圖的度數對之研究

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摘要

簡單圖 $G$ 上一點 $v$ 的平均二度數定義為與 $v$ 相鄰之點的度數平均。度數列和平均二度數列在最大拉普拉斯特徵值上界的應用，已有許多研究成果。若 $G$ 中所有點的平均二度數皆為 $k$，則 $G$ 稱為擬 $k$ 正則圖。在此論文中，我們證明若 $G$ 爲擬 $k$ 正則圖，則 $k$ 是整數；進而找出所有擬正則樹。我們也考慮了當 $G$ 的最大度數為 $k^2 - k$ 的情形，並給出一些基本的結果。最後，我們對於擬 3 正則圖給出了更多的結果。並且刻畫出所有十個點之內非正則的擬 3 正則圖。

關鍵字：圖，鄰接矩陣，拉普拉斯矩陣，度數，平均二度數，擬 $k$ 正則。
The Degree Pairs of a Graph

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Abstract

Let \( v \) be a vertex in a simple graph \( G \). The \textit{average 2-degree} of \( v \) is the average of degrees of vertices adjacent to \( v \). The applications of the degree and average 2-degree sequences on the upper bounds for the maximum eigenvalue of Laplacian matrix of a graph is studied by many authors. The graph \( G \) is called \textit{pseudo} \( k \)-regular if each vertex in \( G \) has average 2-degree \( k \). We prove that if \( G \) is pseudo \( k \)-regular then \( k \) is integral. Moreover, all pseudo regular trees are given in this thesis. We also consider the case when the maximum degree of \( G \) is \( k^2 - k \), and give some basic results. In the end, we give more results of pseudo 3-regular graphs. And characterize all the pseudo 3-regular graph within ten vertices but not regular.

\textbf{Keywords}: Graph, adjacency matrix, Laplacian matrix, degree, average 2-degree, pseudo \( k \)-regular.
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Chapter 1

Introduction

Let $G$ be a graph with vertex set $V_G = \{1, 2, \ldots, n\}$ and edge set $E_G$. Let $d_i$ be the degree of the vertex $i \in V_G$, defined as follows:

$$d_i := |G_1(i)|,$$

where $G_1(i)$ means the set $\{j \in V_G \mid ji \in E_G\}$ of neighbors of $i$.

Let \( m_i \) be the average 2-degree of the vertex \( i \in VG \), defined as follows.

\[
m_i := \frac{1}{d_i} \sum_{j \in EG} d_j.
\]

And the sequence \( \{m_i\}_{i \in VG} \) of \( G \) is called a \textbf{average 2-degree sequence} of \( G \). We shall give a survey of average 2-degree sequence of a graph.

Let \( G \) be a simple graph. The \textbf{adjacency matrix} of \( G \) is the 0-1 matrix \( A \) indexed by \( VG \) such that \( A_{xy} = 1 \) if and only if \( xy \in EG \). The \textbf{degree matrix} of \( G \) is the diagonal matrix \( D \) indexed by \( VG \) such that \( D_{xx} \) is the degree \( d_x \) of \( x \in VG \). The average 2-degree sequence appears often in the study of maximum eigenvalue \( \ell_1(G) \) of the \textbf{Laplacian matrix} \( L = D - A \) associated with \( G \), where \( D \) is the degree matrix and \( A \) is the adjacency matrix of \( G \). The following results are about the upper bounds of \( \ell_1(G) \):

1. In 1998, Merris gave the following bound [15]:

\[
\ell_1(G) \leq \max_{i \in VG} \{d_i + m_i\}.
\]

2. Also in 1998, Li and Zhang gave the following bound [14]:

\[
\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \right\}.
\]

3. In 2001, Li and Pan gave the following bound [13]:

\[
\ell_1(G) \leq \max_{i \in VG} \left\{ \sqrt{2d_i(d_i + m_i)} \right\}.
\]

4. In 2004, Das gave the following bound [4]:

\[
\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_im_j}}{2} \right\}.
\]
5. Also in 2004, Zhang gave the following bounds [21]:

(a) \[
\ell_1(G) \leq \max_{ij \in E G} \left\{ 2 + \sqrt{d_i(d_i + m_i - 4)} + d_j(d_j + m_j - 4) + 4 \right\}.
\]

(b) \[
\ell_1(G) \leq \max_{i \in V G} \left\{ d_i + \sqrt{d_i m_i} \right\}.
\]

(c) \[
\ell_1(G) \leq \max_{ij \in E G} \left\{ \sqrt{d_i(d_i + m_i)} + d_j(d_j + m_j) \right\}.
\]

As everyone knows, a graph \( G \) is \textbf{k-regular} if \( d_i = k \) for all vertices \( i \in V G \). If \( m_i = k \) for all vertices \( i \in V G \), \( G \) is called \textbf{pseudo k-regular} in [20]. For convenience, we rearrange the vertices of \( G \) by \( 1, 2, \ldots, n \) such that \( m_1 \geq m_2 \geq \cdots \geq m_n \). Let \( a_1(G) \) be the maximum eigenvalue of adjacency matrix \( A \) associated with \( G \), and we have following.

Let \( B = D^{-1} A D \), where \( D \) is the degree matrix and \( A \) is the adjacency matrix of \( G \). Then \( B \) is a nonnegative irreducible \( n \times n \) matrix. By Perron-Frobenius Theorem in [16], we have \( a_1(G) \leq m_1 \) with equality if and only if \( G \) is a pseudo \( k \)-regular graph.

In 2011, Chen, Pan and Zhang [3] proved the following.

**Theorem 1.1.** Let \( a := \max \{d_i/d_j \mid 1 \leq i, j \leq n\} \). Then

\[
a_1(G) \leq \frac{m_2 - a + \sqrt{(m_2 + a)^2 + 4a(m_1 - m_2)}}{2}
\]

with equality if and only if \( G \) is a pseudo \( k \)-regular graph.
And in 2014, Huang and Weng [12] proved the following.

**Theorem 1.2.** For any \( b \geq \max \{d_i/d_j \mid ij \in EG \} \) and \( 1 \leq l \leq n \),

\[
a_1(G) \leq \frac{m_l - b + \sqrt{(m_l + b)^2 + 4b \sum_{i=1}^{l-1} (m_i - m_l)}}{2}
\]

with equality if and only if \( G \) is a pseudo \( k \)-regular graph.

This thesis studies degree sequence together with average 2-degree sequence of a graph. Thus we define the sequence \( \{(d_i, m_i)\}_{i \in V_G} \) of pairs as a degree pairs.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig1.png}
\caption{Two graphs with different sequences of degree pairs \((d_i, m_i)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig2.png}
\caption{Two graphs with the same sequence of degree pairs \((d_i, m_i)\).}
\end{figure}
This thesis is organized as follows. In Chapter 2, we introduce some basic results about degree pairs. In Chapter 3, we prove that if $G$ is pseudo $k$-regular then $k \in \mathbb{N}$, and give a family of pseudo $k$-regular graphs $T_k$. Furthermore, we prove that $T_k$ is the only pseudo $k$-regular tree for each $k$. We also consider the case when the maximum degree of $G$ is $k^2 - k$, and give some basic results. In the end, we give more results of pseudo 3-regular graphs. And characterize all the pseudo 3-regular graph within ten vertices but not regular.
Chapter 2

Degree pairs

Let $G$ be a simple graph with vertex set $VG = \{1, 2, \ldots, n\}$, edge set $EG$, and sequence $\{(d_i, m_i)\}_{i \in VG}$ degree pairs. The following lemma provides a feasible condition of degree pairs.

**Lemma 2.1.**

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2.$$

**Proof.**

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i \sum_{j \in EG}d_j = \sum_{i \in VG} \sum_{j \in EG} d_j = \sum_{i \in VG} d_i = \sum_{i \in VG} d_i^2.$$

We give a sequence $A = \{(1, 3), (1, 3), (2, 3), (3, 2), (3, 2)\}$, and a sequence $B = \{(1, 4), (3, 2), (3, 3), (3, 3), (4, 2)\}$. Observe that sequence $A$ matches the condition in Lemma 2.1, and is a sequence of degree pairs of the graph as shown in Figure 2.1. But sequence $B$ does not match the condition in Lemma 2.1, so its not a sequence of degree pairs of any graph.
Here is another feasible condition for degree pairs.

**Lemma 2.2.** There are even number of odd values $d_im_i$ among $i \in VG$.

**Proof.** Since $\sum_{i \in VG} d_i$ is even, there are even number of odd $d_i$, and so does $d_i^2$. Hence $\sum_{i \in VG} d_im_i = \sum_{i \in VG} d_i^2$ is even. \hspace{1cm} \Box

**Corollary 2.3.**

$$\sum_{i \in VG} m_i^2 \geq \sum_{i \in VG} d_i^2$$

with equality if and only if $m_i = d_i = k$ for all $i$.

**Proof.**

$$(\sum_{i \in VG} d_i^2)(\sum_{i \in VG} m_i^2) \geq (\sum_{i \in VG} d_im_i)^2 = (\sum_{i \in VG} d_i^2)^2$$

and equality if and only if $m_i = cd_i$ for all $i \in VG$, where $c = 1$ by the Lemma 2.1. This is also equivalent to that all neighbors of a vertex of minimum degree $k$ also have degree $k$. \hspace{1cm} \Box

Degree sequence gives hints of graph properties. For example, the well-known fact $|EG| = \frac{1}{2} \sum_{i \in VG} d_i$ expressed the number of edges of a graph as a sum its degree sequence.

The sequence of degree pairs give more hints of graph structure. In general, $d_im_i \geq |G_1(i)| + |G_2(i)|$, and there are at least $(d_im_i - n)/2$ triangles based on the vertex $i$. 7
Proposition 2.4. If $\max_{i \in VG} d_i m_i \geq n$ then the graph has girth at most 4.

Proof. If the graph has girth at least 5 then

$$n - 1 = |VG| - 1 \geq |G_1(i) + G_2(i)| = d_i m_i.$$ 

for any $i \in VG$. \hfill \Box

Figure 2.2: A graph has girth at most 4.

In Figure 2.2, we observe that $\max_{i \in VG} d_i m_i = 8 \geq 6 = |VG|$

The distance $d(x, y)$ between two vertices $x$ and $y$ of a graph is the minimum length of the paths connecting them. Let $G^2$ be the square of $G$, denote the graph with $VG^2 = VG$ and $EG^2 = \{xy \mid d(x, y) \leq 2\}$. The independence number of $G$ is $\alpha(G) = \max\{|S| \mid S \subseteq VG, S \text{ is the independent set of } G\}$.

Proposition 2.5.

$$\alpha(G^2) \geq \sum_{i \in VG} \frac{1}{1 + d_i m_i},$$

where $\alpha(G^2)$ is the independence number of the square of $G$.

Proof. If a vertex is picked equally in random then the probability of a vertex $i$ appears before those vertices in $G_1(i) \cap G_2(i)$ is $(1 + |G_1(i)| + |G_2(i)|)^{-1}$. Hence the expected size of a set consisting of these $i$ is $\sum_{i \in VG} (1 + |G_1(i)| + |G_2(i)|)^{-1}$, which is at least $\sum_{i \in VG} \frac{1}{1 + d_i m_i}$, \hfill \Box

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The following lemma will be used later.

**Lemma 2.6.** \( d_i \leq m_i(m_j - 1) + 1 \) for any \( j \) with \( ji \in EG \) and \( d_j \leq m_i \).
Moreover the above equality holds if and only if \( d_j = m_i \) and all neighbors of \( j \) excluding \( i \) have degree 1.

**Proof.** Pick \( j \) such that \( ji \in EG \) and \( d_j \leq m_i \). Then \( d_jm_j \geq d_i + (d_j - 1) \cdot 1 \).
Hence
\[
m_i(m_j - 1) + 1 \geq d_j(m_j - 1) + 1 \geq d_i.
\]
\( \square \)
Chapter 3

Pseudo $k$-regular graphs

We now turn to the study of pseudo $k$-regular graphs, i.e. $m_i = k$ for all $i$. We try to give some theories for pseudo $k$-regular graphs.

From the definition of pseudo $k$-regular graphs, $k \in \mathbb{Q}$, but indeed we have the following.

**Proposition 3.1.** If $G$ is pseudo $k$-regular then $k \in \mathbb{N}$.

**Proof.** Let $A$ be the adjacency matrix of $G$, and note that

$$(d_1, d_2, \ldots, d_n)A = k(d_1, d_2, \ldots, d_n).$$

Being a zero of the characteristic polynomial of $A$, $k$ is an algebraic integer. Since $k$ is also a positive rational number, $k$ is indeed a positive integer. □

Obviously, any $k$-regular graph is a pseudo $k$-regular graph. However, a pseudo $k$-regular graph may not be a regular graph. An interesting problem is to characterize all the non-regular pseudo $k$-regular graphs. There are some examples in [12] of pseudo $k$-regular graphs that are not regular in the following Example 3.2.
Example 3.2. The graphs in Figure 3.1, 3.2, and 3.3 are pseudo $k$-regular but not regular.

Figure 3.1: A graph with $m_i = 2$.

Figure 3.2: A graph with $m_i = 3$.

Figure 3.3: A graph with $m_i = 4$.

It is natural to ask when a pseudo $k$-regular graph attains the maximum number of edges when the order $n$ of a graph is given.
**Theorem 3.3.** A pseudo $k$-regular graph has at most $nk/2$ edges, and the maximum is obtained if and only if the graph is regular.

**Proof.** From

$$2k|EG| = \sum_{i \in V_G} d_i m_i = \sum_{i \in V_G} d_i^2 \geq (\sum_{i \in V_G} d_i)^2 / n = 4|EG|^2 / n,$$

we have $|EG| \leq nk/2$ and equality is obtained if and only if $d_i$ is a constant. \qed

We shall study the connected pseudo $k$-regular graphs of order $n$ which attain the minimum number of edges, i.e. pseudo $k$-regular trees. We also want to study connected pseudo $k$-regular graphs of order $n$ with maximal degree among such graphs.

**Definition 3.4.** Let $T_k$ be the tree of order $k^3 - k^2 + k + 1$ whose root has degree $k^2 - k + 1$ and each neighbor of the root has $k - 1$ children as leafs.

![Figure 3.4: The tree $T_2$.](image)

**Figure 3.4:** The tree $T_2$.

![Figure 3.5: The tree $T_3$.](image)

**Figure 3.5:** The tree $T_3$. 

Note that \( T_1 \) is exactly the complete graph \( K_2 \). For each \( k \geq 2 \), \( T_k \) exists and provides an example for a non-regular pseudo \( k \)-regular graph.

Let \( \Delta(G) = \max\{d_i \mid i \in VG\} \) be the maximal degree of \( G \). We have the following result.

**Theorem 3.5.** Let \( G \) be a connected graph with \( m_i \leq k \) for all \( i \in VG \) and some \( k \in \mathbb{N} \). Then \( \Delta(G) \leq k^2 - k + 1 \). Moreover the following (i)-(ii) are equivalent.

(i) \( \Delta(G) = k^2 - k + 1 \).

(ii) \( G \) is the tree \( T_k \).

**Proof.** Choose \( i \) such that \( d_i = \Delta(G) \). Then by Proposition 2.6, \( \Delta(G) = d_i \leq m_i(m_j - 1) + 1 = k^2 - k + 1 \) for any \( j \) with \( ji \in EG \) and \( d_j \leq m_i \). Moreover \( \Delta(G) = k^2 - k + 1 \) if and only if \( d_j = m_j = m_i = k \) and \( d_z = 1 \) for all neighbors \( z \neq i \) of \( j \). Hence (i) and (ii) are equivalent. \( \square \)

We have seen that the degree of a neighbor of maximum degree vertex is \( k \) in \( T_k \). We are interested in what other vertices have this property.

**Lemma 3.6.** Let \( G \) be a pseudo \( k \)-regular graph. Then the following (i)-(ii) hold.

(i) If \( z \) is a vertex of degree 1 then \( k \) is the degree of the neighbor of \( z \).

(ii) If \( ij \) is an edge with \( 2 \leq d_j < k \) then \( 2 \leq d_i \leq k^2 - 3k + 4 \), with the second equality if and only if all neighbors of \( j \) except \( i \) have degree 2.
Proof. (i) is clear. To prove (ii), note that $d_i \neq 1$, otherwise $d_j = k$, a contradiction. Indeed $d_z \neq 1$ for any neighbors $z$ of $j$. Hence

$$d_i + 2(d_j - 1) \leq d_j m_j = d_j k.$$ 

Hence

$$d_i \leq d_j (k - 2) + 2 \leq k^2 - 3k + 4.$$ 

\[\square\]

**Corollary 3.7.** Let $G$ be a pseudo $k$-regular graph of order $n$ with a vertex of degree $d_i \geq k^2 - 3k + 5$. Then

(i) Any neighbor $j$ of $i$ has degree $d_j = k$;

(ii) The order of $G$ is at least $f(k) := \lceil (5k^4 - 31k^3 + 94k^2 - 140k + 100)/k^2 \rceil$.

Proof. (i) From Lemma 3.6 (i) $d_j \neq 1$, and from Lemma 3.6 (ii) $d_j \geq k$. This is true for all neighbors $j$ of $i$. Hence $d_j = k$.

(ii) From Lemma 2.1 $\sum_{w \in VG} d_w^2 = \sum_{w \in VG} d_w m_w$,

$$d_i^2 + d_i k^2 + \sum_{w \not\in \{i\} \cup G_1(i)} d_w^2 = kd_i + k^2 d_i + \sum_{w \not\in \{i\} \cup G_1(i)} kd_w.$$ 

Hence

$$k^4 - 7k^3 + 22k^2 - 35k + 25 \leq \sum_{w \not\in \{i\} \cup G_1(i)} d_w (k - d_w) \leq \left(\frac{k}{2}\right)^2 (n - 1 - (k^2 - 3k + 5)).$$ 

\[\square\]
Note that for \( k = 3 \), \( k^2 - 3k + 5 = 5 \) and \( f(3) = 11 \).

Now we try to characterize the pseudo \( k \)-regular graphs. It is easily seen that a graph is pseudo \( k \)-regular if and only if each component of it is pseudo \( k \)-regular. Hence we just focus on the characterization of connected pseudo \( k \)-regular graphs.

The first two cases of pseudo \( k \)-regular graphs are easy to settle.

**Lemma 3.8.** If \( G \) is connected pseudo 1-regular then \( G \) is \( K_2 \).

**Lemma 3.9.** If \( G \) is connected pseudo 2-regular then \( G \) is a cycle or \( T_2 \).

**Proof.** Note that \( \Delta(G) = 2 \) or 3, and the first implies that \( G \) is a cycle and the latter implies that \( G = T_2 \).

Pseudo \( k \)-regular graphs is also called harmonic graphs [8], and finite harmonic tree are already given. But for the complete of this thesis we reprove the Theorem as follow.

**Theorem 3.10.** [8, Theorem 2.1] If \( G \) is a pseudo \( k \)-regular tree, then \( G = T_k \).

**Proof.** By Lemma 3.8 and Lemma 3.9, the assumption holds for each \( k \leq 2 \). Let \( G = (VG, EG) \) be a pseudo \( k \)-regular tree with \( k \geq 3 \). Pick any \( v \in VG \) with \( d_v \geq 2 \) as a root. Since a star is not pseudo \( k \)-regular, there exists a leaf \( x \) with parent \( y \neq v \), such that all children of \( y \) are leaves. Then \( y \) has degree \( k \) by Lemma 3.6 and has \( k - 1 \) children as leaves. Hence the degree of root \( d_v = km_y - (k - 1) = k^2 - k + 1 \). This concludes that \( G = T_k \) by Definition 3.4.
We shall study pseudo \( k \)-regular graph with the second largest degree \( k^2 - k \).

**Definition 3.11.** Let \( U_k \) be the tree of order \( k^3 - k^2 + 1 \) whose root has degree \( k^2 - k \) and each neighbor of the root has \( k - 1 \) children as leaves.

![Figure 3.6: The graph \( U_3 \) with type A vertices.](image)

We shall select some vertices from a graph and call them **type A** vertices. In general a type A vertex has degree 1 and its unique neighbor \( j \) has \( d_j = k \) and \( m_j = (k^2 - t)/k \), where \( t \) is the number of type A neighbors of \( j \) (in \( U_k \), \( t = 1 \)).

Let \( M_k \) be the graph obtained from \( U_k \) by identifying \( (k^2 - k)/2 \) pairs of type A vertices into \( (k^2 - k)/2 \) vertices. Then \( M_k \) gives a pseudo \( k \)-regular graphs with maximum degree \( k^2 - k \) for each \( k \geq 3 \).

![Figure 3.7: The graph \( M_3 \).](image)
Proposition 3.12. If $G$ is a pseudo $k$-regular graph with a vertex $x$ of degree $k^2 - k$, then the subgraph induced on $\{x\} \cup G_1(x) \cup G_2(x)$ is $U_k$ with possibly even number of vertices in type $A$ being identified in pairs. Moreover a type $A$ vertex not been identified with another one has degree 2 in $G$.

Proof. Let $y$ be a neighbor of $x$. Then $y$ has degree $d_y = k$ by Corollary 3.7(i), and has a neighbor $z \neq x$ of degree $d_z \geq 2$ by Theorem 3.5. Hence $k^2 = d_y m_y \geq d_x + d_z + (d_y - 2) \geq (k^2 - k) + 2 + (k - 2) = k^2$. This implies that $d_z = 2$ and the remaining vertices $w \not\in \{x, z\}$ of $y$ have degree $d_w = 1$. Note that $z, w$ have distance two to $x$. As one neighbor of $z$ has degree $k$, the other neighbor of $z$ also has degree $k$. Hence the vertex $z$ might adjacent to some neighbor of $x$ or to some vertex of degree $k$ and at distance 3 to $x$. \(\square\)

Let $\mathcal{E}_k$ be a family of graphs constructed as the following. Firstly pick a bipartite $(k-1)$-regular graph of order $2(2k-1)$ with bipartition $X \cup Y$, where $|X| = |Y| = 2k - 1$. Then add a new vertex connecting to all vertices of $X$. One can check that graphs in $\mathcal{E}_k$ are pseudo $k$-regular of order $4k-1$ with maximum degree $2k-1$.

\[\begin{array}{c}
\text{Figure 3.8: The graphs in $\mathcal{E}_k$.}
\end{array}\]
By a **switching** on $G$, we mean a process to obtain a new graph $G'$ by removing two edges $xy$ and $uv$ such that $d_x = d_u$ and $d_y = d_v$ and adding two new edges $xv$ and $yu$ to form a new graph, where $xv$ and $yu$ are not edges in $G$. In this case $G$ and $G'$ are called **switching equivalent**.

![Figure 3.9: Switching.](image)

![Figure 3.10: The graph $E_3 \in \mathcal{E}_3$.](image)

Every graph in $\mathcal{E}_3$ is switching equivalent to $E_3$.

From Corollary 3.7 (ii), we know a pseudo 3-regular graph with maximum degree at least 5 has at least $f(3) = 11$ vertices. All the graphs in $\mathcal{E}_k$ are extremal for this property.

Let $\mathcal{F}_k$ be a family of graphs constructed as the following. Firstly pick any $(k-2)$-regular graph $H$ of order $(2k-1)(k-1)$, not necessary connected.
Secondly add \((2k - 1)(k - 1)\) new vertices of degree 1 by connecting them to vertices of \(H\) one by one. Finally partition the vertex set of \(H\) into \(k - 1\) blocks of equal size \(2k - 1\) and connect all vertices in a block to a new vertex to make it degree \(2k - 1\). One can check that graphs in \(\mathcal{F}_k\) are pseudo \(k\)-regular with maximum degree \(2k - 1\).

![Figure 3.11: The graphs in \(\mathcal{F}_3\).](image)

![Figure 3.12: The graph \(F_3 \in \mathcal{F}_3\).](image)

Every graph in \(\mathcal{F}_3\) is switching equivalent to \(F_3\).

Now we restrict our attention to pseudo 3-regular graph \(G\).
Note that the maximum degree $3 \leq \Delta(G) \leq k^2 - k + 1 = 7$ and the case $\Delta(G) = 7$ is solved by Theorem 3.5 and Theorem 3.10.

The local structure of a maximum degree $\Delta(G) = 6$ is obtained in Proposition 3.12 for $k = 3$.

The following lemma is immediate from Corollary 3.7.

**Lemma 3.13.** Let $G$ be a pseudo $3$-regular graph with a vertex $i$ of degree $d_i = 5$. Then all neighbors $j$ of $i$ have degree $d_j = 3$, and the neighbors of $j$ have degree sequence $(5, 2, 2)$ or $(5, 3, 1)$. □

**Proposition 3.14.** If $G$ is a pseudo $3$-regular graph with a vertex $i$ of degree $5$, then the subgraph induced on $G_1(i)$ is union of disjoint edges or isolated vertices, and each endpoint of an edge is adjacent to a vertex of degree $1$ in $G_2(i)$ and each isolated vertex is adjacent to two vertices in $G_2(i)$ with degrees $(3, 1)$ or $(2, 2)$. □

![Figure 3.13: Graphs with $\Delta(G) = 5$.](image)

Now we study the local structure of a vertex of degree $4$ in a pseudo $k$-regular graph.

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Lemma 3.15. Let $G$ be a pseudo 3-regular graph. Then the neighbor degree sequence of a vertex of degree 4 is $(3, 3, 3, 3)$, $(4, 3, 3, 2)$, or $(4, 4, 2, 2)$.

Proof. Let $(a, b, c, d)$ be a degree sequence of the neighbors of a vertex $i$ of degree $d_i = 4$, where $a \geq b \geq c \geq d$. Note that $a \leq 4$ otherwise $d_i = 3$ by Corollary 3.7 (i). Then $a + b + c + d = d_i \cdot 3 = 12$. By checking all possible such sequences $(a, b, c, d)$, we find these are as listed in the lemma or $(4, 4, 3, 1)$, which is impossible since the neighbor of a leaf must have degree 3. \qed

Proposition 3.16. If $G$ is a pseudo 3-regular graph with a vertex $i$ of degree 4 and the neighbor degree sequence of $i$ is $(3, 3, 3, 3)$, then the subgraph induced on $G_1(i)$ is union of disjoint edges or isolated vertices, and each endpoint of an edge is adjacent to a vertex of degree 2 in $G_2(i)$ (possibly identified in pairs) and each isolated vertex is adjacent to two vertices in $G_2(i)$ with degrees 2, 3 or degrees 1, 4. \qed

Figure 3.14: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree 4 is $(3, 3, 3, 3)$.

In Figure 3.14 we have $1 + |G_1(i)| + |G_2(i)| \geq 7$. 21
Figure 3.15: The graph has $\Delta(G) = 4$ with degree sequence $(3, 3, 3, 3)$.

Proposition 3.17. If $G$ is a pseudo $3$-regular graph with a vertex $i$ of degree $4$ and the neighbor degree sequence of $i$ is $(4, 3, 3, 2)$, then the neighbor of $i$ with degree $2$ in $G$ is isolated in $G_1(i)$, and the neighbor of $i$ with degree $3$ in $G$ has at most one neighbor in $G_1(i)$.

Figure 3.16: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree $4$ is $(4, 3, 3, 2)$.

In Figure 3.16 we have $1 + |G_1(i)| + |G_2(i)| \geq 8$.

Figure 3.17: The graph has $\Delta(G) = 4$ with degree sequence $(4, 3, 3, 2)$.
Proposition 3.18. If $G$ is a pseudo 3-regular graph with a vertex $i$ of degree 4 and the neighbor degree sequence of $i$ is $(4, 4, 2, 2)$, then the neighbor of $i$ with degree 2 in $G$ is not connected to a neighbor of $i$ with degree 4 in $G$. □

Figure 3.18: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree 4 is $(4, 4, 2, 2)$.

In Figure 3.18 we have $1 + |G_1(i)| + |G_2(i)| \geq 9$.

Figure 3.19: The graph has $\Delta(G) = 4$ with degree sequence $(4, 4, 2, 2)$.

We will list all pseudo 3-regular graphs which are not regular of order within 10. From Corollary 3.7(ii), such graphs have maximum degree 4.
Lemma 3.19. Let $G$ be a connected pseudo $3$-regular graph with $\Delta(G) = 4$ and $a_j := |\{i \mid d_i = j\}|$ for $j = 1, 2, 3, 4$. Then

(i) $a_1 + a_2 = 2a_4$,

(ii) $|VG| = a_3 + 3a_4$,

(iii) $a_1 \leq a_3$,

(iv) $a_1, a_2, a_3$ have same parity.

Proof. (i) and (ii) follow from solving

$$0 = \sum_{i \in VG} (m_i - d_i)d_i = \sum_{i \in VG} (3 - d_i)d_i = a_1 \cdot 2 + a_2 \cdot 2 + a_4 (-4).$$

(iii) follows since there exists an injection from the set of degree one vertices into set of degree 3 vertices. Since there are even number of vertices of odd degrees, $a_1 + a_3$ is even. The remaining follows from (i) and (ii). This proves (iv). \qed

From the above lemma, the following is the possible sequence of $(n, a_4, a_3, a_2, a_1)$ for a connected pseudo $3$-regular graph of order $n$ with $\Delta(G) = 4$ and $7 \leq n \leq 10$.

$$(n, a_4, a_3, a_2, a_1)$$

$$=(10, 3, 1, 5, 1), (10, 2, 4, 4, 0), (10, 2, 4, 2, 2), (10, 2, 4, 0, 4), (10, 1, 7, 1, 1)$$

$$=(9, 3, 0, 6, 0), (9, 2, 3, 3, 1), (9, 2, 3, 1, 3), (9, 1, 6, 2, 0), (9, 1, 6, 0, 2)$$

$$=(8, 2, 2, 4, 0), (8, 2, 2, 2, 2), (8, 1, 5, 1, 1)$$

$$=(7, 2, 1, 3, 1), (7, 1, 4, 2, 0), (7, 1, 4, 0, 2).$$

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One can check directly that there is no graph whose corresponding sequence \((n, a_4, a_3, a_2, a_1)\) is \((10, 3, 1, 5, 1), (10, 2, 4, 2, 2), (10, 1, 7, 1, 1), (9, 2, 3, 1, 3), (9, 1, 6, 0, 2), (8, 2, 2, 4, 0), (8, 1, 5, 1, 1), (7, 2, 1, 3, 1), \) or \((7, 1, 4, 0, 2)\).

Small pseudo 3-regular but not 3-regular graphs are listed as follows.

\[ |VG| = 7: \]

\[
\begin{array}{c}
\text{Figure 3.20: Graphs with sequence } (n, a_4, a_3, a_2, a_1) = (7, 1, 4, 2, 0). \\
\end{array}
\]

\[ |VG| = 8: \]

\[
\begin{array}{c}
\text{Figure 3.21: The graph with sequence } (n, a_4, a_3, a_2, a_1) = (8, 2, 2, 2, 2). \\
\end{array}
\]
$|VG| = 9$:

Figure 3.22: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (9, 3, 0, 6, 0)$.

Figure 3.23: The graph with sequence $(n, a_4, a_3, a_2, a_1) = (9, 2, 3, 3, 1)$.

Figure 3.24: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (9, 1, 6, 2, 0)$.
$|VG| = 10$:

(Switching equivalent)

**Figure 3.25:** Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (10, 2, 4, 4, 0)$.

**Figure 3.26:** The graph with sequence $(n, a_4, a_3, a_2, a_1) = (10, 2, 4, 0, 4)$.

Under what kind of partial information of the pairs $(d_i, m_i)$, one can conclude the diameter of $G$ is at most 6.

In our study of pseudo $k$-regular graph with a vertex of the maximum degree $k^2 - k + 1$, the obtained graph $T_k$ has diameter 4.

The vertices with large degrees should also play an important role in other graphs.
Bibliography


