1$, and suppose $\Gamma$ does not contain parallelograms of any length up to $d + 1$. Let $x$ be a vertex of $\Gamma$. Then there is no pentagon of shape $i, i, i, i, i + 1$ with respect to $x$ for $1 \leq i \leq d$.

**Proof.** Let $u_1u_2u_3u_4u_5$ be a pentagon of shape $i, i, i, i, i + 1$ with respect to $x$. We derive a contradiction by induction on $i$. The case $i = 1$ is impossible otherwise $u_1xu_2u_3$ is a parallelogram of length 2. Suppose $i \geq 2$. Note that $B(x, u_1) = B(x, u_2) = B(x, u_3) = B(x, u_4)$ by Lemma 3.1. We shall prove $C(x, u_1) = C(x, u_2) = C(x, u_3) = C(x, u_4)$. First we prove $C(x, u_1) = C(x, u_2)$. It suffices to show $C(x, u_2) \subseteq C(x, u_1)$ since both sets have the same size $c_i$. To the contrary suppose there exists $v \in C(x, u_2) - C(x, u_1)$. Note that $v \in A(x, u_1)$ as $B(x, u_1) = B(x, u_2)$. Then $B(u_1, x) = B(u_1, v)$ by Lemma 3.1 and hence $\partial(v, u_5) = i + 1$ since $u_5 \in B(u_1, x)$. Now $u_3u_4u_5u_3$ has shape $i - 1, i, i + 1, i, i$ with respect to $v$ by Theorem 2.3, a contradiction since $v \notin B(x, u_4)$. This proves $C(x, u_2) \subseteq C(x, u_1)$ as desired. By symmetry, $C(x, u_3) = C(x, u_4)$. It remains to show $C(x, u_3) \subseteq C(x, u_4)$. To the contrary suppose there exists $u \in C(x, u_2) - C(x, u_4)$. Note that $u \in A(x, u_4)$ as $B(x, u_2) = B(x, u_4)$. Then $B(u_4, x) = B(u_4, u)$ by Lemma 3.1 and hence $\partial(u, u_5) = i + 1$ since $u_5 \in B(u_4, x)$. Hence $u_2u_1u_5u_4u_3$ has shape $i - 1, i, i + 1, i, i$ with respect to $u$ by Theorem 2.3, a contradiction since $u \notin B(x, u_4)$. Pick a vertex $v \in C(x, u_1) = C(x, u_2) = C(x, u_3) = C(x, u_4)$. Then $u_1u_2u_3u_4u_5$ is a pentagon of shape $i - 1, i - 1, i - 1, i - 1, i$ with respect to $v$, a contradiction to the inductive hypothesis. □
**Proposition 3.3.** Fix integers $1 \leq i \leq d \leq D - 1$, and suppose $\Gamma$ does not contain parallelograms of any length up to $d + 1$. Let $x$ be a vertex and $u_1 u_2 u_3 u_4 u_5$ be a pentagon of shape $i, i-1, i, i-1, i$ or of shape $i, i-1, i, i-1, i-1$ with respect to $x$ for $1 \leq i \leq d$. Then $B(x, u_1) = B(x, u_3)$.

**Proof.** It suffices to show $B(x, u_3) \subseteq B(x, u_1)$ since both sets have the same size $b_i$. Pick $u \in B(x, u_3)$. Then $\partial(u, u_3) = i + 1$, $\partial(u, u_4) = i$ and $\partial(u, u_2) = i$. Note that $\partial(u, u_4) \neq i - 1$, otherwise by Theorem 2.3, the pentagon $u_1 u_2 u_3 u_4 u_5$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $u$, a contradiction. Suppose $\partial(u, u_1) = i$ for this moment. Then to avoid obtaining a pentagon $u_1 u_2 u_3 u_4 u_5$ of type $i, i, i, i, i + 1$ with respect to $u$ we must have $\partial(u, u_5) = i + 1$ by Lemma 3.2. Then $\partial(x, u_5) = i$ by construction. Now $u_5 u_1 x u$ is a parallelogram of length $i + 1$, a contradiction. Hence $\partial(u, u_1) = i + 1$ or equivalently $u \in B(x, u_1)$. This proves $B(x, u_3) \subseteq B(x, u_1)$ as desired. \[ \Box \]

**Lemma 3.4.** Fix integers $1 \leq i \leq d \leq D - 1$, and suppose $\Gamma$ does not contain parallelograms of any length up to $d + 1$. Let $x$ be a vertex. Then there is no pentagon of shape $i, i, i, i + 1, i + 1$ with respect to $x$ for $1 \leq i \leq d$.

**Proof.** Suppose that $u_2 u_3 u_4 u_5 u_1$ is a pentagon of shape $i, i, i, i + 1, i + 1$ with respect to $x$. We derive a contradiction by induction on $i$. The case $i = 1$ is impossible otherwise $u_2 x u_4 u_5$ is a parallelogram of length 2. Suppose $i \geq 2$. Pick $v \in C(x, u_2)$ and note that $\partial(v, u_1) = i$ by construction. In particular $v \not\in B(x, u_2)$ and $B(x, u_2) = B(x, u_3) = B(x, u_4)$ by Lemma 3.1, so $v \in C(x, u_4) \cup A(x, u_4)$. In fact $v \in C(x, u_4)$; otherwise $\partial(v, u_4) = i$, $\partial(v, u_5) = i$ by Theorem 2.3, and then $x v u_4 u_5$ is a parallelogram of length $i + 1$, a contradiction. We also have $\partial(v, u_5) = i$ by construction. Note that $\partial(v, u_3) = i$; otherwise $\partial(v, u_3) = i - 1$ and $u_2 u_3 u_4 u_5 u_1$ is a pentagon of shape $i - 1, i - 1, i - 1, i, i$ with respect to $v$, a contradiction to inductive hypothesis. Now as $x = v$ in Proposition 3.3, we have $B(v, u_1) = B(v, u_3)$, a contradiction since $x \in B(v, u_1) - B(v, u_3)$. \[ \Box \]

4 Proof of Theorem 1.3

Let $\Gamma = (X, R)$ denote a distance-regular graph with intersection numbers $a_1 = 0$, $a_2 \neq 0$ and diameter $D \geq 3$. Fix an integer $1 \leq d \leq D - 1$. Suppose $\Gamma$ contains no parallelograms of length up to $d + 1$. We shall prove $\Gamma$ is $d$-bounded in this section. We first give a definition.
**Definition 4.1.** For any vertex \( x \in X \) and any subset \( \Pi \subseteq X \), define \([x, \Pi]\) to be the set
\[
\{ v \in X \mid \text{there exists } y' \in \Pi, \text{ such that the sequence } x, v, y' \text{ is geodetic } \}.
\]

For any \( x, y \in X \) with \( \partial(x, y) = d \), set
\[
\Pi_{xy} := \{ y' \in \Gamma_d(x) \mid B(x, y) = B(x, y') \}
\]  
(4.1)
and
\[
\Delta(x, y) = [x, \Pi_{xy}].
\]  
(4.2)

We shall prove that for any vertices \( x, y \in X \) with \( \partial(x, y) = d \) the following statement \( B_d \) holds.

\( (B_d) \) \( \Delta(x, y) \) is regular weak-geodetically closed with valency \( a_d + c_d \).

By referring to Theorem 2.1, \( (B_d) \) is equivalent to the following statements \( (W_d) \) and \( (R_d) \).

\( (W_d) \) \( \Delta(x, y) \) is weak-geodetically closed with respect to \( x \), and

\( (R_d) \) the subgraph induced on \( \Delta(x, y) \) is regular with valency \( a_d + c_d \)
for any vertices \( x, y \in X \) with \( \partial(x, y) = d \).

We prove \( (W_d) \) and \( (R_d) \) by induction on \( d \). Since \( a_1 = 0 \), there is no edges in \( \Gamma_1(x) \) for any vertex \( x \in X \). If \( d = 1 \) in Definition 4.1, then \( \Pi_{xy} = \{ y \} \), and consequently \( \Delta(x, y) = \{ x, y \} \) is an edge; in particular \( \Delta(x, y) \) is regular with valency \( 1 = a_1 + c_1 \) and is weak-geodetically closed with respect to \( x \) since \( a_1 = 0 \). This proves \( (R_1) \) and \( (W_1) \). We now assume \( d \geq 2 \). By inductive hypothesis \( (W_i), (R_i) \) and \( (B_i) \) are assumed throughout this section for \( 1 \leq i \leq d - 1 \). The following proposition proves the statement \( (W_d) \).

**Proposition 4.2.** For any vertices \( x, y \in X \) with \( \partial(x, y) = d \) and for any vertex \( z \in \Delta(x, y) \cap \Gamma_i(x) \), where \( 1 \leq i \leq d \), we have the following (i), (ii).

(i) \( A(z, x) \subseteq \Delta(x, y) \).

(ii) For any vertex \( w \in \Gamma_i(x) \cap \Gamma_2(z) \) with \( B(x, w) = B(x, z) \), we have \( w \in \Delta(x, y) \).
In particular $(W_d)$ holds.

Proof. We prove (i), (ii) by induction on $d - i$. The case $i = d$ follows from the construction of $\Delta(x,y)$ in Definition 4.1 and by Lemma 3.1. Suppose $i < d$.

To prove (i) we note that if $i = 1$ then $A(z,x)$ is an empty set, clearly contained in $\Delta(x,y)$. Hence we suppose $2 \leq i < d$ in this case. We pick a vertex $v \in A(z,x)$ and show $v \in \Delta(x,y)$. Pick $u \in \Delta(x,y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$. Note that (i), (ii) hold if we use $u$ to replace $z$ by inductive hypothesis. Let $uu_2u_3vz$ be a pentagon of $\Gamma$ for some $u_2, u_3 \in X$. Note that $uu_2u_3vz$ can not have shape $i + 1, i, i - 1, i, i$, shape $i + 1, i + 2, i + 1, i, i$ by Theorem 2.3, can not have shape $i + 1, i + 1, i, i, i$ by Lemma 3.2, and can not have shape $i + 1, i + 1, i, i, i$ by Lemma 3.4 with respect to $x$. Hence $uu_2u_3vz$ has shape $i + 1, i + 1, i, i, i$ with respect to $x$. In the first case we have $u_2 \in A(u,x)$, $u_3 \in A(u_2,x)$, and this implies $u_2, u_3 \in \Delta(x,y)$ by the inductive hypothesis of (i), and $v \in \Delta(x,y)$ by construction. In the latter case we have $B(x,u) = B(x,u_3)$ by Lemma 3.3, and consequently $u_3 \in \Delta(x,y)$ by inductive hypothesis of (ii), $v \in \Delta(x,y)$ by construction.

To prove (ii) let $zzv_2wv_4v_5$ be a pentagon for some $v_2, v_4, v_5 \in X$. Note that $\Delta(x,z)$ is a regular weak-geodetically closed subgraph of diameter $i$ by $(B_i)$, and $\Delta(x,z) = \Delta(x,w)$ by construction in Definition 4.1 and since $B(x,w) = B(x,z)$; in particular $v_2, v_4, v_5 \in \Delta(x,z)$ and $v_2, v_4, v_5 \notin \Gamma_{i+1}(x)$. If $v_2 \in A(z,x)$ then $v_2, w \in \Delta(x,y)$ by (i) that we just proved. Hence we assume $zzv_2wv_4v_5$ has shape $i, i - 1, i, a, b$ with respect to $x$ for integers $a, b \in \{i - 1, i\}$. Pick $u \in \Delta(x,y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$. Let $zzuv_2y_3y_4$ be a pentagon for some $y_3, y_4 \in X$. Then $zzuv_2y_3y_4$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$ by Theorem 2.3. Let $zzuv_2w_3w_4w$ be a pentagon for some $w_3, w_4 \in X$. If $w_4 \in A(w,x) \cup C(w,x)$ then $w_4 \in \Delta(x,w) = \Delta(x,z)$ and this forces $y_4 \in \Delta(x,y)$ as $z, y_4 \in \Delta(x,z)$. We have a contradiction since $\Delta(x,z)$ has diameter $i$ and $\partial(x,y_4) = i + 1 > i = \text{diam} \Delta(x,z)$. Hence $\partial(x,w_4) = i + 1$ and $zzuv_2w_3w_4w$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$ by Theorem 2.3 as shown in Figure 1. Note that $B(x,u) = B(x,y_3)$ and $B(x,w_4) = B(x,w_4)$ by Lemma 3.1. If $B(x,y_3) = B(x,w_3)$ then by (i) the inductive hypothesis of (ii) we have $y_3, w_3 \in \Delta(x,y)$ in the order, and $w \in \Delta(x,y)$ by the construction in (4.2) to complete the proof. Suppose $B(x,y_3) \neq B(x,w_3)$ in the remaining. Let $y_4y_3y_3y_4w_3$ be a pentagon for some $p_3, p_4 \in X$. By Lemma 3.1, Lemma 3.3 and Theorem 2.3, the pentagon
$y_3y_3p_3p_4w_3$ has shape $i, i+1, i+2, i+2, i+1$ with respect to $x$. Now we have three pentagons and their shapes with respect to $x$ as shown in Figure 1. Note that $B(x, y_4) \neq B(x, z)$, otherwise $\Delta(x, y_4) = \Delta(x, z)$ and $y_3 \in \Delta(x, z)$, a contradiction as before. Pick $p \in B(x, y_4) - B(x, z)$. Then $\partial(p, y_4) = i+1$ and $\partial(p, z) = i - 1$ or $i$. Suppose for this moment $\partial(p, z) = i - 1$. Then $zuyp_4v_2$ is a pentagon of shape $i - 1, i, i+1, i+1, i$ with respect to $p$ by Theorem 2.3. Note that $\partial(p, p_3) = i + 2$, otherwise $\partial(p, p_3) = i + 1$ and $xpyp_3$ is a parallelogram of length $i + 2 \leq d + 1$, a contradiction. Now by applying Lemma 3.2, Lemma 3.4, we have $\partial(p, w_3) = i + 2$ and consequently $v_2y_4w_3w_4w$ is a pentagon of shape $i, i+1, i+2, i+2, i+1$ with respect to $p$ by Theorem 2.3. That is $p \in B(x, w)$, a contradiction to $B(x, z) = B(x, w)$. By symmetry, we also have $\partial(p, w) \neq i - 1$. We suppose in the last case $\partial(p, z) = \partial(p, w) = i$. As $p \in A(x, z)$, we have $B(z, x) = B(z, p)$ by Lemma 3.1, in particular $\partial(p, u) = i + 1$. By symmetry, $\partial(p, w_4) = i + 1$. As $p \notin B(x, u) = B(x, y_3)$, we have $\partial(p, y_3) = i$ or $i + 1$. We shall prove $\partial(p, y_3) = i$, and by symmetry $\partial(p, w_3) = i$. Suppose to the contrary we have $\partial(p, y_3) = i + 1$. As $p_3 \in B(y_3, x) = B(y_3, p)$, $\partial(p, p_3) = i + 2$. Applying Lemma 3.2, Lemma 3.4 to the pentagon $w_3y_3y_3p_3p_4$ and considering its shape with respect to $p$, we find $\partial(p, w_3) \neq i + 1$, and applying Theorem 2.3 to find $\partial(p, w_3) \neq i$. Now $\partial(p, w_3) = i + 2$ and $pxw_4w_3$ is a parallelogram of length $i + 2 \leq d + 1$, a contradiction. We conclude that $y_3y_3p_3p_4w_3$ is a pentagon of shape $i + 1, i, a, i + 1, i$ or of shape $i + 1, i, i + 1, b, i$ with respect to $p$ for $a, b \in \{i, i + 1\}$ by Lemma 2.3, Lemma 3.2, and this implies $x \in B(p, p_4) = B(p, y_4)$ in the first case or $x \in B(p, p_3) = B(p, y_4)$ in the latter case by Proposition 3.3, a contradiction since $x \in C(p, y_4)$.

$\Delta(x, y)$ is clear to be weak-geodetically closed with respect to $x$ by (2.2) and (i).  \[ \square \]
The following proposition proves \((R_d)\) and hence completes the proof of Theorem 1.3.

**Proposition 4.3.** For any vertices \(x, y \in X\) with \(\partial(x, y) = d\), \(\Delta(x, y)\) is regular with valency \(a_d + c_d\).

**Proof.** Set \(\Delta = \Delta(x, y)\). Clearly from the construction and Proposition 4.2, 
\[ |\Gamma_1(y') \cap \Delta| = a_d + c_d \text{ for any } y' \in \Pi_{xy}. \] First we show 
\[ |\Gamma_1(x) \cap \Delta| = a_d + c_d. \]
Note that \(y \in \Delta \cap \Gamma_d(x)\) by construction of \(\Delta\). For any \(z \in C(x, y) \cup A(x, y)\), 
\[ \partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1. \]
This implies \(z \in \Delta\) since \(\Delta\) is weak-geodetically closed with respect to \(x\) by Proposition 4.2. Hence \(C(x, y) \cup A(x, y) \subseteq \Delta\). Suppose \(B(x, y) \cap \Delta \neq \emptyset\). Choose \(t \in B(x, y) \cap \Delta\). Then there exists \(y' \in \Pi_{xy}\) such that \(t \in C(x, y')\), a contradiction to \(B(x, y) = B(x, y')\). Hence \(B(x, y) \cap \Delta = \emptyset\) and \(\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)\). This proves 
\[ |\Gamma_1(x) \cap \Delta| = a_d + c_d. \]
Since each vertex in $\Delta$ appears in a sequence of vertices $x = x_0, x_1, \ldots, x_d$ in $\Delta$, where $\partial(x, x_j) = j$, $\partial(x_{j-1}, x_j) = 1$ for $1 \leq j \leq d$, and $x_d \in \Pi_{xy}$, it suffices to show
\[ |\Gamma_1(x_i) \cap \Delta| = a_d + c_d \] (4.3)
for $1 \leq i \leq d - 1$. For each integer $1 \leq i \leq d$, we show
\[ |\Gamma_1(x_{i-1}) \setminus \Delta| \leq |\Gamma_1(x_i) \setminus \Delta| \] (4.4)
by the 2-way counting of the number of the pairs $(z, s)$ for $z \in \Gamma_1(x_{i-1}) \setminus \Delta$, $s \in \Gamma_1(x_i) \setminus \Delta$ and $\partial(z, s) = 2$. For a fixed $s \in \Gamma_1(x_i) \setminus \Delta$, we have $\partial(x, s) = i + 1$ and $\partial(x_{i-1}, s) = 2$ since $\Delta$ is weak-geodetically closed with respect to $x$ by Proposition 4.2. Hence $z \in A(x_{i-1}, s)$. The number of such pairs $(z, s)$ is at most $|\Gamma_1(x_{i-1}) \setminus \Delta|a_2$.

On the other hand, we show this number is $|\Gamma_1(x_{i-1}) \setminus \Delta|a_2$ exactly. Fix an $z \in \Gamma_1(x_{i-1}) \setminus \Delta$. Note that $\partial(x, z) = i$ by Proposition 4.2, and $\partial(x_{i-1}, z) = 2$ since $a_1 = 0$. Pick any $s \in A(x_i, z)$. We shall prove $s \notin \Delta$. Suppose to the contrary $s \in \Delta$ in the below arguments and choose any $w \in C(s, z)$. Note that $\partial(x, s) \leq i$, otherwise $\partial(x, s) = i + 1$ and the pentagon $x_{i-1}x_iswz$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$ by Theorem 2.3 to force $z \notin \Delta$ by Proposition 4.2(i) and construction of $\Delta$, a contradiction. Similarly $\partial(x, w) \leq i$. If $s \in A(x_i, x)$, $w \in A(s, x)$ and $z \in A(w, x)$, then $z \notin \Delta$ by Proposition 4.2(i), a contradiction. Applying Proposition 3.3 in the remaining cases we have $B(x, z) = B(x, x_i)$ and then $z \notin \Delta$ by Proposition 4.2(ii), a contradiction.

From the above counting, we have
\[ |\Gamma_1(x_{i-1}) \setminus \Delta|a_2 \leq |\Gamma_1(x_i) \setminus \Delta|a_2 \] (4.5)
for $1 \leq i \leq d$. Eliminating $a_2$ from (4.5), we find (4.4) or equivalently
\[ |\Gamma_1(x_{i-1}) \cap \Delta| \geq |\Gamma_1(x_i) \cap \Delta| \] (4.6)
for $1 \leq i \leq d$. We have known previously $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_d) \cap \Delta| = a_d + c_d$. Hence (4.3) follows from (4.6).

\[ \square \]
5 Classical parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. $\Gamma$ is said to have classical parameters $(D, b, \alpha, \beta)$ whenever the intersection numbers of $\Gamma$ satisfy

$$
c_i = \left[ i \right] \left( 1 + \alpha \left[ i - 1 \right] \right) \quad \text{for } 0 \leq i \leq D, \quad (5.1)
$$

$$
b_i = \left( \left[ D \right] - \left[ i \right] \right) \left( \beta - \alpha \left[ i \right] \right) \quad \text{for } 0 \leq i \leq D, \quad (5.2)
$$

where

$$
\left[ i \right] := 1 + b + b^2 + \cdots + b^{i-1}. \quad (5.3)
$$

Applying (2.1) with (5.1), (5.2), we have

$$
a_i = \left[ i \right] \left( \beta - 1 + \alpha \left( \left[ D \right] - \left[ i \right] - \left[ i - 1 \right] \right) \right) \quad (5.4)
$$

$$
a_i = \left[ i \right] \left( a_1 - \alpha \left[ i \right] + \left[ i - 1 \right] - 1 \right) \quad (5.5)
$$

for $1 \leq i \leq D$.

Suppose $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ and $D \geq 3$. Then $b$ is an integer, $b \neq 0$ and $b \neq -1$ [2, p. 195]. To apply Theorem 1.4 we need the following lemma.

Lemma 5.1. ([15, Theorem 2.12], [17, Lemma 7.3(ii)]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$, $b < -1$ and $D \geq 3$. Then $\Gamma$ contains no parallelograms of any length. \hfill \Box

More general version of Lemma 5.1 can be found in [16, 8, 9].

Theorem 5.2. ([18, Theorem 4.2]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $b < -1$. Suppose that $\Gamma$ is $D$-bounded with $D \geq 4$. Then

$$
\beta = \alpha \frac{1 + b^D}{1 - b}. \quad (5.6)
$$
Proof of Theorem 1.5. Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, where $D \geq 4$. Then $\Gamma$ contains no parallelograms of any length by Lemma 5.1. By (5.1), (5.4), we have $c_2 = 1$ and $a_2 = 2 > 0 = a_1$. Hence $\Gamma$ is $D$-bounded by Theorem 1.4 and since $b_1 > b_2$. By (5.6), $\beta = ((-2)^{D+1} - 2)/3$, a contradiction. \hfill \Box

We quote a few previous results in the study of distance-regular graphs with classical parameters and $c_2 = 1$ for later use.

**Lemma 5.3.** ([18, Corollary 6.3]) There is no distance-regular graph $\Gamma$ with classical parameters $(D, b, \alpha, \beta)$, $D \geq 4$, $c_2 = 1$ and $a_2 > a_1 > 1$. \hfill \Box

**Lemma 5.4.** ([11, Theorem 2.2]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $D \geq 3$. Assume the intersection numbers $a_1 = 0$, $a_2 \neq 0$, and $c_2 = 1$. Then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$. \hfill \Box

**Lemma 5.5.** ([15, Theorem 2.11], [17, Lemma 7.3(ii)]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $D \geq 3$. Suppose $\Gamma$ contains no parallelograms of length 2. Then $\Gamma$ contains no parallelograms of any length. \hfill \Box

**Proof of Corollary 1.6.** Since $c_2 = 1$, $\Gamma$ contains no parallelograms of length 2 and then contains no parallelogram of any length by Lemma 5.5. By Lemma 5.3, Lemma 5.4, Theorem 1.5, only the case $a_2 > a_1 = 1$ and the case $a_2 = a_1$ remain. The first case is impossible by Friendship Theorem as mentioned in the introduction. For the latter case, we have $\alpha = -b/(1 + b)$ since $c_2 = 1$ and by (5.1). Applying this to (5.5) we find the impossibility of $a_2 = a_1 = 0$.

We close this section by proposing the following conjecture.

**Conjecture 5.6.** There is no distance-regular graph $\Gamma$ with classical parameters $(D, b, \alpha, \beta)$, $D \geq 4$, and $c_2 = 1$.

There is a mistake in [2, Proposition 6.1.2] which proves the above conjecture. This mistake is corrected in [3].

**Remark 5.7.** (See [2, p. 194]) The Triality graph $^3D_{4,2}(q)$ is a distance-regular graph with classical parameters $(3, -q, q/(1 - q), q^2 + q)$, $c_2 = 1$ and $a_1 = a_2 = q - 1$. Hence the assumption $D \geq 4$ in Conjecture 5.6 is necessary. Note that the Triality graph $^3D_{4,2}(q)$ is not 3-bounded by Theorem 1.4 since $b_1 = b_2$.  

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6 Classical parameters with $b < -1$

Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$, $b < -1$ and $D \geq 3$. We survey the progress on the classification of such $\Gamma$ in this section. Two main classes of such examples are the dual polar graphs $^2A_{2D-1}(-b)$ and the Hermitian forms graphs $\text{Her}_-b(D)$ as listed in [2, Tabel 6.1]. A.A. Ivanov and S.V. Shpectorov show that if $\Gamma$ has the same intersection numbers as the dual polar graph $^2A_{2D-1}(-b)$ then $\Gamma$ is the dual polar graph $^2A_{2D-1}(-b)$ [5]. They also show that if $\Gamma$ does not contain parallelograms of length 2 and has the same intersection numbers as the Hermitian forms graph $\text{Her}_-b(D)$ then $\Gamma$ is the Hermitian forms graph $\text{Her}_-b(D)$ [6, 7]. P. Terwilliger shows that in fact $\Gamma$ does not contains parallelograms of any length [15] as also stated in Lemma 5.1. According to different assumptions on the intersection numbers of $\Gamma$, the $D$-bounded property of $\Gamma$ are proved by different authors as stated in the introduction. Putting all these results together, if $\Gamma$ has intersection numbers $b_1 > b_2$ and $a_2 \neq 0$ then $\Gamma$ is $D$-bounded as also stated in Theorem 1.4.

We assume $b_1 > b_2$ and $a_2 \neq 0$ in $\Gamma$ thereinafter. The third author shows that if $D \geq 4$ then

$$\beta = \frac{\alpha + b^D}{1 - b}$$

in [18] as also stated in (5.6), and use this to conclude in [19] that if $\Gamma$ is not the dual polar graph $^2A_{2D-1}(-b)$ and not the Hermitian forms graph $\text{Her}_-b(D)$ then

$$\alpha = (b - 1)/2, \quad \beta = -(1 + b^D)/2,$$

where $-b$ is a power of an odd prime.

There are some results of $\Gamma$ in the assumption $D \geq 3$, $a_1 = 0$ and $a_2 \neq 0$. For example in [11], the second author and the third author show that $c_2 \leq 2$, and in the case $c_2 = 1$, it must be

$$(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3).$$

Note that if $D \geq 4$, (6.3) does not hold by (6.2). This is essentially the proof of Theorem 1.5. Hence we have the following conjecture about the case $D = 3$. 

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Conjecture 6.1. There is no distance-regular graphs with classical parameters \((D, b, \alpha, \beta) = (3, -2, -2, 5)\).

Also in [4] A. Hiraki assume that \(D \geq 3\), \(a_1 = 0\), \(a_2 \neq 0\), \(c_2 > 1\) and show that \(\Gamma\) is either the Hermitian forms graph \(\text{Her}_2(D)\) or \(\alpha, \beta\) satisfy (6.2) with \(b = -3\). Hence the following conjecture is the first step to study the unknown case of (6.2).

Conjecture 6.2. There is no distance-regular graph with classical parameters \((D, b, \alpha, \beta) = (3, -3, -2, 13)\).

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