THE SPATIAL ENTROPY OF TWO-DIMENSIONAL SUBSHIFTS OF FINITE TYPE*

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In this paper, two recursive formulas for computing the spatial entropy of two-dimensional subshifts of finite type are given. The exact entropy of a nontrivial example arising in cellular neural networks is obtained by using such formulas. We also establish some general theory concerning the spatial entropy of two-dimensional subshifts of finite type. In particular, we show that if either of the transition matrices is rank-one, then the associated exact entropy can be explicitly obtained. The generalization of our results to higher dimension can be similarly obtained. Furthermore, these formulas can be used numerically for estimating the spatial entropy.

1. Introduction

The dynamical properties of one-dimensional subshifts with finite type are well understood. However, not much is known for a general theory of higher dimensional subshifts. For instance, the spatial entropy of subshifts of finite type is known to be the logarithm of the largest eigenvalue of its corresponding transition matrix. On the other hand, very little is known about spatial entropy of higher dimensional subshifts. In this paper, two recursive formulas for computing the spatial entropy of two-dimensional subshifts of finite type are given. The exact entropy of a nontrivial example arising in cellular neural networks is obtained by using such formulas. We also establish some general theory concerning the spatial entropy of two-dimensional subshifts of finite type. In particular, we show that if either of the transition matrices is rank-one, then the associated exact entropy can be explicitly obtained. The generalization of our results to higher dimension can be similarly obtained. Furthermore, these formulas can be used numerically for estimating the spatial entropy. We conclude this section by introducing some notations, definitions and well-known results.

Let $S = \{1, 2, \ldots, n\}$. Denote by $\mathbb{Z}^d$ the integer lattice on $\mathbb{R}^d$. Here $d \geq 1$ is a positive integer representing the lattice dimension. The set of all functions $u: \mathbb{Z}^d \to S$ is denoted by $S^{\mathbb{Z}^d}$. For $\alpha \in \mathbb{Z}^d$, we write $u(\alpha)$ as $u_\alpha$. The shift operator $\sigma_k$ is defined by
\[ (\sigma_k u)_\alpha = u_{\alpha + e_k} \quad \text{for } \alpha \in \mathbb{Z}^d, \tag{1} \]
where $e_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ is the usual...
unit vector in the direction of the \( k \)th coordinate. A set \( \Sigma \subset S^{2d} \) is called translation invariant if \( \sigma_k(\Sigma) = \Sigma \) for each \( 1 \leq k \leq d \), where \( \sigma_k \) is a shift operator given in (1). For \( d = 1 \), \( S^2 \coloneqq \Sigma_1 \) is the space of all (two-sided) sequences with symbols in the set \( S \). The shift operator \( \sigma_1 =: \sigma \) is defined by \[
(\sigma u)_\alpha = u_{\alpha+1}, \quad \text{for} \; \alpha \in \mathbb{Z},
\] where \( u \in \Sigma_1 \). In general, a subset \( \Sigma' \subset \Sigma_1 \) is called a subshift provided that it is closed and invariant by the shift operator \( \sigma \). The restriction of the shifts on a subshift is called a symbolic dynamical system. Let \( A = (a_{i,j})_{n \times n} \) be an \( n \times n \) matrix whose entries \( a_{i,j} \) are either zeros or ones. Such a matrix is called a transition matrix. Let \[
\Sigma(A) = \{ u \in \Sigma_1 : a_{u_\alpha, u_{\alpha+1}} = 1 \; \text{for} \; \alpha \in \mathbb{Z} \}.
\] In other words, the transition matrix \( A \) determines all admissible transitions between the symbols, 1, 2, \ldots, \( n \). The set \( \Sigma(A) \) is obviously translation invariant and closed with a suitable defined metric.

**Definition 1.1.** The restriction \( \sigma|_{\Sigma(A)} =: \sigma_A \) is called the subshift of finite type determined by the matrix \( A \). Sometimes \( \sigma_A \) is also called the Markov shift of finite type.

For \( d > 1 \), a subshift of finite type can be defined accordingly. Specifically, let \( A_k = (a_{i,j}^k)_{n \times n} \), \( 1 \leq k \leq d \), be transition matrices of size \( n \times n \). Let \[
\Sigma(A_1, A_2, \ldots, A_d) = \{ u \in S^{2d} : a_{u_\alpha, u_{\alpha+1+k}} = 1, \quad \text{for all} \; \alpha \in \mathbb{Z}, 1 \leq k \leq d \}.
\]

We shall write \( \Sigma(A_1, A_2, \ldots, A_d) \) as \( \Sigma_d \) provided no confusion arises. Note that \( \Sigma_d \) consists of “patterns” on \( \mathbb{Z}^d \) with their admissible transitions, in the direction of \( k \)th coordinate, \( 1 \leq k \leq d \), being determined by the transition matrix \( A_k \). It is clear that \( \Sigma_d \) is closed and translation invariant.

**Definition 1.2.** The restriction of the shift operators \( \sigma_k, 1 \leq k \leq d \), on \( \Sigma_d \) is called the subshifts of the finite type.

To measure the complexity of a translation invariant set \( \Sigma_d \subset S^{2d} \), we compute the growth rate of the number of patterns on a parallelepiped of the size \( N_1 \times N_2 \times N_3 \times \cdots \times N_d \) in the lattice as \( N_1, N_2, \ldots, N_d \) go to infinity. Now we recall the following definition (see e.g. [Chow et al., 1996; Robinson, 1995]).

**Definition 1.3.** The spatial entropy \( h(\Sigma_d) \) is defined by
\[
h(\Sigma_d) = \lim_{N_1, N_2, \ldots, N_d \to \infty} \frac{ \log \Gamma_N(\Sigma_d) }{ N_1 N_2 \cdots N_d }.
\]
Here \( \Gamma_N(\Sigma_d) \) is the number of distinct patterns that one observes among the elements of \( \Sigma_d \) by restricting one’s observation to a parallelepiped of size \( N_1 \times N_2 \times \cdots \times N_d \) in the lattice. Note that (2) is well-defined and exists (see e.g. [Chow et al., 1996]).

On the other hand, topological entropy is a quantitative measurement of how chaotic the map is. In fact, it is determined by how many “different orbits” there are for a given map (or flow). In the following, we recall the definition of topological entropy for a map (see e.g. [Robinson, 1995]). Let \( f : X \to X \) be a continuous map on the space \( X \) with metric \( d \). Set the distance \[
d_{n,f}(x, y) = \sup_{0 \leq j < n} d(f^j(x), f^j(y)).
\]

A set \( S \subset X \) is \( (n, \epsilon) \)-separated for \( f \) provided \( d_{n,f}(x, y) > \epsilon \) for every pair of distinct points \( x, y \in S \), \( x \neq y \). The number of different orbits of length \( n \) (as measured by \( \epsilon \)) is defined by \[
r(n, \epsilon, f) = \max \{ z(S) : S \subset X \text{ is a} \quad (n, \epsilon)\text{-separated set for } f \},
\]
where \( z(S) \) is the number of elements in \( S \). We want to measure the growth rate of \( r(n, \epsilon, f) \) as \( n \) increases, so we define
\[
h(\epsilon, f) = \lim_{n \to \infty} \frac{ \log(r(n, \epsilon, f)) }{ n }.
\]

**Definition 1.4.** The topological entropy of \( f \) is defined to be
\[
h(f) = \lim_{\epsilon \to 0^+} h(\epsilon, f).
\]

**Theorem 1.5** (see e.g. Theorem 8.1.9 of [Robinson, 1995]). For \( d = 1 \), let \( A \) be a transition matrix.
Let $\sigma : \Sigma(A) \to \Sigma(A)$ be the associated subshift of finite type. Then $h(\sigma) = h(\Sigma_1) = \log(\lambda_1)$, where $\lambda_1$ is the largest eigenvalue of $A$.

The generalization of Theorem 1.5 for $d \geq 2$ is still open. Our effort here is to seek the relationship (if any) between $h(\Sigma_2)$ and the transition matrices $A_1$ and $A_2$.

2. Two Recursive Formulas

In this section, we shall derive two recursive formulas for computing the spatial entropy of $\Sigma_2$. In the following, we first introduce some notations and concepts.

Finite strings of symbols are called words. For instance, $(1 \ 2 \ 4 \ 2)$ is a word of length 4, and $(\alpha \ \beta \ \gamma)$ is a word of length 3. The words can also be arranged in a vertical fashion. For instance, $(1 \ 2 \ 3)$ is a word of length 3. Given a transition matrix $A = (a_{i,j})_{n \times n}$, a word $\omega = (\omega_0 \omega_1 \cdots \omega_k-1)$ of length $k$ is called admissible if $a_{\omega_j-1, \omega_j} = 1$ for $j = 1, 2, \ldots, k-1$. Let $A$ be a transition matrix. The set of admissible words of length $m$ whose first symbol is $\omega_0$ is to be denoted by $\omega(\omega_0, m; A)$. Set

$$\omega(m; A) = \text{set of all admissible words of length } m \quad \text{such that } \omega_0 \leq n.$$  

To save notation, the transition matrices $A_1$ and $A_2$ introduced in Sec. 1 will be denoted by $H = (h_{i,j})_{n \times n}$ and $V = (v_{i,j})_{n \times n}$, respectively, called horizontal and vertical transition matrices. Then $\text{Card}(\omega(m; H)) = \sum_{i,j=1}^n (H^{m-1})_{i,j} =: N_m$. Here $H^0$ = identity matrix. Using these $N_m$ symbols, we may define a transition matrix $T^{(m)}_{H,V} = (t^{(m)}_{i,j})$ of size $N_m \times N_m$ as follows. We begin by giving a lexicographic order for elements in $\omega(m; H)$. Specifically, let $s = (s_1 s_2 \cdots s_m)$ and $p = (p_1 p_2 \cdots p_m) \in \omega(m; H)$, and suppose that $j$ is the smallest index for which $s_j \neq p_j$, then we define

$$s < p \quad \text{if } s_j < p_j. \quad (4)$$

With such ordering, the sets $\omega(m; H)$ and $\{1, 2, 3, \ldots, N_m\}$ can have an association that is one to one, onto and order preserving. Now, if $s$ and $p$ in $\omega(m; H)$ are associated with positive integers $k$ and $l$, where $1 \leq k, l \leq N_m$ respectively, then we define the $(k, l)$-entry or $(s, p)$-entry of $T^{(m)}_{H,V}$ as

$$t^{(m)}_{s,p} = t^{(m)}_{k,l} = v_{s_1,p_1} \cdot v_{s_2,p_2} \cdot v_{s_m,p_m} := \prod_{i=1}^m v_{s_i,p_i}. \quad (5)$$

i.e. $t^{(m)}_{k,l} = 1$ provided that for all $1 \leq i \leq m$, the words $(s_i)_{p_i}$ are admissible with respect to $V$. Otherwise, $t^{(m)}_{k,l} = 0$. For convenience, we shall use $t^{(m)}_{s,p}$ to denote $t^{(m)}_{s,p}$. We shall call $T^{(m)}_{H,V}$ the $m$-transition matrices with respect to the horizontal and vertical transition matrices $H$ and $V$, or for short, the $m$-transition matrix. If we start out with a lexicographic order for elements in $\omega(m; V)$, we shall obtain the so-called $m$-transition matrix $T^{(m)}_{V,H}$ with respect to $V$ and $H$. The relationship between $m$-transition matrix $T^{(m)}_{H,V}$ and $h(\Sigma(H, V))$ is given in the following.

**Proposition 2.1.** Let $T^{(m)}_{H,V}$ be the $m$-transition matrix with respect to $H$ and $V$. Let $\rho(T^{(m)}_{H,V})$ be the maximal eigenvalue of $T^{(m)}_{H,V} = (t^{(m)}_{s,p})$, where $t^{(m)}_{s,p}$ are given in (5). Then

$$h(\Sigma_2) = \lim_{m \to \infty} \frac{\log \rho(T^{(m)}_{H,V})}{m}, \quad (6)$$

where $\Sigma_2 = \Sigma(H, V)$.

**Proof.** For each $m$, it follows from Theorem 1.5 that

$$\lim_{k \to \infty} \frac{\log \Gamma(k,m)(\Sigma_2)}{k} = \log \rho(T^{(m)}_{H,V}).$$

This is to say that for fixed $m$, the admissible words of symbols are determined by $T^{(m)}_{H,V}$. Upon using the fact that the double limit in (2) is well-defined and exists, we conclude that the iterated limit

$$\lim_{m \to \infty} \frac{1}{m} \lim_{k \to \infty} \frac{\log \Gamma(k,m)(\Sigma_2)}{k}$$
numbers are k! in the lexicographic order defined in (4), there exist s and 1
Card(T) where T

Let us next derive a recursive formula for constructing T(m) i

where T(m) T is a matrix of size Card(ω(i, m; H)) × Card(ω(j, m; H)). Let 1 ≤ k ≤ Card(ω(i, m; H)) and 1 ≤ l ≤ Card(ω(j, m; H)). Via the lexicographic order defined in (4), there exist s ∈ ω(i, m; H) and p ∈ ω(j, m; H) whose associated numbers are k and l, respectively. Then the (k, l)-entry, or simply (s, p)-entry, of the matrix T(m) is 1 provided that for all 1 ≤ r ≤ m, \begin{array}{c}
         \frac{\rho(T(m))}{r} \end{array}
is an admissible word of size 2 with respect to vertical transition matrix V. Otherwise, the entry is zero. We are now ready to state the following result.

Theorem 2.2. Let T(m+1) H,V and T(m) H, V be, respectively, (m + 1)- and m-transition matrices with respect to horizontal and vertical transition matrices H = (h(i, j)) and V = (v(i, j)). Let α(i) = {q ∈ N : 1 ≤ q ≤ n, hi,q = 1} and Card(α(i)) = αi. Moreover, we set α(i) = {i1, i2, . . . , iα} in the following order i1 ≤ i2 ≤ · · · ≤ iα. Then T(m) H, V can be defined recursively as follows:

\begin{equation}
T(1) H, V = V, \tag{8a}
\end{equation}

and

\begin{equation}
T(m+1) H, V = (T(m+1) T)_{k,l} \tag{8b}
\end{equation}

Here the block matrices T(m+1) k,l are of following form

\begin{equation}
\begin{pmatrix}
T(m+1)_{k1,1} & T(m+1)_{k1,2} & \cdots & T(m+1)_{k1,l1} \\
T(m+1)_{k2,1} & T(m+1)_{k2,2} & \cdots & T(m+1)_{k2,l1} \\
\vdots & \vdots & \ddots & \vdots \\
T(m+1)_{kn,1} & T(m+1)_{kn,2} & \cdots & T(m+1)_{kn,l1}
\end{pmatrix}
\end{equation}

where k_i = α(k), l_i = α(l), T^{m+1}_{k,l} and T^{m}_{k,l}, 1 ≤ v ≤ k and 1 ≤ q ≤ α_i, are defined as in (7).

Proof. The results follow from (5) and (7). We illustrate the case for T(2) m H,V. Let s = (s_1, s_2) and p = (p_1, p_2) be in ω(2; H), i.e. h_{s_1,s_2} = 1 and h_{p_1,p_2} = 1. We may assume that s_1 = k and p_1 = l for some 1 ≤ k, l ≤ n. Clearly, s_2 ∈ (α(k) and p_2 ∈ (α(l). Hence, s_2 = k and p_2 = l_q, for some 1 ≤ v ≤ α_k, and some 1 ≤ q ≤ α_l. Thus, the (s, p)-entry of T(2) k,l ,

\begin{equation}
((T^{(2)} k,l)_{s,p}) = (v_{s_1,p_1})(v_{s_2,p_2}) = v_{k,l}(v_{k,v})(v_{k,l}).
\end{equation}

Hence, T(2) m holds as claimed. We thus complete the proof of the theorem for m = 2. An inductive argument similar to that of m = 2 leads to the assertion of the theorem.

To give a better understanding of (8), we consider the following example for which the exact entropy of Σ(H, V) can be computed via the recursive formula (8).

Example 2.3. We consider the following horizontal and vertical transition matrices

\begin{equation}
H = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\end{equation}

\begin{equation}
V = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} =: \text{diag} \begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix}, A.
\end{equation}

Such pair of transition matrices describes some stable defect patterns (see [Juang & Lin, preprint]) generated by Cellular Neural Networks (see e.g. [Chua & Yang, 1988; Juang & Lin, 2000] and the work cited there in). To compute h(Σ_2),
we see, via (8b), that
\[
T_{H,V}^{(m+1)} = \begin{pmatrix}
T_{1,1}^{(m+1)} & 0 & 0 & 0 \\
T_{2,1}^{(m+1)} & T_{2,2}^{(m+1)} & 0 & 0 \\
0 & 0 & T_{3,4}^{(m+1)} & 0 \\
0 & 0 & 0 & T_{4,4}^{(m+1)} \\
0 & 0 & T_{5,3}^{(m+1)} & 0 & 0
\end{pmatrix}
= \text{diag} \begin{pmatrix}
\frac{T_{1,1}^{(m+1)}}{T_{2,1}^{(m+1)}} & 0 & 0 & 0 \\
\frac{T_{2,2}^{(m+1)}}{T_{2,2}^{(m+1)}} & \Lambda^{(m)} \\
0 & 0 & \Lambda^{(m)} & 0 \\
T_{5,3}^{(m+1)} & 0 & 0 & \Lambda^{(m)}
\end{pmatrix}.
\] (10)

Let
\[
\Lambda^{(m)} = \begin{pmatrix}
T_{2,2}^{(m+1)} & 0 & 0 & 0 \\
0 & 0 & T_{3,4}^{(m+1)} & 0 \\
0 & 0 & T_{3,4}^{(m+1)} & T_{4,5}^{(m+1)} \\
0 & T_{5,3}^{(m+1)} & 0 & 0
\end{pmatrix}.
\] (11)

We get
\[
T_{H,V}^{(m+1)} = \text{diag} \left( \frac{T_{2,2}^{(m+1)}}{T_{2,2}^{(m+1)}}, \Lambda^{(m)} \right),
\] (12a)

and
\[
\Lambda^{(m+1)} = \begin{pmatrix}
\Lambda^{(m)} & O \\
O & A \otimes T_{1,1}^{(m)}
\end{pmatrix}.
\] (12b)

Here \( \otimes \) denotes the Kronecker product (see e.g. [Bellman, 1970]). Moreover,
\[
T_{H,V}^{(m+2)} = \text{diag} \left( \frac{\Lambda^{(m)}}{\Lambda^{(m)}}, \frac{O}{O}, \frac{A \otimes T_{1,1}^{(m)}}{A \otimes T_{2,2}^{(m)}} \right),
\]

\[
A \otimes T_{1,1}^{(m)}
\]

and
\[
T_{H,V}^{(m+3)} = \text{diag} \begin{pmatrix}
\Lambda^{(m)} & 0 & 0 & 0 \\
0 & A \otimes T_{1,1}^{(m)} & \Lambda^{(m)} & 0 \\
0 & 0 & A \otimes T_{1,1}^{(m)} & 0 \\
0 & 0 & 0 & A \otimes T_{2,2}^{(m)}
\end{pmatrix},
\] (13)

It follows from (11) and (12) that \( \rho(T_{2,2}^{(m)}) \leq \rho(\Lambda^{(m)}) \), and that
\[
\rho(T_{1,1}^{(m)}) = \rho(T_{2,2}^{(m-1)}) \leq \rho(\Lambda^{(m-1)})
\leq \rho(\Lambda^{(m)}).
\]

Here \( \rho(A) \) is the spectrum of \( A \). Consequently,
\[
\max\{\rho(A \otimes T_{1,1}^{(m)}), \rho(A \otimes T_{2,2}^{(m)})\}
\leq \rho(A \otimes \Lambda^{(m)}).
\]

We thus see that
\[
\rho(T_{H,V}^{(m+3)}) = \rho(A \otimes \Lambda^{(m)}) = \rho(A) \rho(\Lambda^{(m)})
\leq \rho(A) \rho(T_{H,V}^{(m)}).
\] (14)

We get inductively, via (14), that
\[
\rho(T_{H,V}^{(3m)}) = \rho(T_{H,V}^{(3)})^m \rho(T_{H,V}^{(3)}).
\]
Thus,
\[ h(\Sigma_2) = \lim_{m \to \infty} \frac{\log(\rho(T_{H,V}^{(2m)})))}{3m} = \frac{\log \rho(A)}{3}. \]
Here \( \rho(A) \) is the largest real root of \(-3^3 + \lambda^2 + 1 = 0\).

We next describe another form of the \( m \)-transition matrix and its corresponding recursive formula. Let \( \overline{w}(i, m) \) denote the set of words of length \( m \), whose first symbol is \( i \), and let
\[ \overline{w}(m) = \bigcup_{1 \leq i \leq n} \overline{w}(i, m). \]
Note that such words may not be admissible. Using these \( n^m \) symbols, we may define an \( m \)-transition matrix \( T_{H,V}^{(m)} \) with respect to \( H \) and \( V \). Specifically, given \( s = (s_1, s_2, \ldots, s_m) \) and \( u = (u_1, u_2, \ldots, u_m) \) in \( \overline{w}(m) \), the \((s, u)\)-entry of \( T_{H,V}^{(m)} \) is
\[ T_{s,u}^{(m)} \equiv \prod_{i=1}^{m-1} (h_{s_i,s_{i+1}}(h_{u_i,u_{i+1}}) \prod_{i=1}^{m} v_{s_i,u_i}. \quad (15a) \]
As in (7), \( T_{H,V}^{(m)} \) can be arranged as an \( n \times n \) block matrix
\[ T_{H,V}^{(m)} = (T_{k,l}^{(m)})_{n \times n}. \quad (15b) \]
We next show that \( T_{H,V}^{(m)} \) and \( T_{H,V}^{(m)} \) have the same maximal eigenvalue.

**Proposition 2.4.** For all \( m \in \mathbb{N} \), \( T_{H,V}^{(m)} \) and \( T_{H,V}^{(m)} \) have the same maximal eigenvalue. Here \( T_{H,V}^{(m)} \) and \( T_{H,V}^{(m)} \) are defined in (15) and (7), respectively.

**Proof.** Let \( k \) and \( l \) be positive integers for which \( 1 \leq k, l \leq n \), and let \( \alpha(j) = \{ j_1, j_2, \ldots, j_{\alpha(j)} \} \), \( j = k \) or \( l \). Here \( \alpha(j) \) is defined as in Theorem 2.2. Set \( S = \{ 1, 2, \ldots, n \} \). Then the \( i \)th columns and \( j \)th rows of \( T_{k,l}^{(2)} \), where \( i \in \mathbb{N} - \alpha(k) \) and \( j \in \mathbb{N} - \alpha(l) \) are zero. In particular, by deleting those zero columns and rows of \( T_{k,l}^{(2)} \), one would get \( T_{H,V}^{(2)} \). Consequently, \( T_{H,V}^{(m)} \) can be obtained by deleting appropriate zero columns and zero rows of \( T_{H,V}^{(2)} \). Hence, the maximal eigenvalues of \( T_{H,V}^{(2)} \) and \( T_{H,V}^{(m)} \) are the same. Similarly, the assertion of the proposition holds for all \( m \). \( \blacksquare \)

To derive recursive formulas for computing \( T_{H,V}^{(m)} \), we need the following notations.

Let \( A = (a_{ij})_{n \times n} \), set
\[
D_A := \text{diag}(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{n1}, \ldots, a_{nn}),
\]
\[ := \begin{pmatrix}
D_{A_1} & 0 & 0 & 0 \\
0 & D_{A_2} & 0 & 0 \\
0 & 0 & \vdots & 0 \\
0 & 0 & 0 & D_{A_n}
\end{pmatrix}
\]
and
\[ A^\otimes n = A \otimes A \otimes \cdots \otimes A. \]
Here \( \otimes \) denotes the Kronecker product. We are now ready to state the following recursive formula.

**Theorem 2.5.** Let \( T_{H,V}^{(m)} \) be transition matrices described in (15). Then
\[ T_{H,V}^{(1)} = V \]
\[ T_{H,V}^{(m+1)} = (D_H \otimes I^{[m-1]}) (V \otimes T^{(m)}) (D_H \otimes I^{[m-1]}) \]
\[ (16a) \]
\[ (D_H (V \otimes V)) D_{s,u} = h_{k,i} h_{l,j} v_{k,i} v_{l,j}. \quad (17) \]
To see this, we first observe that the \((k,l)\)-block of \( D_H (V \otimes V) D_H \) is \( (v_{k,i} D_{H_{k,l}} V D_{H_l}) \). Moreover, the \((i,j)\)-entry of \( v_{k,i} D_{H_{k,l}} V D_{H_l} \) is \( v_{k,i} h_{k,i} v_{l,j} h_{l,j} \). Hence, (17) holds as claimed. For any \( s, u \in \overline{w}(m) \), we suppose that the \((s, u)\)-entry of
\[ (D_H \otimes I^{[m-1]} (V \otimes T^{(m-1)})(D_H \otimes I^{[m-2]}) \]
is given as in (15a). Let \( s = (s_1, s_2, \ldots, s_{m+1}) \) and \( u = (u_1, u_2, \ldots, u_{m+1}) \) be in \( \overline{w}(m+1) \). Then the \((s_1, u_1)\)-block of the matrix \( (D_H \otimes I^{[m-1]} (V \otimes T^{(m)}))(D_H \otimes I^{[m-1]}) \) is
\[ (D_{H_{s_1}} \otimes I^{[m-1]} (v_{s_1,u_1} T^{(m)})(D_{H_{u_1}} \otimes I^{[m-1]}). \quad (18) \]
Set \( \tilde{s} = (s_2, s_3, \ldots, s_{m+1}) \) and \( \tilde{u} = (u_2, u_3, \ldots, u_{m+1}) \). Then the \((s, u)\)-entry of the matrix in (18) is \( (h_{s_1,s_2}(h_{u_1,u_2}) (v_{s_1,u_1}) T^{(m)}_{s,u} \) i.e.
\[ T_{s,u}^{(m+1)} = \prod_{i=1}^{m} \left((h_{s_i,s_{i+1}}(h_{u_i,u_{i+1}}) \prod_{i=1}^{m} v_{s_i,u_i} \right). \]
The last equality above is justified by the induction hypothesis. We thus complete the proof of the theorem.

The following result is thus a direct consequence of Propositions 2.1 and 2.4.

**Theorem 2.6.** Let \( p(T_{H,V}^{(m)}) \) be the maximal eigenvalues of \( T_{H,V}^{(m)} \). Then

\[
h(\Sigma_2) = \lim_{m \to \infty} \frac{\log p(T_{H,V}^{(m)})}{m}.
\]

**Remark 2.7.** The recursive formulas for \( m \) transition matrices \( T_{V,H}^{(m)} \) and the corresponding results can be similarly derived.

## 3. Two-Dimensional Entropy

Using the recursive formulas (8) and (15), we can derive the exact entropy of certain subshifts of finite type.

**Proposition 3.1.** Let \( H = (h_{i,j}) \) and \( V = (v_{i,j}) \) be, respectively, horizontal and vertical transition matrices. If \( h_{i,j} = 1 \) (resp. \( v_{i,j} = 1 \)) for all \( i, j \), then

\[
h(\Sigma_2) = \log \rho(V) \quad \text{(resp., } \log \rho(H)\).
\]

Here \( \rho(A) \) denotes the maximal eigenvalue of \( A \).

**Proof.** Using (15b), we see that

\[
T_{H,V}^{(m+1)} = V \otimes T_{H,V}^{(m)}.
\]

Consequently, \( \rho(T_{H,V}^{(m+1)}) = \rho(V) \rho(T_{H,V}^{(m)}) \), and hence,

\[
\rho(T_{H,V}^{(m)}) = (\rho(V))^m.
\]

The first assertion of the proposition now follows from Theorem 2.6. The second assertion can be similarly obtained.

**Proposition 3.2.** \( h(\Sigma_2) \leq \min \{ \log \rho(H), \log \rho(V) \} \).

Consequently, If \( \rho(V) = 1 \) or \( \rho(H) = 1 \), then \( h(\Sigma_2) = 0 \).

**Proof.** Let \( E = (e_{i,j}) \), and \( e_{i,j} = 1 \) for all \( i, j \). We have, upon using Proposition 3.1, that

\[
h(\Sigma_2) = h(\Sigma(H, V)) \leq h(\Sigma(H, E)) = \log \rho(H),
\]

and

\[
h(\Sigma_2) = h(\Sigma(H, V)) \leq h(\Sigma(E, H)) = \log \rho(V).
\]

We thus complete the proof of the proposition.

We next state the main result of the paper.

**Theorem 3.3.** Let \( H = ab^T \) be a horizontal transition matrix of rank-one. Here \( a = (a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n \) and \( b = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n \). Set \( D_a = \text{diag}(a_1, a_2, \ldots, a_n), \ D_b = \text{diag}(b_1, b_2, \ldots, b_n), \) and \( \rho(D_a D_b V D_a D_b) := \lambda_{a,b} \).

Then

\[
h(\Sigma_2) = \log \lambda_{a,b}.
\]

**Proof.** If \( H = ab^T \), then \( D_H = D_a \otimes D_b \). Using (16), we see that

\[
T_{H,V}^{(2)} = (D_a \otimes D_b)(V \otimes V)(D_a \otimes D_b)
\]

= \( (D_a V D_a) \otimes (D_b V D_b) \),

and

\[
T_{H,V}^{(3)} = (D_a \otimes D_b \otimes I)(V \otimes D_a V D_a \otimes D_b V D_b)
\]

\[
\times (D_a \otimes D_b \otimes I)
\]

= \( (D_a V D_a) \otimes (D_a D_b V D_a D_b) \otimes (D_b V D_b) \).

An inductive approach yields that

\[
T_{H,V}^{(m)} = (D_a V D_a) \otimes (D_b V D_a D_a D_b) \otimes (D_b V D_b) \quad \text{(19)}
\]

Using (19) and Theorem 2.6, we get that

\[
h(\Sigma_2) = \log \lambda_{a,b}
\]

as claimed.

**Remark 3.4.** All the results obtained here can be easily generalized to an \( n \)-dimensional problem. Here \( n \geq 3 \).

**References**

