NUMERICAL COMPUTATION OF THE MINIMAL $H_\infty$ NORM OF THE DISCRETE-TIME OUTPUT FEEDBACK CONTROL PROBLEM*

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Abstract. Numerical computation of the minimal $H_\infty$ norm for the discrete-time output feedback control problem is considered. First of all, a lower bound is established in terms of the $H_\infty$ norm of certain stable transfer functions. Since the computational work in the evaluation of the $H_\infty$ norm of a stable transfer function involves the determination of unimodular eigenvalues of the associated parameterized symplectic pencil of matrices, we discuss in detail how to get a numerically reliable solution when the pencil becomes singular as the parameter varies. Next, by exploiting the stable deflating subspaces of the two parameterized symplectic pencils derived by Iglesias and Glover in 1991, we characterize the critical points such that the corresponding two discrete-time Riccati equations (with parameter $r$) have stabilizing positive semidefinite solutions and satisfy certain inertia conditions. This characterization makes some kind of secant method applicable for finding these critical points. Finally, using the maximum of the two critical points as the starting point, we then devise an algorithm for computing the optimal (minimal) $H_\infty$ norm by considering a secant method applied to the spectral radius (function of $r$) of the product of the corresponding two Riccati solutions. Numerical aspects are addressed throughout. In addition, some algebraic verifiable examples are given.

Key words. discrete-time, $H_\infty$ control, output feedback, optimal $H_\infty$ norm, symplectic, deflating subspace, singular pencil

AMS subject classifications. 93C05, 93C35, 93C45, 93C55, 93B36, 93B40, 93B52

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1. Introduction. Consider the output feedback control system given in Figure 1, where $G$ is the plant to be controlled and $K$ is the output feedback controller. The signal $w$ contains all external inputs; the output $z$ is an error signal; $y$ is the measured variable; and $u$ is the control input. We denote the resulting closed-loop transfer function from $w$ to $z$ by $T_{zw}$.

![Block Diagram](Fig. 1. The block diagram.)

In discrete-time $H_\infty$ control theory, the performance measure of a stable linear time-invariant controlled system is the $H_\infty$ norm of its closed-loop transfer function $T_{zw}$:

$$\|T_{zw}\|_\infty := \max_{\theta \in [0, 2\pi]} \|T_{zw}(e^{i\theta})\|_2,$$

where $i$ is the principal square root of $-1$ and $\|\cdot\|_2$ denotes the matrix 2-norm. In 1991, Iglesias and Glover [14] used a state-space approach to solve the finite dimensional

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linear time-invariant discrete-time systems. For a given number \( r > 0 \), they gave a characterization of all controllers such that \( \| T_{zw} \|_\infty < r \). The characterization uses the solutions of two discrete-time Riccati equations arising from the stable deflating subspaces of two symplectic pencils. In this paper, we present a numerically stable method for finding the optimum (minimum) of such \( r \) by exploiting the stable deflating subspaces.

To capture the essence of the deflating subspace method, in the following we only consider the simpler case that the realization of \( G \) is of the form

\[
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix},
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B_1 \in \mathbb{R}^{n \times m_1} \), \( B_2 \in \mathbb{R}^{n \times m_2} \), \( C_1 \in \mathbb{R}^{p_1 \times n} \), \( C_2 \in \mathbb{R}^{p_2 \times n} \), \( D_{12} \in \mathbb{R}^{p_1 \times m_2} \), \( D_{21} \in \mathbb{R}^{p_2 \times m_1} \). We assume further that the following conditions are satisfied:

(A1) \((A, B_1)\) is stabilizable (i.e., for all \( \lambda \) and \( x \neq 0 \) such that \( x^* A = x^* \lambda \) and \(|\lambda| \geq 1, x^* B_1 \neq 0 \)) and \((C_1, A)\) is detectable (i.e., \((A^T, C_1^T)\) is stabilizable);
(A2) \((A, B_2)\) is stabilizable and \((C_2, A)\) is detectable;
(A3) \(D_{12}^T C_1 D_{12} = [0 \ I]\);
(A4) \(D_{21} B_1^T D_{21}^T = [0 \ I]\).

The general case presented in [14] can also be similarly treated without any difficulty except for some tedious algebraic manipulations.

Define

\[
S_\infty(r) = (S_{\infty_1}(r), S_{\infty_2}(r)) := \begin{pmatrix}
A & 0 \\
-C_1^T C_1 & I
\end{pmatrix}, \begin{pmatrix}
I & B_2 B_1^T - \frac{1}{2} B_1 B_1^T \\
0 & A^T
\end{pmatrix},
\]

\[
T_\infty(r) = (T_{\infty_1}(r), T_{\infty_2}(r)) := \begin{pmatrix}
A^T & 0 \\
-B_1 B_1^T & I
\end{pmatrix}, \begin{pmatrix}
I & C_1^T C_2 - \frac{1}{2} C_1^T C_1 \\
0 & A
\end{pmatrix}.
\]

Under assumptions (A1)–(A4), [14] then shows that there exists an admissible controller \( K \) that acts on the plant \( G \) such that the closed-loop transfer function from \( w \) to \( z \) satisfies \( \| T_{zw} \|_\infty < r \) if and only if the following three conditions hold:

(C1) \( S_\infty(r) \in \text{dom}(\text{Ric}) \), \( X(r) := \text{Ric}(S_\infty(r)) \geq 0 \) and 
\[
-2 I_{m_1} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} X(r) [B_1 \ B_2] \text{ is congruent to } -I_{m_1} \begin{bmatrix} 0 & I_{m_2} \end{bmatrix};
\]
(C2) \( T_\infty(r) \in \text{dom}(\text{Ric}) \), \( Y(r) := \text{Ric}(T_\infty(r)) \geq 0 \) and 
\[
-2 I_{p_1} \begin{bmatrix} C_1 & C_2 \end{bmatrix} Y(r) [C_1^T \ C_2^T] \text{ is congruent to } -I_{p_1} \begin{bmatrix} 0 & I_{p_2} \end{bmatrix};
\]
(C3) \( \rho(X(r)Y(r)) < r^2 \).

In this paper, we consider the problem of finding the infimum of such \( r \), under assumptions (A1)–(A4). We divide the problem into three subproblems. First, we determine \( r_x \), the infimum of \( r \) such that (C1) holds. To do this, we first establish a lower bound of \( r_x \) in terms of the \( H_\infty \) norm of a certain stable transfer function. This helps the evaluation of the lower bound become easy and fast and precludes the troublesome fact shown in section 2 that the \( r \)-intervals in which \( S_\infty(r) \in \text{dom}(\text{Ric}) \) are disconnected. After this, we characterize \( r_x \) in terms of the upper half of the matrix stacked up by the basis vectors of the stable deflating subspace of \( S_\infty(r) \) (we use the notation \( S_\infty(r) = (S_{\infty_1}(r), S_{\infty_2}(r)) \) to denote the pencil \( \lambda S_{\infty_2}(r) - S_{\infty_1}(r) \) throughout when there is no confusion; a similar situation applies to \( T_\infty(r) \)). Then we use this
to reformulate (C1) and apply the secant method to obtain \( r_x \). Next, by applying the same technique, we can find \( r_y \), the infimum of \( r \) such that (C2) holds. Finally, with \( \max\{r_x, r_y\} \) as the starting point, we use the monotonicity of \( \rho(X(r)Y(r)) \) to devise a secant method to find \( r^* \), the infimum of \( r \) such that conditions (C1)–(C3) hold simultaneously.

Since we establish a lower bound of \( r_x \) in terms of the \( H_\infty \) norm of a stable transfer function and the globally quadratically convergent algorithm mentioned in [19] for computing the \( H_\infty \) norm involves the determination of unimodular eigenvalues of the associated parametrized symplectic pencil of matrices, we discuss in detail how to get a numerically reliable solution when the symplectic pencil under consideration becomes a singular pencil as the parameter varies. Unlike [19], our concern is not only for the \( H_\infty \) norm but also for the reliability of the controller obtained numerically. The controllers given in [14] are expressed in terms of the Riccati solutions \( X(r) \) and \( Y(r) \) in (C1)–(C2). When one of \( S_\infty(r) \) and \( T_\infty(r) \) is a singular (even merely nearly singular) pencil, it is difficult to compute the corresponding Riccati solutions correctly. In this case, a little loss in the accuracy of \( r \) is needed in order to guarantee a reliable controller numerically available. Thus, in section 2, we slightly modify the procedure for computing the \( H_\infty \) norm to fit our main problem. The computed \( H_\infty \) norm will be larger than the true one when the singular case does occur. The proposed algorithms are not mainly to compute the \( H_\infty \) norm itself but to generate a feasible preliminary lower bound of our main problem.

The following notation and definitions are used throughout the paper:

- \( A^T, A^H, A^{-1} \): the transpose, the conjugate transpose, and the inverse of \( A \), respectively;
- \( \text{Im}(A) \): the column space of \( A \);
- \( \Lambda(S_1, S_2), \Lambda_-(S_1, S_2) \): the set of eigenvalues of the pencil \( \lambda S_2 - S_1 \) and the set of eigenvalues of the pencil \( \lambda S_2 - S_1 \) that lie in the open unit disk, respectively;
- \( \rho(A) \): the spectral radius of \( A \);
- \( \sigma_{\min}(A), \sigma_{\max}(A) \): the minimum and maximum singular values of \( A \), respectively;
- \( 0, I \): the zero matrix and the identity matrix with suitable dimensions, respectively;
- \( I_p \): the \( p \) by \( p \) identity matrix;
- \( \mathbb{R}^{r \times s}, \mathbb{C}^{r \times s} \): the set of all real and complex \( r \times s \) matrices, respectively;
- \( \partial D, \mathring{D} \): the unit circle and the open unit disk, respectively;
- \( \phi \): the empty set.

A square matrix \( A \) is said to be stable if the set of eigenvalues of \( A \) is contained in \( \mathring{D} \). A real symmetric matrix \( X \) that satisfies the real discrete-time algebraic Riccati equation

\[
X = A^T X (I + R X)^{-1} A + Q
\]

is said to be stabilizing if \( (I + R X)^{-1} A \) is stable. A Hermitian \( n \times n \) matrix \( A \) is said to be positive semidefinite, denoted as \( A \geq 0 \), if \( x^H A x \geq 0 \) for all \( x \in \mathbb{C}^n \); and \( A \) is said to be positive definite, denoted as \( A > 0 \), if \( x^H A x > 0 \) for all nonzero \( x \in \mathbb{C}^n \). Let \( A \) and \( B \) be two Hermitian matrices; we write \( A \geq B \) if \( A - B \geq 0 \). We denote by \( A \neq 0 \) if \( A \) is not positive semidefinite. The inertia of a square matrix \( A \) is the ordered triple \( \text{In}(A) = (\nu_A, \zeta_A, \pi_A) \), where \( \nu_A, \zeta_A, \) and \( \pi_A \) are, respectively, the number of eigenvalues of \( A \) counting multiplicities with negative, zero, and positive
real parts [15]. A symmetric matrix $A$ is said to have a $J_{m_1,m_2}$ factorization if $In(A) = (m_1, 0, m_2)$. We denote by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = C(zI - A)^{-1}B + D$$

the transfer function matrix of a discrete-time linear system with state-space realization

$$\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \\
y_k &= Cx_k + Du_k.
\end{align*}$$

Let $A, Q$, and $R$ be real $n \times n$ matrices with $Q$ and $R$ symmetric. Define the ordered pair of $2n \times 2n$ matrices

$$S = (S_1, S_2) := \left( \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix}, \begin{bmatrix} I & R \\ 0 & A^T \end{bmatrix} \right).$$

Then $\lambda S_2 - S_1$ is a symplectic pencil, i.e., $S_1JS_1^T = S_2JS_2^T$, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Assume $\Lambda(S_1, S_2) \cap \partial \mathcal{D} = \emptyset$. Then the deflating subspace $\mathcal{X}_-(S)$ of the pencil $\lambda S_2 - S_1$ corresponding to eigenvalues in the open unit disk $\mathcal{D}$ is $n$-dimensional. Suppose that

$$\mathcal{X}_-(S) = \text{Im} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

where $T_1$ and $T_2 \in \mathbb{R}^{n \times n}$. If $T_1$ is nonsingular, we can set $X := T_2T_1^{-1}$. Then it is known ([14, Lemma 2.1]) that $X$ is a symmetric matrix uniquely determined by $S$; write $X = \text{Ric}(S)$. We say $S \in \text{dom}(\text{Ric})$ if $X := \text{Ric}(S)$ exists and $I + RX$ is invertible. Under the circumstances, $X$ satisfies the discrete-time algebraic Riccati equation (1.3).

### 2. The $H_\infty$ norm of a stable transfer function.

This section first reviews some basic relationships between the transfer function of a discrete-time linear system and the associated parameterized symplectic pencil of matrices. Then special attention is paid to the singularity of the symplectic pencil. A modified algorithm for computing the “$H_\infty$-norm” that fits the main purpose of this work is presented. Notice that the modified version differs from the ordinary ones, such as those in [19] and the references therein, only in the case when the singularity does occur.

Let

$$H(z) = C(zI - A)^{-1}B + D$$

be the transfer function matrix of a discrete-time stable (i.e., $\rho(A) < 1$) real linear system:

$$\begin{align*}
x_{k+1} &= Ax_k + Bw_k, \\
z_k &= Cx_k + Dw_k.
\end{align*}$$
and let $\lambda L_r - M_r$ with

$$
(2.2) \quad M_r = \begin{bmatrix} A + BD_r^{-1}D^T C & 0 \\ -C^T(I + DD_r^{-1}D^T)C & I \end{bmatrix}, \quad L_r = \begin{bmatrix} I & -BD_r^{-1}B^T \\ 0 & (A + BD_r^{-1}D^T C)^T \end{bmatrix}
$$

be the associated symplectic pencil (parameterized by $r > 0$), where $D_r = r^2 I - D^T D$. The $H_\infty$ norm of $H$ is defined as

$$
\|H\|_\infty = \max_{\theta \in [0, 2\pi]} \|H(e^{i\theta})\|_2 = \max_{\theta \in [0, 2\pi]} \|C(e^{i\theta} I - A)^{-1} B + D\|_2.
$$

It is known [25, p. 548] that $\|H\|_\infty \geq \|D\|_2$. And the following theorem gives the relationship between $H(z)$ and $(M_r, L_r)$.

**Theorem 2.1.** Assume that $A$ has no unimodular eigenvalue, that $r > 0$ is not a singular value of $D$, and that $\theta$ belongs to $[0, 2\pi]$. Then, $r$ is a singular value of $H(e^{i\theta})$ if and only if $\det(M_r - e^{i\theta} L_r) = 0$.

**Proof.** See [19]. □

According to Theorem 2.1, one can easily follow the results of [3], [4], [5], [12], [13], [18], etc., to establish algorithms for computing $\|H\|_\infty$. Here we do not want to restate the standard procedure but to deal with how to compute a feasible one for future use when the singular case occurs.

**Definition 2.2.** A pencil of square matrices $\lambda L - M$ is called regular if the determinant $\det(\lambda L - M)$ does not vanish identically. The pencil $\lambda L - M$ is said to be singular if it is not regular. We say that the matrix pair $(M, L)$ is regular/singular if the pencil $\lambda L - M$ is regular/singular [7].

From (2.2) we see $(M_r, L_r)$ is a symplectic pair. A regular symplectic pair has the property [17] that if $\lambda$ is an eigenvalue then $\frac{1}{\lambda}$ is also an eigenvalue with the same multiplicity as that of $\lambda$. When $A$ is an $n \times n$ matrix and $(M_r, L_r)$ is regular, there are exactly $2n$ (generalized) eigenvalues counting multiplicities in the set $\Lambda(M_r, L_r)$ [20]. Thus in particular there are at most a finite number of $\theta$’s in $[0, 2\pi]$ such that $\det(M_r - e^{i\theta} L_r) = 0$, provided $(M_r, L_r)$ is regular. So for any $r$ for which $(M_r, L_r)$ is regular, Theorem 2.1 shows there are at most a finite number of $\theta$’s in $[0, 2\pi]$ such that $r = \sigma_k(H(e^{i\theta}))$, where $\sigma_k(\cdot)$ denotes the $k$th largest singular value of a matrix. This means that the horizontal of height $r$ intersects the $\theta-r$ plot of the continuous function

$$
(2.3) \quad f_k(\theta) := \sigma_k(H(e^{i\theta}))
$$

at a finite number of points in $\theta \in [0, 2\pi]$ for each fixed $k$. However, when $(M_{r_0}, L_{r_0})$ is singular, all $\theta$’s in $[0, 2\pi]$ satisfy $\det(M_{r_0} - e^{i\theta} L_{r_0}) = 0$. Under the circumstances we have the next theorem.

**Theorem 2.3.** Let $r_0 > 0$ be such that $(M_{r_0}, L_{r_0})$ is a singular pencil. Then

$$
 f_1(\theta) \geq r_0 \quad \text{for all} \ \theta \ \text{in} \ [0, 2\pi],
$$

where $f_1$ is defined in (2.3).

**Proof.** Given any fixed $\theta_0$ in $[0, 2\pi]$. Since $(M_{r_0}, L_{r_0})$ is singular, we have $\det(M_{r_0} - e^{i\theta_0} L_{r_0}) = 0$. Theorem 2.1 then implies $r_0 = f_k(\theta_0)$ for some $k$, where $f_k(\cdot)$ is defined in (2.3). It follows that $f_1(\theta_0) \geq f_k(\theta_0) = r_0$, since $f_1 \geq f_k$ for every $k$. Notice that $\theta_0 \in [0, 2\pi]$ is arbitrary. This proves the theorem. □

**Theorem 2.4.** Let $r_0$ lie in the range of $f_1$ with $r_0$ equaling none of the singular values of $D$. Suppose that $(M_{r_0}, L_{r_0})$ is singular. Then for all $r_1 > r_0$ with $r_1$ equaling

$$
(2.4) \quad f_1(\theta) \geq r_1 \quad \text{for all} \ \theta \ \text{in} \ [0, 2\pi],
$$

where $f_1$ is defined in (2.3).
none of the singular values of \( D \), \((M_{r_1},L_{r_1})\) is regular and, further, \( r_1 \) lies in the range of \( f_1 \) if and only if the regular pencil \( \lambda L_{r_1} - M_{r_1} \) has a unimodular eigenvalue.

Proof. Assume on the contrary that \((M_{r_1},L_{r_1})\) is singular for some \( r_1 > r_0 \) with \( r_1 \) not a singular value of \( D \). According to Theorem 2.1, we have \( r_1 \leq \|H\|_\infty \). Otherwise, \((M_{r_1},L_{r_1})\) will have no unimodular eigenvalues and consequently cannot be singular.

Since \( r_0 \) lies in the range of \( f_1 \) and \((M_{r_1},L_{r_1})\) is singular, Theorem 2.3 implies \( r_0 \geq r_1 \). But this contradicts \( r_0 < r_1 \leq \|H\|_\infty \). Therefore \((M_{r_1},L_{r_1})\) must be regular. The remainder of the theorem is then an easy consequence of Theorem 2.1.

Theorem 2.4 indicates that the range of \( f_1 \) contains at most one \( r_0 \) such that \((M_{r_0},L_{r_0})\) is singular. This property helps us avoid much trouble when singularity is encountered. For instance, one need not worry about how many singular points one might encounter once one keeps the iteration \( r_1 \) larger than the singularity \( r_0 \), since it is impossible for \( r_1 \) to be a singularity again. Moreover, the second assertion of Theorem 2.4 provides a criterion of judging whether a computed \( r_1 > r_0 \) is in the range of \( f_1 \) (i.e., whether \( r_1 \) is a lower bound of \( \|H\|_\infty \)). If none of the eigenvalues of \( \lambda L_{r_1} - M_{r_1} \) is unimodular, then \( r_1 \) is an upper bound of \( \|H\|_\infty \); otherwise \( r_1 \) is a lower bound. Note that this bisection property does not hold when \( r_0 \) is not in the range of \( f_1 \). When \( r_0 \) does not lie in the range of \( f_1 \), we can still assert by Theorem 2.1 that \( r_1 \) is a lower bound of \( \|H\|_\infty \) provided \((M_{r_1},L_{r_1})\) has a unimodular eigenvalue, but we cannot conclude that \( r_1 \) is an upper bound of \( \|H\|_\infty \) if \((M_{r_1},L_{r_1})\) has no unimodular eigenvalues. This is because the \( r \)-intervals in which \((M_r,L_r)\) has no unimodular eigenvalues are disconnected, even within the interval \( \|D\|_2, \infty \) (recall that \( \|H\|_\infty \geq \|D\|_2 \)). All examples given later in the section illustrate this situation. Their respective \( \|H\|_\infty \)'s all separate their respective \( r \)-intervals in which \((M_r,L_r)\) has no unimodular eigenvalues. So the bisection property mentioned above does not hold in the general situation. This is the most important reason to consider the range of \( f_1 \) in Theorem 2.4.

Below, we list two algorithms for computing \( \|H\|_\infty \). The first one, which is the main one, is standard except for a slight modification to deal with the singularity. Readers are referred to the continuous-time counterparts [4] and [5] for the geometric explanation and convergency of step 3 of the following algorithm.

Algorithm 2.1. Main algorithm for computing \( \|H\|_\infty \). Given \( H(z) = D + C(zI - A)^{-1}B \) with \( \rho(A) < 1 \), this algorithm computes

\[
\begin{align*}
    r_\infty &:= \|H\|_\infty = \max_{\theta \in [0,2\pi]} \|H(e^{i\theta})\|_2 \\
    \text{or gives a slightly larger value when the singularity is encountered. In the latter case, the algorithm generates } &r_\infty + \delta \text{ with } \delta > 0 \text{ such that } (M_r,L_r) \text{ given in (2.2) is not nearly singular if } r \geq r_\infty + \delta, \text{ and that } \delta > 0 \text{ is the smallest possible such one.}
\end{align*}
\]

1. Initialization:
   1. Set \( r_\infty = \max\{\|H(1)\|_2, \|H(-1)\|_2, \|H(i)\|_2, \|H(-i)\|_2\} \).

2. Check for singularity:
   1. Use [23] to check whether the pencil \( \lambda L_{r_\infty} - M_{r_\infty} \) is nearly singular. If yes, call Algorithm 2.2.

3. Repeat: (comment: \((M_{r_\infty},L_{r_\infty})\) is already not nearly singular.)
   1. Compute \( \Lambda(L_{r_\infty},L_{r_\infty}) \), using [1], and determine the unimodular ones, say, \( \lambda_1, \lambda_2, \ldots, \lambda_q \).
   2. Solve \( \theta_k \in [0,2\pi] \) in equation \( \lambda_k = e^{i\theta_k} \), for \( k = 1,2,\ldots,q \).
3.3 Pick all \( \theta_k \)'s that satisfy \( \theta_k \leq \pi \) and
\[
\Re I - H^* (e^{i\theta_k}) H (e^{i\theta_k}) \geq 0.
\]

3.4 Reorder the \( \theta_k \)'s chosen from step 3.3 in increasing order, say,
\[
0 \leq \theta_1 \leq \theta_2 \cdots \leq \theta_q \leq \pi.
\]

3.5 For \( k = 1, 2, \ldots, q - 1 \),
- Set \( \theta = (\theta_k + \theta_{k+1})/2 \).
- Set \( r_\infty = \max \{ r_\infty, \| H (e^{i\theta}) \|_2 \} \).

End

3.6 Go to step 3 until converge.

4. Stop:
   Output \( \| H \|_\infty \approx r_\infty \).

In Algorithm 2.1, step 1 generates a starting point that lies in the range of \( f_1 \) (recall \( f_1(\theta) = \sigma_1(H(e^{i\theta})) = \| H(e^{i\theta}) \|_2 \) ), while step 3 generates an increasing sequence of iterations \( r_\infty \). So according to Theorem 2.4, we need only to take care of the singularity of the starting point. Once the singularity of \( (M_{r_\infty}, L_{r_\infty}) \) at the starting point is tackled in step 2, the successive iterations in step 3 do not have to make any effort to deal with the singularity again.

If \( (M_{r_u}, L_{r_u}) \) is nearly singular at the starting point, we consider \( r_u = r_\infty + \Delta \), where \( \Delta \) is sufficiently large so that \( (M_{r_u}, L_{r_u}) \) is not nearly singular. Practically, \( \Delta = 1 \) is large enough. Then we check whether \( r_u \) is in the range of \( f_1 \) by using Theorem 2.4: \( r_u \) lies in the range of \( f_1 \) if and only if \( (M_{r_u}, L_{r_u}) \) has an unimodular eigenvalue. If \( r_u \) is in the range of \( f_1 \), one may apply step 3 of Algorithm 2.1 to compute \( \| H \|_\infty \). On the other hand, if \( r_u \) does not fall into the range of \( f_1 \), it must be greater than \( \| H \|_\infty \). Then we use the following bisection algorithm to obtain an adjusted \( \| H \|_\infty \).

**Algorithm 2.2. Bisection for singular case.** Given nearly singular \( (M_{r_{\infty}}, L_{r_{\infty}}) \), this algorithm first lifts \( r_{\infty} \) by a positive integer \( N \) so that \( (M_{r_{\infty}+N}, L_{r_{\infty}+N}) \) is not nearly singular. Then it splits into two types of process according to whether \( (M_{r_{\infty}+N}, L_{r_{\infty}+N}) \) has unimodular eigenvalues. If \( (M_{r_{\infty}+N}, L_{r_{\infty}+N}) \) has a unimodular eigenvalue, the algorithm returns to Algorithm 2.1 with \( r_{\infty} \) replaced by \( r_{\infty}+N \). Otherwise, it adopts a bisection strategy and repeats the above procedure. If the algorithm does not return to Algorithm 2.1 from end to end, it produces an answer \( r_u \) with \( r_u = \| H \|_\infty + \delta \), where \( \delta > 0 \) is the smallest possible value such that \( (M_r, L_r) \) is not nearly singular whenever \( r \geq \| H \|_\infty + \delta \). \( Tol \) is a given error tolerance.

1. Initialization:
   - Set \( r_\ell = r_{\infty} \) and \( \Delta = 1 \).
2. Lift until \( (M_{r_u}, L_{r_u}) \) is not nearly singular:
   2.1 Set \( r_u = r_\ell + \Delta \). \( (M_{r_u}, L_{r_u}) \) is nearly singular.
   2.2 Use [23] to check whether the pencil \( AL_{r_u} - M_{r_u} \) is nearly singular. If yes, set \( r_\ell = r_u \) and repeat step 2.
3. Compute \( \Lambda(M_{r_u}, L_{r_u}) \), using [1]. Set \( \Delta = r_u - r_\ell \).
4. Process when \( r_u \) lies in the range of \( f_1 \):
   - If \( \Lambda(M_{r_u}, L_{r_u}) \cap \partial D \neq \phi \), set \( r_\infty = r_u \) and return to Algorithm 2.1.
5. Process when \( r_u \) does not fall into the range of \( f_1 \):
   - While \( \Delta > Tol \) do
     - Set \( \Delta \leftarrow \Delta/2 \).
Set \( r = r_\ell + \Delta \).
Use [23] to check whether the pencil \( \lambda M_r - L_r \) is nearly singular. If yes, set \( r_\ell = r \); else set \( r_u = r \) and go to step 3.
End.

6. Exhibit an “adjusted” \( \| H \|_\infty \):
Output \( \| H \|_\infty \approx r_u \).
Break.

Algorithm 2.2 provides a robust method for dealing with the singular case. It is designed not mainly to give an exact \( \| H \|_\infty \) itself but to preclude the singularity of the final answer in advance. Its advantages will become more clear in the remainder of the paper. Roughly speaking, Algorithm 2.2 generates an answer that guarantees a reliable controller numerically available when applied to control problems.

Numerical aspects of Algorithms 2.1 and 2.2, such as the computation of \( \Lambda(M_r, L_r) \), the determination of the unimodular eigenvalues, and the stop criteria, etc., will be discussed later in the next section.

Now we close this section with four examples. These examples stand for four different situations that occur when we consider the eigenvalue curves of \( (M_r, L_r) \). Notice that the \( r \)-intervals in which \( (M_r, L_r) \) has no unimodular eigenvalues are disconnected for each example. Because the computation of the \( H_\infty \) norm is not the main topic of this work, here we will not address the numerical details for obtaining \( \| H \|_\infty \).

Example 2.1. Let
\[
A = \frac{9 - \sqrt{65}}{4}, \quad B = \frac{\sqrt{13} - \sqrt{5}}{2}, \quad C = 1, \quad D = 0.
\]
Then
\[
\det (M_r - \lambda L_r) = \left( \frac{9 - \sqrt{65}}{8} \right) \left( 2\lambda^2 + \left( \frac{4}{r^2} - 9 \right) \lambda + 2 \right).
\]
The roots are
\[
\lambda(r) = \frac{(9 - \frac{4}{r^2}) \pm \sqrt{(9 - \frac{4}{r^2})^2 - 16}}{4} \in \begin{cases} R & \text{if } r \in \left(0, \frac{4}{\sqrt{13}}\right) \cup \left[\sqrt{\frac{2}{5}}, \infty\right); \\
\partial D & \text{if } r \in \left[\sqrt{\frac{4}{13}}, \sqrt{\frac{2}{5}}\right].\end{cases}
\]
The eigenvalue curves are shown in Figure 2. In the example,
\[\| H \|_\infty = \| H(1) \|_2 = \sqrt{\frac{4}{5}} \approx 8.94427190999916 \times 10^{-1}.\]

Example 2.2. Let
\[
A = \begin{bmatrix} 0 & \frac{-3 + \sqrt{5}}{2} \\ \frac{2 - \sqrt{5}}{2} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\sqrt{5} - 1}{2} & 0 \\ 0 & \frac{\sqrt{5} + 1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Then
\[
\det (M_r - \lambda L_r) = \left( \frac{7 - 3\sqrt{5}}{2r^4} \right) \left( r^4 \lambda^4 + (7r^4 - 6r^2 + 1)\lambda^2 + r^4 \right).
\]
The roots are

$$\lambda(r) = \pm \frac{1}{2r^2} \left( \sqrt{-(5r^2 - 1)(r^2 - 1)} \pm |3r^2 - 1| i \right)$$

$$\in \begin{cases} C_0 & \text{if } r \in (0, \sqrt{\frac{3}{5}}) \cup [1, \infty), \\ \partial D & \text{if } r \in \left[\sqrt{\frac{5}{7}}, 1\right]. \end{cases}$$

where $C_0$ denotes the imaginary axis. The eigenvalue curves are shown in Figure 3.

In the example,

$$\|H\|_\infty = \|H(i)\|_2 = 1.$$ 

Example 2.3. Let

$$A = \frac{3 - \sqrt{5}}{4} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \sqrt{5 - \sqrt{5}} \cdot \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{\sqrt{10}}{10\sqrt{2}} & \frac{1}{\sqrt{10}} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Then

$$\det(M_r - \lambda L_r) = \frac{\lambda}{8r^4} \left( \sqrt{5} - 3 \right) \cdot (2r^2 - 1) \left( 2r^2 \lambda^2 + (1 - 6r^2) \lambda + 2r^2 \right).$$

Thus $(M_r, L_r)$ is singular if and only if $r = \sqrt{\frac{1}{2}}$. When $r \neq \sqrt{\frac{1}{2}}$, the eigenvalues of $(M_r, L_r)$ are 0, $\infty$, and $\lambda(r)$, where

$$\lambda(r) = \frac{6r^2 - 1 \pm \sqrt{(2r^2 - 1)(10r^2 - 1)}}{4r^2}$$

$$\in \begin{cases} \mathbb{R} & \text{if } r \in (0, \sqrt{\frac{3}{10}}) \cup \left( \sqrt{\frac{1}{2}}, \infty \right), \\ \partial D & \text{if } r \in \left[\sqrt{\frac{1}{10}}, \sqrt{\frac{1}{2}}\right]. \end{cases}$$
The eigenvalue curves are shown in Figure 4. In the example,
\[ \|H\|_\infty = \|H(1)\|_2 = \sqrt{\frac{1}{2}} \approx 0.7071067811865476 \times 10^{-1}. \]
Since \( \lim_{r \to \frac{1}{\sqrt{2}}} \lambda(r) = \lim_{r \to \frac{1}{\sqrt{2}}} \lambda(r) = 1 \), we see two eigenvalue curves are continuous and attach to the unit circle at the singularity \( r = \frac{1}{\sqrt{2}} \).

**Example 2.4.** Let
\[ A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = 3, \quad D = 1. \]
Then
\[ \det(M_r - \lambda L_r) = \frac{1}{4}(2\lambda - 1)(\lambda - 2)(r - 2) \left( \frac{r + 2}{r^2 - 1} \right). \]
Thus \((M_r, L_r)\) is singular if and only if \( r = 2 \). When \( r \neq 2 \), the eigenvalues of \((M_r, L_r)\) are \( \frac{1}{2} \) and 2, independent of \( r \). The eigenvalue curves are shown in Figure 5. In the example
\[ \|H\|_\infty = \|H(e^{i\theta})\|_2 = 2 \geq \|D\|_2, \]
where \( \theta \in [0, 2\pi] \) is arbitrary. Unlike Example 2.3, the eigenvalue curves do not attach to the unit circle continuously because they are fixed at \( \frac{1}{2} \) and 2, respectively, when \( r \neq 2 \).

**3. Solution for the first two conditions.** Recall the conditions (C1) and (C2) in section 1:
(C1) \( S_\infty(r) \in \text{dom}(\text{Ric}), X(r) := \text{Ric}(S_\infty(r)) \geq 0 \)

\[ \begin{bmatrix} -r^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} X(r) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

has a \( J_{m_1,m_2} \) factorization;

(C2) \( T_\infty(r) \in \text{dom}(\text{Ric}), Y(r) := \text{Ric}(T_\infty(r)) \geq 0 \)

\[ \begin{bmatrix} -r^2 I_{p_1} & 0 \\ 0 & I_{p_2} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} Y(r) \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \]

has a \( J_{p_1,p_2} \) factorization,

where \( S_\infty(r) \) and \( T_\infty(r) \) are given in (1.1) and (1.2), respectively. The two conditions are mutually dual. So in order to solve them, it suffices to consider the solution of (C1) in the following discussion.

To find \( r_x \), the infimum of \( r \) such that condition (C1) holds, we first analyze the condition \( S_\infty(r) \in \text{dom}(\text{Ric}) \). This condition involves three subconditions:
Fig. 5. The eigenvalue curves of Example 2.4.

(S1) \( \Lambda(S_\infty^1(r), S_\infty^2(r)) \cap \partial D = \phi; \)
(S2) \( T_1(r) \in \mathbb{R}^{n \times n} \) is nonsingular, where

\[
\text{Im} \left( \begin{bmatrix} T_1(r) \\ T_2(r) \end{bmatrix} \right) = \mathcal{X}_-(S_\infty(r));
\]

(S3) \( I + (B_2B_2^T - \frac{1}{r^2}B_1B_1^T)X(r) \) is invertible, where \( X(r) = \text{Ric}(S_\infty(r)). \)

Below, we treat the three subconditions separately.

Consider the symplectic pair

\[
S_2 = (S_{21}, S_{22}) := \begin{bmatrix} A^T & 0 \\ -B_2B_2^T & I \end{bmatrix}, \begin{bmatrix} I & C_1^T C_1 \\ 0 & A \end{bmatrix}.
\]

According to assumptions (A1)–(A2) of section 1, we get that \((A^T, C_1^T)\) is stabilizable and that \((B_2^T, A)\) is detectable. So, by [17], \( S_2 \in \text{dom}(\text{Ric}) \) and \( Z := \text{Ric}(S_2) \geq 0. \)

Moreover, \( Z \) satisfies

\[
Z = AZ(I + C_1^T C_1)Z^{-1} A^T + B_2B_2^T. \tag{3.1}
\]

Thus we have the following theorem.

**Theorem 3.1.** Let \( Z \) be the stabilizing solution of (3.1), let \( S_\infty(r) = (S_{\infty_1}(r), S_{\infty_2}(r)) \) be the symplectic pair defined by (1.1), and let

\[
S_{Z1}(r) = \begin{bmatrix} (I + C_1^T C_1)Z^{-1} A^T \\ 0 \\ -B_1B_1^T \end{bmatrix}, \quad S_{Z2}(r) = \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} - \frac{1}{r^2} C_1^T(I + C_1ZC_1^T)^{-1} C_1.
\]

Then

\[
S_{\infty_1}(r) = V_1(r)S_{Z2}(r)V_2(r)^{-1} \quad \text{and} \quad S_{\infty_2}(r) = V_1(r)S_{Z1}(r)V_2(r)^{-1},
\]

where

\[
V_1(r) = \begin{bmatrix} -AZ \\ I + C_1^T C_1 \\ \frac{1}{r^2} I \end{bmatrix}, \quad V_2(r) = \begin{bmatrix} -Z \\ I \\ 0 \end{bmatrix}.
\]

**Proof.** By direct expansion and using (3.1), one can easily verify that

\[
S_{\infty_1}(r)V_2(r) = V_1(r)S_{Z2}(r) \quad \text{and} \quad S_{\infty_2}(r)V_2(r) = V_1(r)S_{Z1}(r).
\]

Thus the theorem is established. \( \square \)

The above theorem indicates that

\[
\Lambda(S_{\infty_1}(r), S_{\infty_2}(r)) = \Lambda(S_{Z2}(r), S_{Z1}(r)).
\]
However,

\[ \Lambda(S_{Z2}(r), S_{Z1}(r)) = \Lambda(S_{Z1}(r), S_{Z2}(r)), \]

since \((S_{Z1}(r), S_{Z2}(r))\) is symplectic. So we have

\[ \Lambda(S_{\infty_1}(r), S_{\infty_2}(r)) = \Lambda(S_{Z1}(r), S_{Z2}(r)). \]

Theorem 2.1 shows that \(\det(S_{Z1}(r) - e^{i\theta}S_{Z2}(r)) = 0\) if and only if \(r\) is a singular value of \(H_Z(e^{i\theta})\), where

\[ H_Z(e^{i\theta}) = B_1^T \left( e^{i\theta} I - (I + C_1^T C_1 Z)^{-1} A^T \right)^{-1} C_1^T (I + C_1 Z C_1^T)^{-\frac{1}{2}}. \]

(Notice that \(Z\) is the stabilizing solution of (3.1), \(\rho((I + C_1^T C_1 Z)^{-1} A^T) < 1\)) Thus, as shown in section 2, the set of \(r\) such that \((S_{Z1}(r), S_{Z2}(r))\), and consequently \((S_{\infty_1}(r), S_{\infty_2}(r))\), has no unimodular eigenvalues is generally the union of disjoint intervals. So it is difficult to determine all the \(r\)-intervals in which the condition (S1) holds. Fortunately, [14, Theorem 3.1(a)] implies that the set of \(r\) that satisfies condition (C1) is a connected interval \([r_x, \infty)\). Therefore \(r_x \geq \|H_Z\|_\infty\), where \(H_Z(\cdot)\) is defined in (3.3). We may apply the two algorithms of section secHinfNorm to compute the lower bound \(\|H_Z\|_\infty\) of \(r_x\). This seems to risk instability since \(Z\) should be computed and \(H_Z(e^{i\theta})\) \((S_{Z1}(r), S_{Z2}(r))\) should be explicitly updated. However, in practice one can use this to quickly get an approximated lower bound of \(r_x\) then apply a bisection or secant method listed in Algorithm 3.1 to improve it. So the instability problem does not affect the final result very much. Instead, this consideration helps the evaluation of the lower bound become easy and fast and precludes the useless \(r\)-intervals in advance.

Algorithm 3.1 contains some details for improving the computed \(\|H_Z\|_\infty\). It consists of a bisection method and a secant method to accelerate the convergence. In general, one could not determine the radius of convergence for the secant method. So a threshold check is needed (which we omit throughout the paper): If the secant method is not convergent within a prescribed number of iteration steps, say seven steps, then bisect the interval (the interval where the secant method started off) several times. And repeat the above procedure until some stop criterion is met. The improved result exhibited by Algorithm 3.1 will be a bit larger than the true result. That is, we obtain \(\|H_Z\|_\infty + \eta > \|H_Z\|_\infty\), for which it could be verified numerically that

\[ \Lambda(S_{\infty_1}(r_{tb} + \eta), S_{\infty_2}(r_{tb} + \eta)) \cap \partial D = \emptyset \]

and that \(X(r_{tb} + \eta) := Ric(S_{\infty_1}(r_{tb} + \eta))\) exists, where \(r_{tb} = \|H_Z\|_\infty\).

For \(r > r_{tb}\), the real symplectic matrix pair \((S_{\infty_1}(r), S_{\infty_2}(r))\) has no eigenvalues on the unit circle. So we can find a set of real orthogonal basis vectors for \(X_-(S_{\infty_1}(r))\), say

\[ X_-(S_{\infty_1}(r)) = I_m \left[ \begin{array}{c} T_1(r) \\ T_2(r) \end{array} \right], \]

where \(T_1(r), T_2(r) \in \mathbb{R}^{n \times n}\). If \(T_1(r)\) is invertible, then

\[ X(r) := Ric(S_{\infty_1}(r)) = T_2(r)T_1^{-1}(r) \]

exists and we have the following sign-changing theorem.
THEOREM 3.2. Let \( r_{th} = \|H_Z\|_\infty \) with \( H_Z \) given by (3.3). For \( r > r_{th} \), let \( X(r) = Ric(S_\infty(r)) \) whenever \( Ric(S_\infty(r)) \) exists. Assume that \( X(r_2) \geq 0 \) and \( X(r_1) \geq 0 \) for some \( r_1, r_2 \) with \( r_2 > r_1 > r_{th} \). Then there exists an \( r_s \in (r_1, r_2) \) such that either \( T_1(r) \) defined by (3.4) is singular at \( r = r_s \), or \( I + (B_2 B_2^T - \frac{1}{r^2} B_1 B_1^T)X(r) \) is not invertible at \( r = r_s \).

Proof. Let \( r > r_{th} \). And let \( \left[ \begin{array}{c} T_i(r) \\ T_2(r) \end{array} \right] \) form a real orthogonal basis for \( \mathcal{X}_r(S_\infty(r)) \). Suppose for all \( r \in (r_1, r_2) \), \( T_1(r) \) is nonsingular and \( I + (B_2 B_2^T - \frac{1}{r^2} B_1 B_1^T)X(r) \) is invertible. Then \( X(r) \) satisfies the discrete-time algebraic Riccati equation (DARE):

\[
X(r) = A^T X(r) \left( I + \left( B_2 B_2^T - \frac{1}{r^2} B_1 B_1^T \right) X(r) \right)^{-1} A + C_1^T C_1
\]

with \( \rho(S(r)) < 1 \), where

\[
S(r) = \left( I + \left( B_2 B_2^T - \frac{1}{r^2} B_1 B_1^T \right) X(r) \right)^{-1} A.
\]

Differentiating (DARE) with respect to \( r \), it is easy to show that \( \frac{d}{dr} X(r) \) satisfies

\[
\frac{d}{dr} X(r) = S^T(r) \left( \frac{d}{dr} X(r) \right) S(r) - \frac{2}{r^3} S^T(r) X(r) B_1 B_1^T X(r) S(r).
\]

So, by standard Lyapunov theorem, we see that \( \frac{d}{dr} X(r) \leq 0 \), which implies \( X(r_2) \leq X(r_1) \) since \( r_2 > r_1 \). But this is impossible, since \( X(r_2) \geq 0 \) and \( X(r_1) \geq 0 \). Thus, either \( T_1(r) \) or \( I + (B_2 B_2^T - \frac{1}{r^2} B_1 B_1^T)X(r) \) must be singular at some \( r_s \in (r_1, r_2) \). □

Theorem 3.2 connects condition (S2) with the condition \( X(r) := Ric(S_\infty(r)) \geq 0 \) coincidently. It provides a criterion for finding the critical points such that \( X(r) \geq 0 \). To be more precise, it indicates that the sign-changing points of \( X(r) \) are points such that \( T_1(r) \) is singular. If we know that \( X(r_2) \geq 0 \) and \( X(r_1) \geq 0 \) in a small neighborhood of \( r_s \) with \( r_1 < r_s < r_2 \) and \( T_1(r_s) \) singular, we may apply the secant method to find a zero of

\[
f(r) := \begin{cases} \sigma_{\min}(T_1(r)) & \text{if } r \geq r_s, \\ -\sigma_{\min}(T_1(r)) & \text{if } r \leq r_s, \end{cases}
\]

which then gives \( r_s \). To distinguish the case \( r > r_s \) from the case \( r < r_s \) at each iteration, we use Theorem 3.2. That is, we determine the inertia of \( X(r) \) first. If \( X(r) \geq 0 \), we assert that \( r > r_s \); if \( X(r) \nless 0 \), we then conclude \( r < r_s \). Notice that \( X(r) := T_2(r) T_1^{-1}(r) \) is congruent to \( T_1^{-T}(r) T_2(r) \). So in order to determine whether \( X(r) \geq 0 \) and to avoid the computation of \( T_1^{-1}(r) \) especially when \( T_1(r) \) is nearly singular, we check the inertia of \( T_1^{-T}(r) T_2(r) \) instead of checking that of \( X(r) \) [16].

Finally, we deal with condition (S3) and the condition that

\[
\begin{bmatrix} -r^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} X(r) [B_1 \ B_2]
\]

has a \( J_{m_1,m_2} \) factorization. For convenience, we denote \( \begin{bmatrix} -r^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \) by \( R(r) \) and \([B_1 \ B_2] \) by \( B \).
When $X(r) \geq 0$, $I + B_2^T X(r) B_2 > 0$. Therefore $R(r) + B^T X(r) B$ is congruent to

$$
\begin{bmatrix}
-v^2 I_{m_1} + B_2^T X(r) (I + B_2 B_2^T X(r))^{-1} B_1 & 0 \\
0 & I_{m_2} + B_2^T X(r) B_2
\end{bmatrix}.
$$

Consequently, $R(r) + B^T X(r) B$ has a $J_{m_1,m_2}$ factorization if and only if

(3.6) $v^2 I_{m_1} - B_1^T X(r) (I + B_2 B_2^T X(r))^{-1} B_1 > 0.$

On the other hand, by the Sherman–Morrison formula [8] we see that

$v^2 I - B_1^T X(r) (I + B_2 B_2^T X(r))^{-1} B_1$

is invertible if and only if

$I + B_2 B_2^T X(r) - \frac{1}{v^2} B_1 B_1^T X(r)$

is invertible. This connects condition (S3) with the $J_{m_1,m_2}$-factorization condition. So it remains to consider inequality (3.6) whenever we know $X(r) \geq 0$.

As before, let $X(r) = T_2(r) T_1^{-1}(r)$. Then (3.6) holds if and only if

$v^2 I - B_1^T T_2(r) (T_1(r) + B_2 B_2^T T_2(r))^{-1} B_1 > 0.$

which holds if and only if

$$
\lambda_{\text{max}} \left( B_1 B_1^T T_2(r) (T_1(r) + B_2 B_2^T T_2(r))^{-1} \right) < v^2,
$$

where $\lambda_{\text{max}}(\cdot)$ denotes the maximum eigenvalue of a matrix. So in order to avoid the computation of $T_1^{-1}(r)$ as well as $(T_1(r) + B_2 B_2^T T_2(r))^{-1}$, we apply the bisection or the secant method to find the zero of

(3.7) $g(r) = \lambda_{\text{max}} \left( B_1 B_1^T T_2(r), T_1(r) + B_2 B_2^T T_2(r) \right) - v^2$

(where $\lambda_{\text{max}}(M_1, M_2)$ denotes the maximum eigenvalue of the generalized eigenvalue problem $M_1 - \lambda M_2$) to get the critical point such that inequality (3.6) holds. Notice that the above method is valid only when it is ensured that the corresponding Riccati solution $X(r)$ is kept positive semidefinite during the iteration.

Based on the discussion above, we briefly summarize the procedure for computing $r_x$ as follows. First of all, we use the two algorithms in section 2 to compute $\|H_Z\|_\infty$ as a preliminary lower bound $r_1$, where $H_Z$ is given by (3.3). Then we choose a sufficiently large upper bound $r_2$ (which will be discussed later in section 3.2) so that condition (C1) holds. Thus $X(r_2) \geq 0$ and inequality (3.6) holds for $r = r_2$. If $X(r_1) \geq 0$ and (3.6) holds for $r = r_1$, then $r_x = r_1$. If $X(r_1) \geq 0$ but inequality (3.6) does not hold for $r = r_1$, then, as long as $X(r_1)$ is kept positive semidefinite during the iteration, apply bisection or the secant method to find the zero of the function $g$ defined in (3.7). If $X(r_1) \not\geq 0$, then, as long as $X(r_1)$ is kept nonpositive, apply the bisection or the secant method to find the zero of the function $f$ defined in (3.5). As to the other cases, because they are entangled with one another, we only list the details in Algorithm 3.1. However, the three cases mentioned above constitute the main frame of the algorithm. Numerical details are discussed in section 3.2.
3.1. Algorithm for solving condition (C1).

Algorithm 3.1. Bisection and Secant method. Given an upper bound \( r_+ \), this algorithm computes \( r_2 \), the infimum of \( r \) such that condition (C1) in section 1 holds. Let \( Tol \) be a threshold for the bisection method to stop and the secant method to start.

Initialization:
\[
r_- \leftarrow \|H_Z\|_\infty, \text{ where } H_Z \text{ is given by (3.3)}.
\]
\[
\tilde{r} \leftarrow \frac{1}{2}(r_- + r_+), \text{ case} \# \leftarrow 0.
\]
Repeat:
1. Compute \( \Lambda(S_\infty(\tilde{r}), S_\infty(\tilde{r})) \).
2. Determine the candidate \([r_-, r_+]\):
   2.1 If \( \Lambda(S_\infty(\tilde{r}), S_\infty(\tilde{r})) \cap \partial D \neq \emptyset \), set \([r_-, r_+] = [\tilde{r}, r_+] \). Go to step 3.
   2.2 Compute a set of real orthogonal basis vectors \([T_1, T_2] \) of \( \mathcal{X}(S_\infty(\tilde{r})) \), where \( T_1, T_2 \in \mathbb{R}^{n \times n} \).
      
      If \( T_1^T T_2 \geq 0 \) and \( \lambda_{\max}(B_1 B_1^T T_2 + B_2 B_2^T T_2 < \tilde{r}^2 \), set \([r_-, r_+] = [r_-, \tilde{r}] \) and go to step 3.
      
      If \( T_1^T T_2 \geq 0 \), set case \# = 1; else set case \# = 2.
Set \([r_-, r_+] = [\tilde{r}, r_+] \).
3. Update the iterate \( \tilde{r} \):
   - If \( r_+ - r_- \geq Tol \), update \( \tilde{r} \) by bisection method, i.e., \( \tilde{r} \leftarrow \frac{1}{2}(r_- + r_+) \).
   - If \( r_+ - r_- < Tol \), update \( \tilde{r} \) by secant method with suitable function considered:
      Set \( \tilde{r} \) by (3.21) if case \# = 0, by (3.22) if case \# = 1, and by (3.23) if case \# = 2.
4. Stop criterion:
   - If any of the stop criterions proposed in section 3.2 holds, then accept \( r_* = r_+ \). Go to step 5.
Else go to Repeat.
5. Double check:
   5.1 Compute \( \Lambda(S_\infty(r_*), S_\infty(r_*)) \).
   5.2 If \( \Lambda(S_\infty(r_*), S_\infty(r_*)) \cap \partial D = \emptyset \), compute a set of real orthogonal basis vectors \([T_1, T_2] \) of \( \mathcal{X}(S_\infty(r_*)) \) with \( T_1, T_2 \in \mathbb{R}^{n \times n} \). If \( T_1^T T_2 \geq 0 \) and \( \lambda_{\max}(B_1 B_1^T T_2 + B_2 B_2^T T_2 < r_*^2 \),
      use (3.25) to find \( \Delta r > 0 \) such that \( T_1(r_* + \Delta r) \) is well conditioned. Accept \( r_* = r_* + \Delta r \). Stop.
   5.3 Set \( r_1 = r_*, r_+ \) sufficiently large, and \( \tilde{r} = \frac{1}{2}(r_- + r_+) \). Go to Repeat.

3.2. Numerical aspects. In this section, we present some numerical aspects and stop criteria for Algorithm 3.1.

First of all, we have to determine a preliminary upper bound \( r_+ \) of \( r_x \). In Algorithm 3.1 we may consider, for example,
\[
r_+ = \max_x \left\{ \frac{x^T B_1 B_1^T x}{x^T B_2 B_2^T x} \mid x^T B_2 B_2^T x \neq 0 \right\},
\]
provided \( B_2 \) is nonzero and the null space of the nonzero matrix \( B_2^T \) is contained in the null space of \( B_1^T \). In this case we have
\[
B_2 B_2^T - \frac{1}{r_+^2} B_1 B_1^T \geq 0
\]
whenever $r > r_+$. Therefore we can expect from [14, Lemma 2.2] that $X(r) := \text{Ric}(S_\infty(r)) \geq 0$ for $r > r_+$, in most cases. Thus, when the null space of $B_1^T$ lies in the null space of $B_1^T$ and $B_2$ is of full row rank, we can first compute the maximum singular value $\sigma_{\text{max}}(B_1)$ of $B_1$ and the minimum singular value $\sigma_{\text{min}}(B_2)$ of $B_2$ then set

$$r_+ = \frac{\sigma_{\text{max}}(B_1)}{\sigma_{\text{min}}(B_2)}.$$ 

(3.8)

This gives an intuitive way of computing an “upper bound” of $r_x$ in general. Of course, this requires an a priori check for feasibility of condition (C1). However, (3.8) is not always valid, especially when $\sigma_{\text{min}}(B_2) = 0$. So in general, we may consider the numerically trivial upper bound (though not a good one)

$$r_+ = \frac{\sigma_{\text{max}}(B_1)}{\sqrt{\epsilon}},$$

where $\epsilon$ is the machine precision.

In step 1 of Algorithm 3.1, we adopt the numerically stable, structure-preserving method proposed in [1] to compute the set $\Lambda(S_\infty(\tilde{r}))$ of eigenvalues of the symplectic matrix pair $S_\infty(\tilde{r}) := (S_{\infty_1}(\tilde{r}), S_{\infty_2}(\tilde{r}))$.

In step 2, we must determine whether $\Lambda(S_\infty(\tilde{r})) \cap \partial D = \emptyset$. Based on the results of a lot of numerical experiments, we assume in this paper that all multiple unimodular singular values of $S_\infty(\tilde{r})$ have linear elementary divisors. Geometrically, this means that for each point on the unit circle, there are at most two eigenvalue curves that meet at that point and then split into two opposite-directional curves along the unit circle. See the similar phenomena shown in Examples 2.1–2.3. So we derive the error bounds of the computed eigenvalues under that assumption and use these bounds as criteria for judging $\Lambda(S_\infty(\tilde{r})) \cap \partial D = \emptyset$ or $\neq \emptyset$ in the general case. All the bounds derived are under the hypothesis that $(S_{\infty_1}(\tilde{r}), S_{\infty_2}(\tilde{r}))$ is regular.

Let $\mathcal{E}(\tilde{r}) = S_{\infty_2}(\tilde{r}) - S_{\infty_1}(\tilde{r})$, $\mathcal{A}(\tilde{r}) = S_{\infty_2}(\tilde{r}) + S_{\infty_1}(\tilde{r})$. Then $\mu \mathcal{E} - \mathcal{A}$ (we drop $\tilde{r}$ here for brevity) is a Hamiltonian pencil [1]. And the eigenvalues of $\mu \mathcal{E} - \mathcal{A}$ are of the form $\mu = \frac{1+\lambda}{1-\lambda}$, where $\lambda$ is an eigenvalue of $(S_{\infty_1}(\tilde{r}), S_{\infty_2}(\tilde{r}))$. Let $\lambda \neq \pm 1$ with $|\lambda| = 1$, and let $\hat{\lambda}$ be the corresponding eigenvalue computed by using the method of [1]. Then by [1, Theorem 10] we have the error bound

$$|\hat{\lambda} - \lambda| \leq \frac{2\epsilon}{|1-\lambda|} \cdot \sqrt{|1-\lambda|^2 + |1+\lambda|^2} \cdot \frac{\|\mathcal{E}, \mathcal{A}\|_2 \cdot (\|\mathcal{E}^T \tilde{y}\|_2 + \|\mathcal{E}^T \tilde{x}\|_2)}{|y^H \mathcal{E} J \mathcal{E}^T \tilde{x}|},$$

where $y, x$ with $\|y\|_2 = \|x\|_2 = 1$ are the left eigenvectors of the Hamiltonian pencil $\mu \mathcal{E} - \mathcal{A}$ corresponding to $\mu = \frac{1+\lambda}{1-\lambda}$ and $-\mu$, respectively, $\tilde{x}$ denotes $(x^H)^T$, and $J$ is given in (1.4). Similarly, by letting $\mathcal{E}(\tilde{r}) = S_{\infty_2}(\tilde{r}) + S_{\infty_1}(\tilde{r})$ and $\mathcal{A}(\tilde{r}) = -S_{\infty_1}(\tilde{r}) + S_{\infty_1}(\tilde{r})$, we have

$$|\hat{\lambda} - \lambda| \leq \frac{2\epsilon}{|1+\lambda|} \cdot \sqrt{|1-\lambda|^2 + |1+\lambda|^2} \cdot \frac{\|\mathcal{E}, \mathcal{A}\|_2 \cdot (\|\mathcal{E}^T \tilde{y}\|_2 + \|\mathcal{E}^T \tilde{x}\|_2)}{|\tilde{y}^H \mathcal{E} J \mathcal{E}^T \tilde{x}|},$$

where $\tilde{y}, \tilde{x}$ with $\|\tilde{y}\|_2 = \|\tilde{x}\|_2 = 1$ are the left eigenvectors of the Hamiltonian pencil $\mu \mathcal{E} - \mathcal{A}$ corresponding to $\hat{\mu} = \frac{1+\lambda}{1+\lambda}$ and $-\hat{\mu}$, respectively. Thus we say that $\hat{\lambda}$ lies on the unit circle (including 1 and $-1$) if

$$|\hat{\lambda}|^2 - 1 < \delta_{\lambda(\tilde{r})},$$

(3.11)
where
\[
\delta_{\lambda(\hat{r})} = 8\epsilon \cdot \sqrt{1 - \lambda^2} + \frac{1 + \lambda^2}{1 - \lambda^2} \cdot \min \left\{ \frac{\| [\mathcal{E}, A] \|_2}{1 - \lambda}, \frac{\| [\mathcal{E}, \hat{A}] \|_2}{1 + \lambda} \right\}.
\]

In step 2.2 of Algorithm 3.1, we use the QZ method to find \([\hat{T}_1 \ 0] \), the basis of \( \chi_-(S_{\infty}(\hat{r})) \). Let \( \hat{U} \) and \( \hat{V} \) be two orthogonal matrices obtained such that
\[
\hat{U}^T S_{\infty}(\hat{r}) \hat{V} = \hat{M} + \Delta \hat{M} \quad \text{and} \quad \hat{U}^T S_{\infty}(\hat{r}) \hat{V} = \hat{L} + \Delta \hat{L},
\]
where
\[
\hat{M} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \quad \Delta \hat{M} = \begin{bmatrix} 0 & 0 \\ M_{21} & 0 \end{bmatrix},
\]
\[
\hat{L} = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}, \quad \Delta \hat{L} = \begin{bmatrix} 0 & 0 \\ L_{21} & 0 \end{bmatrix}.
\]

Then the computed \([\hat{T}_1 \ \hat{T}_2] \) has the perturbation bound [20, p. 307]
\[
\left\| \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{bmatrix} - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right\|_F \leq 2 \cdot \max \left\{ \| M_{21} \|_F, \| L_{21} \|_F \right\} \frac{\delta(\hat{r})}{\delta(\hat{r})},
\]
where \( \left\| \cdot \right\|_F \) stands for the Frobenius norm and \( \delta(\hat{r}) = \text{dif}[[M_{11}, L_{11}], (M_{22}, L_{22})] \) is the separation between the stable deflating subspace and the unstable one of \( S_{\infty}(\hat{r}) \) [20]. Since \( \hat{U} \) and \( \hat{V} \) arise from the QZ method, we have
\[
\max \left\{ \| M_{21} \|_F, \| L_{21} \|_F \right\} \leq \epsilon \cdot \| S_{\infty}(\hat{r}) \|_F,
\]
where \( \| S_{\infty}(\hat{r}) \|_F = \max \left\{ \| S_{\infty_1}(\hat{r}) \|_F, \| S_{\infty_2}(\hat{r}) \|_F \right\} \). Thus,
\[
(3.12) \quad \left\| \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{bmatrix} - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right\|_F \leq \frac{2 \cdot \| S_{\infty}(\hat{r}) \|_F}{\delta(\hat{r})} \cdot \epsilon.
\]

Now we consider the problem of judging the positive semidefiniteness of \( T_2 T_1^{-1} \).
First, notice that \( T_2 T_1^{-1} \) and \( T_1 T_2 \) are congruent. So, to avoid the computation of \( T_1^{-1} \) when \( T_1 \) is near to being singular, we check \( T_1^T T_2 \) instead of checking \( T_2 T_1^{-1} \). Below, we list the detailed procedures for doing this.

**Algorithm 3.2. Subroutine for the determination of \( T_2 T_1^{-1} \geq 0 \) (see [16]).** Let \([\hat{T}_1 \ \hat{T}_2] \) be computed such that \( \text{Im}(\hat{T}_1) = \chi_-(S_{\infty}(\hat{r})) \) and \( \| \hat{T}_1 \|_2 = 1 \).

1. Find orthogonal matrices \( U, V \) such that
   \[
   U^T T_1^T V = R_1 = \begin{bmatrix} & & \end{bmatrix} \quad \text{(upper triangular)},
   \]
   \[
   V^T T_2 U = R_2 = \begin{bmatrix} & \end{bmatrix} \quad \text{(upper Hessenberg)}.
   \]

Let
\[
\Omega = U^T T_1^T T_2 U = R_1 R_2 = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \end{bmatrix} \quad \text{(tridiagonal)}.
\]
(Note that \( T_1^T T_2 = T_2^T T_1 \) is symmetric.)
2. Find an orthogonal matrix \( W \) such that
\[
W^T \Omega W = D = \begin{pmatrix} \cdots & \vdots & \cdots \\ \end{pmatrix} \text{ (diagonal)}.
\]

Determine \( \text{In}(\Omega) = (\nu_1, \zeta_1, \pi_1) \), the inertia of \( \Omega \).

3. If \( \nu_1 \neq 0 \), then \( D \neq 0 \) and hence \( T_2 T_1^{-1} \neq 0 \).
   If \( \nu_1 = 0 \), then \( D \geq 0 \) and hence \( T_2 T_1^{-1} \geq 0 \).

In the following, we analyze the errors associated with the above procedures step by step and discuss how to distinguish numerically \( T_2 T_1^{-1} \geq 0 \) from \( T_2 T_1^{-1} \neq 0 \).

**Error analysis of Algorithm 3.2 for the determination of \( \text{In}(T_2 T_1^{-1}) \).**

In numerical implementation, we see from (3.12) that \( \tilde{T}_1 \) and \( \tilde{T}_2 \) may be written as

\[
\tilde{T}_1 = T_1 + \Delta T_1 \quad \text{and} \quad \tilde{T}_2 = T_2 + \Delta T_2
\]

with

\[
\| \Delta T_1 \|_2, \| \Delta T_2 \|_2 \leq \frac{2 \sqrt{n} \| S_{\infty}(\hat{r}) \|_2}{\delta(\hat{r})} \cdot \epsilon.
\]  
(3.13)

Thus we have the following perturbation result on \( R_1 \):

\[
\hat{R}_1 \equiv U^T \hat{T}_1^T V + \Delta \hat{T}_1 = R_1 + U^T (\Delta T_1)^T V + \Delta \hat{T}_1
\]

and

\[
\| \Delta \hat{T}_1 \|_2 \leq c_1(n) \| \hat{T}_1 \|_2 \epsilon \leq c_1(n) (\| T_1 \|_2 + \| \Delta T_1 \|_2) \epsilon,
\]  
(3.14)

where \( c_1(n) \) is a polynomial in \( n \) of low degree [9]. Since \( \| [T_1] \|_2 = 1 \), from (3.13), (3.14), and (3.15) we have

\[
\| \hat{R}_1 - R_1 \| \lesssim c_1(n) \left( 1 + \frac{2 \sqrt{n} \| S_{\infty}(\hat{r}) \|_2}{\delta(\hat{r})} \epsilon \right) \cdot \epsilon + \frac{2 \sqrt{n} \| S_{\infty}(\hat{r}) \|_2}{\delta(\hat{r})} \cdot \epsilon.
\]

Similarly, if we let \( \hat{R}_2 = V^T \hat{T}_2 U + \Delta \hat{T}_2 \), then we have the error bound on \( \hat{R}_2 \):

\[
\| \hat{R}_2 - R_2 \| \leq c_2(n) \left( 1 + \frac{2 \sqrt{n} \| S_{\infty}(\hat{r}) \|_2}{\delta(\hat{r})} \epsilon \right) \cdot \epsilon + \frac{2 \sqrt{n} \| S_{\infty}(\hat{r}) \|_2}{\delta(\hat{r})} \cdot \epsilon,
\]

where \( c_2(n) \) is a polynomial in \( n \) of low degree, and

\[
\| \Delta \hat{T}_2 \|_2 \leq c_2(n) (\| T_2 \|_2 + \| \Delta T_2 \|_2) \epsilon.
\]

Let \( \hat{\Omega} \) be the computed \( \Omega \) in step 1 of Algorithm 3.2; then

\[
\hat{\Omega} = (U^T \hat{T}_1^T V + \Delta \hat{T}_1)(V^T \hat{T}_2 U + \Delta \hat{T}_2) + \Delta \hat{\Omega} = \Omega + \hat{R}_1 (\Delta \hat{T}_2 + V^T \Delta T_2 U) + (\Delta \hat{T}_1 + U^T (\Delta T_1)^T V) R_2 + \Delta \Omega + O(\epsilon^2).
\]

This implies

\[
\| \hat{\Omega} - \Omega \| \leq \| R_1 \|_2 (\| \Delta T_2 \|_2 + \| \Delta T_2 \|_2) + (\| \Delta \hat{T}_1 \|_2 + \| \Delta T_1 \|_2) \| R_2 \|_2 + \| \Delta \Omega \|_2 + O(\epsilon^2).
\]
But now \(\|R_1\|_2 = \|U^T T_1^T V\|_2 = \|T_1^T\|_2 \leq 1\) and, similarly, \(\|R_2\|_2 \leq 1\). Moreover, we have \(\|\Delta \Omega\|_2 \leq c_3(n)\epsilon\), where \(c_3(n) \approx 6\sqrt{n}\). Therefore,
\[
\|\hat{\Omega} - \Omega\|_2 \leq (c_1(n) + c_2(n)) \left( 1 + \frac{2\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \cdot \epsilon \right) + 2 \cdot \frac{2\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \cdot \epsilon + c_5(n)\epsilon + O(\epsilon^2)
\]
\[
= \left( c_4(n) + \frac{4\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \right) \cdot \epsilon + O(\epsilon^2),
\]
where \(c_4(n) = c_1(n) + c_2(n) + c_3(n)\).

Similarly, if we let \(\hat{D}\) be the computed
\[
\hat{D} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}
\]
in step 2 of Algorithm 3.2, then
\[
\hat{D} = W^T \hat{\Omega} W + \Delta_D = W^T (\Omega + \hat{\Omega} - \Omega) W + \Delta_D
\]
where \(\|\Delta_D\|_2 \leq c_5(n)\cdot \epsilon\), and \(c_5(n)\) is a polynomial in \(n\) of low degree. Thus, we have
\[
\|\hat{D} - D\|_2 \leq \left( c_6(n) + \frac{4\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \right) \cdot \epsilon + O(\epsilon^2),
\]
where \(c_6(n) = c_4(n) + c_5(n)\). So, to determine whether \(D \geq 0\), we exploit the perturbation bound (3.16). We say \(D \not\geq 0\) if for some \(k \in \{1, 2, \ldots, n\}\),
\[
d_k < - \left( c_6(n) + \frac{4\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \right) \epsilon.
\]
That is, we say \(T_2 T_1^{-1} \not\geq 0\) if (3.17) holds. If for all \(j \in \{1, 2, \ldots, n\}\),
\[
d_j \geq - \left( c_6(n) + \frac{4\sqrt{n}\|S_\infty(\tilde{r})\|_2}{\delta(\tilde{r})} \right) \epsilon,
\]
then we say \(D \geq 0\). Notice that we may replace \(\delta(\tilde{r})\) with \(\min_{\lambda, \mu \in \Lambda_- (S_\infty(\tilde{r}))} (|\frac{1}{\lambda}| - |\mu|)\) and \(c_6(n)\) with \(n\) in practical use.

Now we consider the problem of how to judge whether
\[
\lambda_{\text{max}} \left( B_1 B_1^T T_2, T_1 + B_2 B_2^T T_2 \right) < \tilde{r}^2
\]
in Step 2.2 of Algorithm 3.1. By the QZ method, the computed eigenvalue \(\hat{\lambda}_{\text{max}}\) of the perturbed matrix pair \((\hat{B}_1 \hat{B}_1^T T_2, \hat{T}_1 + \hat{B}_2 \hat{B}_2^T T_2)\) satisfies
\[
\left| \hat{\lambda}_{\text{max}} \left( \hat{B}_1 \hat{B}_1^T T_2, \hat{T}_1 + \hat{B}_2 \hat{B}_2^T T_2 \right) - \lambda_M \right| \leq \epsilon \cdot \frac{\|B_1 B_1^T T_2\|_2 + |\lambda_M| \cdot \|T_1 + B_2 B_2^T T_2\|_2}{\text{cond} (\lambda_M)},
\]
where $\lambda_M = \lambda_{\max}(\widetilde{B}_1 B_1^T T_2, T_1 + \widetilde{B}_2 B_2^T T_2)$, and $\text{cond}(\lambda_M)$ means the condition number of $\lambda_M$ [24]. Notice that

$$B_1 B_1^T T_2 = (B_1 + \Delta_{B_1}) (B_1 + \Delta_{B_1})^T (T_2 + \Delta T_2) = B_1 B_1^T T_2 + \Delta J_1,$$

$$T_1 + \widetilde{B}_2 B_2^T T_2 = (T_1 + \Delta T_1) + (\widetilde{B}_2 + \Delta_{B_2}) (B_2 + \Delta_B) (T_2 + \Delta T_2) = T_1 + \widetilde{B}_2 B_2^T T_2 + \Delta J_2,$$

where

$$\|\Delta_{B_1}\|_2 \leq \|B_1\|_2 \cdot \epsilon, \quad \|\Delta_{B_2}\|_2 \leq \|B_2\|_2 \cdot \epsilon,$$

and $\|\Delta_{T_1}\|_2$, $\|\Delta_{T_2}\|_2$ satisfy (3.13). Therefore,

$$\|\Delta J_1\|_2 \leq 2 \|B_1\|_2 \cdot \left(\|T_2\|_2 + \frac{\sqrt{n}}{\delta(\hat{r})} \cdot \|S_\infty(\hat{r})\|_2\right) \cdot \epsilon + O(\epsilon^2)$$

and

$$\|\Delta J_2\|_2 \leq 2 \left(\|B_2\|_2 \cdot \|T_2\|_2 + \left(1 + \|B_2\|_2^2\right) \cdot \frac{\sqrt{n}}{\delta(\hat{r})} \cdot \|S_\infty(\hat{r})\|_2\right) \cdot \epsilon + O(\epsilon^2).$$

Thus,

$$\hat{\lambda}_{\max}(\widetilde{B}_1 B_1^T T_2, T_1 + \widetilde{B}_2 B_2^T T_2) - \lambda_M \lesssim \epsilon \cdot \delta_1 + O(\epsilon^2),$$

where

$$\delta_1 = \frac{\|B_1 B_1^T T_2\|_2 + \|B_2\|_2 \cdot \|T_1 + T_2 B_2^T T_2\|_2}{\text{cond}(\lambda_M)}.$$

And

$$\lambda_{\max}(\widetilde{B}_1 B_1^T T_2, T_1 + \widetilde{B}_2 B_2^T T_2) - \lambda_{\max}(B_1 B_1^T T_2, T_1 + \tilde{B}_2 B_2^T T_2) \lesssim \frac{\|\Delta_{J_1}\|_2 + \|\Delta_{J_2}\|_2}{\text{cond}(\lambda_M)} + O(\epsilon^2) \leq \epsilon \cdot \delta_2 + O(\epsilon^2),$$

where

$$\delta_2 = 2 \cdot \frac{\|T_2\|_2 \cdot (\|B_1\|_2^2 + \|B_2\|_2^2) + (\|B_1\|_2^2 + \|B_2\|_2^2 + 1) \cdot \frac{\sqrt{n}}{\delta(\hat{r})} \cdot \|S_\infty(\hat{r})\|_2}{\text{cond}(\lambda_M)}.$$

So, by (3.18) and (3.19), we have

$$\lambda_{\max}(B_1 B_1^T T_2, T_1 + \tilde{B}_2 B_2^T T_2) \sim \hat{\lambda}_{\max}(B_1 B_1^T T_2, T_1 + \tilde{B}_2 B_2^T T_2) \sim \hat{\epsilon}^2.$$

Thus we say $\lambda_{\max}(B_1 B_1^T T_2, T_1 + \tilde{B}_2 B_2^T T_2) < \hat{\epsilon}^2$ if

$$\hat{\epsilon}^2 - \lambda_{\max}(B_1 B_1^T T_2, T_1 + \tilde{B}_2 B_2^T T_2) > -(\delta_1 + \delta_2) \cdot \epsilon.$$
Notice that we may replace \( \text{cond}(\lambda_M) \) with \( \|B_1B_1^T T_2\|_2 + \|T_1 + B_2B_2^T T_2\|_2 \) in practical use.

In step 3 of Algorithm 3.1, we update the iteration \( \tilde{r} \) as follows. If the bisection method is adopted, we let

\[
\tilde{r} = \frac{r_- + r_+}{2}.
\]

If the secant method is adopted, we separate the discussion into three cases. For the first case with \textit{case}# = 0, we have \( r_x = \|H_Z\|_\infty \) (where \( H_Z \) is defined in (3.3)) and we consider

\[
\tilde{r} = r_+ + \frac{(r_+ - r_-) \cdot \min_{\theta_j \neq \theta_k : \theta_j, \theta_k \in \ell_0(r_-)} |\theta_j - \theta_k|}{\min_{\lambda \in \Lambda(S_\infty(r_+))} \{ |\lambda|^2 - 1 \} + \min_{\theta_j \neq \theta_k : \theta_j, \theta_k \in \ell_0(r_-)} |\theta_j - \theta_k|},
\]

which is the secant method applied to the function

\[
h(r) = \begin{cases} 
- \min_{\theta_j \neq \theta_k : \theta_j, \theta_k \in \ell_0(r)} |\theta_j - \theta_k| & \text{if } r \leq \|H_Z\|_\infty, \\
\min_{\lambda \in \Lambda(S_\infty(r))} \{ |\lambda|^2 - 1 \} & \text{if } r > \|H_Z\|_\infty,
\end{cases}
\]

where

\[
\ell_0(r) = \left\{ \theta = \tan^{-1} \frac{b}{a} \mid a + bi \in \Lambda(S_\infty(r)) \cap \partial D, a, b \in \mathbb{R} \right\}.
\]

There are two reasons for choosing \( h \) in this form. The first one is that there is an eigenvalue curve of \( S_\infty(r) \) that attaches to the unit circle as \( r \) decreases from \( \infty \) to \( \|H_Z\|_\infty \) (notice that we have assumed \( S_\infty(r) \) is not singular throughout the section, so that the situation that occurred in Example 2.4 could not happen here). The second reason that the angle between the nearest two unimodular eigenvalues of \( S_\infty(r) \) approaches zero as \( r \) increases to \( \|H_Z\|_\infty \) because there are multiple unimodular eigenvalues for the real symplectic pair \( S_\infty(r) \) at \( r = \|H_Z\|_\infty \).

When \textit{case}# = 1, we expect that \( r_x = r_s \) (where \( r_s \) is defined in Theorem 3.2) and we consider

\[
\tilde{r} = r_+ + \frac{(r_+ - r_-) \sigma_{\min}(T_1(r_-))}{\sigma_{\min}(T_1(r_+)) + \sigma_{\min}(T_1(r_-))},
\]

which is the secant method applied to the function \( f \) defined in (3.5). Note that even though (3.22) involves the computation of the smallest singular value of \( T_1 \), we can use the cheaper URV decomposition [21] to achieve this to avoid the costly singular value decomposition. When \textit{case}# = 2, we consider

\[
\tilde{r} = r_- + \frac{(r_+ - r_-) g(r_-)}{g(r_+) + g(r_-)},
\]

where \( g \) is defined in (3.7).

Now we give stop criteria for step 4 of Algorithm 3.1.

If \textit{case}# = 0, (3.21) is adopted to get the new \( \tilde{r} \). When \( r < \|H_Z\|_\infty \), let

\[
\lambda(r) = \cos \theta(r) + i \sin \theta(r) \in \Lambda(S_\infty(r)) \cap \partial D.
\]
Then we have \(|\dot{\theta}(r)| = |\dot{\lambda}(r)|\), and the computed \(\dot{\theta}_j\) and \(\dot{\theta}_k\) satisfy (if smoothness conditions are imposed)

\[
\left| \left( \dot{\theta}_j(r_-) - \dot{\theta}_k(r_-) \right) - (\theta_j(r_-) - \theta_k(r_-)) \right| \leq \left( \left| \dot{\theta}_j(r_-) \right| + \left| \dot{\theta}_k(r_-) \right| \right) \cdot \epsilon + O(\epsilon^2)
\]

\[
= \left( \left| \dot{\lambda}_j(r_-) \right| + \left| \dot{\lambda}_k(r_-) \right| \right) \cdot \epsilon + O(\epsilon^2).
\]

Therefore, in view of the definition of \(h(r)\), (3.9), (3.10), and (3.11) suggest the stop criterion: If

\[
h(\tilde{r}) < \min_{\theta_j \neq \theta_k; \theta_j, \theta_k \in h_0(r_-)} \left\{ 4\epsilon \cdot (\delta_{\theta_j} + \delta_{\theta_k}) \right\}
\]

or

\[
h(\tilde{r}) < \min_{\lambda(r_+) \in \Lambda(S_\infty(r_+))} \delta_{\lambda(r_+)},
\]

then accept \(r_+ = r_+\), where

\[
\delta_{\theta_j} = \sqrt{\left| 1 - e^{i\theta_j} \right|^2 + \left| 1 + e^{i\theta_j} \right|^2} \cdot \min \left\{ \frac{\| [E(r_-), A(r_-)] \|_2}{\| 1 - e^{i\theta_j} \| \cdot \| E(r_-) \|_2}, \frac{\| [\hat{E}(r_-), \hat{A}(r_-)] \|_2}{\| 1 + e^{i\theta_j} \| \cdot \| \hat{E}(r_-) \|_2} \right\}
\]

for each \(j\).

If case \#1, (3.22) is adopted to compute the next iteration \(\tilde{r}\). Let \(\tilde{T}_1 = T_1 + \Delta T_1\) be the computed \(T_1\). Consider the singular value decomposition of \(\tilde{T}_1\):

\[
\tilde{U}_{T_1} \tilde{T}_1 \tilde{V}_{T_1} + \Delta_{SVD} = \tilde{U}_{T_1} (T_1 + \Delta_{T_1}) \tilde{V}_{T_1} + \Delta_{SVD}.
\]

The round-off error inherited is less than or equal to \(\| \Delta_{T_1} \|_2 + \| \Delta_{SVD} \|_2\). By (3.13), we have

\[
\| \Delta_{T_1} \|_2 + \| \Delta_{SVD} \|_2 \leq \left( c_7(n) + \frac{2\sqrt{n} \| S_\infty(\tilde{r}) \|_2}{\delta(\tilde{r})} \right) \cdot \epsilon,
\]

where \(c_7(n)\) is a polynomial in \(n\) of low degree. Thus, in practical implementation, we may accept \(r_+ = r_+\) if

\[
\sigma_{\min} (\tilde{T}_1(r_+)) < \left( n + \frac{2\sqrt{n} \| S_\infty(r_+) \|_2}{\delta(r_+)} \right) \cdot \epsilon
\]

or

\[
\sigma_{\min} (\tilde{T}_1(r_-)) < \left( n + \frac{2\sqrt{n} \| S_\infty(r_-) \|_2}{\delta(r_-)} \right) \cdot \epsilon.
\]

If case \#2, (3.23) is used to update \(\tilde{r}\). So according to (3.20) we may accept \(r_+ = r_+\) if

\[
|\tilde{r}^2 - \lambda_{\max} \left( B_1 \tilde{B}_1^T T_2, T_1 + B_2 \tilde{B}_2^T T_2 \right)| < (\delta_1 + \delta_2) \cdot \epsilon.
\]

Finally, we do double check in step 5 of Algorithm 3.1 for pushing the final result away from \(r_+\) obtained in step 4. At \(r = r_+\), \(T_1(r_+)\) may be ill-conditioned (in case
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case $n=1$) and therefore the corresponding Riccati solution $X(r) := T_2(r)T_1^{-1}(r)$ will become undesirable in engineering. So, instead of obtaining the true solution, we further find $\Delta r \geq 0$ such that $T_1(r_\ast + \Delta r)$ is well-conditioned and export $r_x \approx r_\ast + \Delta r$. In view of (3.24), we say that $T_1(r)$ is well-conditioned if

$$
\sigma_{\min}(T_1(r)) \geq L \equiv \sqrt{n + \frac{2\sqrt{n} \|S_\infty(r_\ast)\|_2}{\delta(r_\ast)}} \cdot \epsilon.
$$

One can easily solve the equation

(3.25)

$$
\sigma_{\min}(T_1(r_\ast + \Delta r)) = L
$$

to get the critical point $\Delta r \geq 0$, provided $\sigma_{\min}(T_1(r_\ast)) < L$.

3.3. Examples. This section gives some algebraic verifiable examples as well as the associated numerical results for the last example. To be more concise and to show the effect of using secant method, we list only the iterations generated by steps 1–4 of Algorithm 3.1 and omit the final double check. Furthermore, to present the key intermediate iterations, we manually assign $\|H_Z\|_\infty$ (which is algebraic available for each of the following examples) in the initialization step. The last example is experimented using PC-MATLAB. The following notations are used throughout:

$$
G(z) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix};
$$

$Z$ is the stabilizing solution of Riccati equation (3.1); $H_Z$ is defined in (3.3); $S_\infty(r) := (S_{\infty_1}(r), S_{\infty_2}(r))$ and $S_Z(r) := (S_{Z_1}(r), S_{Z_2}(r))$ are defined in (1.1) and (3.2), respectively; $X(r) := \text{Ric}(S_\infty(r))$ whenever it exists; $R(r) = \begin{bmatrix} -r^2 I_m & 0 \\ 0 & t_{m2} \end{bmatrix}$ and $B = [B_1 B_2]$; $r_x$ denotes the infimum of $r$ such that condition (C1) in section 1 holds.

Example 3.1. We first look at the simple example with

$$
G(z) = \begin{bmatrix}
1/2 & [1 0 1] \\
[1 0 0] & [0 1 0]
\end{bmatrix},
$$

which is open-loop stable. It can be shown that $Z = (\sqrt{65} + 1)/8$, and

$$
S_Z(r) = \begin{bmatrix}
\frac{9 - \sqrt{65}}{4} & 0 \\
-1 & 1
\end{bmatrix} \begin{bmatrix} 1 & \frac{1}{r^2} \left(\frac{\sqrt{13} - \sqrt{5}}{2}\right)^2 \\
0 & \frac{9 - \sqrt{65}}{4}
\end{bmatrix}.
$$

Therefore, according to Example 2.1, we have

$$
\|H_Z\|_\infty = \sqrt{\frac{4}{5}} \approx 8.94427190999916 \times 10^{-1}.
$$

Direct computation shows that

$$
X(r) = \begin{cases}
\frac{8}{-1 + \frac{1}{r^2} + \sqrt{(9 - \frac{1}{r^2})^2 - 16}} > 0 & \text{if } r > \sqrt{\frac{4}{5}}, \\
\frac{8}{-1 + \frac{1}{r^2} - \sqrt{(9 - \frac{1}{r^2})^2 - 16}} > 0 & \text{if } 0 < r < \sqrt{\frac{4}{13}}.
\end{cases}
$$
When \( X(r) > 0, R(r) + B^T X(r) B \) is congruent to
\[
\text{diag} \left\{ \begin{array}{c} -r^2, \frac{X(r)}{X(r) + 1} - r^2, X(r) + 1 \end{array} \right\}.
\]
Notice that
\[
\frac{X(r)}{X(r) + 1} - r^2 \begin{cases} < 0 & \text{if } r > \sqrt{\frac{3}{5}}, \\ > 0 & \text{if } 0 < r < \sqrt{\frac{4}{13}}. \end{cases}
\]
Thus we see from the definition of \( r_x \) that
\[
r_x = \sqrt{\frac{4}{5}} \approx 8.94427190999916 \times 10^{-1}.
\]
This example illustrates the importance of the lower bound \( \| H_Z \|_\infty \). It may happen that \( r_x = \| H_Z \|_\infty \) as in this example.

**Example 3.2.** Consider the open-loop unstable system
\[
G(z) = \begin{bmatrix} A & [I \ 0] & I \\ [I] & 0 & 0 \\ [0 \ I] & 0 \end{bmatrix}
\]
with
\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
It can be shown that \( Z = \frac{\sqrt{5} - 1}{3 - \sqrt{5}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \), and
\[
S_Z(r) = \begin{bmatrix} A_Z & 0 \\ -I & I \end{bmatrix}, \quad \begin{bmatrix} I & -\frac{1}{r^2} B_Z B_Z^T \\ 0 & A_Z^T \end{bmatrix},
\]
where
\[
A_Z = \left( \frac{3 - \sqrt{5}}{2} \right) \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_Z = \left( \frac{\sqrt{5} - 1}{2} \right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Therefore, according to Example 2.2, we have
\[
\| H_Z \|_\infty = 1.
\]
Direct computation shows that
\[
X(r) = \begin{cases} \frac{2 r^2}{1 - r^2 + \sqrt{5} r^2 + 1} \cdot I > 0 & \text{if } r > 1, \\ \frac{2 r^2}{1 - r^2 - \sqrt{5} r^2 + 1} \cdot I > 0 & \text{if } 0 < r < \frac{1}{\sqrt{5}}. \end{cases}
\]
When $X(r) > 0$, $R(r) + B^T X(r) B$ is congruent to
$$\text{diag}\left\{ -r^2 I, (1 - r^2)X(r) - r^2 I, X(r) + I \right\}.$$ Notice that
$$\begin{cases} (1 - r^2) X(r) - r^2 I < 0 & \text{if } r > 1, \\ > 0 & \text{if } 0 < r < \frac{1}{\sqrt{5}} \end{cases}.$$ Thus we have
$$r_x = 1 = \|H_Z\|_{\infty}.$$ This example illustrates that it still may happen that $r_x = \|H_Z\|_{\infty}$ even for an open-loop unstable system.

Example 3.3. Now we replace the matrix $A$ in Example 3.2 with
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ and keep the other system matrices the same. Then it can be shown that
$$Z = \begin{bmatrix} \frac{\sqrt{5} + 3}{2} & \frac{\sqrt{5} + 1}{2} \\ \frac{\sqrt{5} + 1}{2} & \frac{\sqrt{5} + 3}{2} \end{bmatrix}$$ and
$$S_Z(r) = \left( \begin{bmatrix} \tilde{A}_Z & 0 \\ -I & I \end{bmatrix}, \begin{bmatrix} I & \frac{1}{r} \tilde{B}_Z \tilde{B}_Z^T \end{bmatrix} \right),$$ where
$$\tilde{A}_Z = \begin{bmatrix} 3 - \sqrt{5} \\ \frac{4}{2} \end{bmatrix}, \quad \tilde{B}_Z = \sqrt{5 - \sqrt{5}} \begin{bmatrix} \frac{1}{2 \sqrt{5}} & 0 \\ \frac{1}{10} \sqrt{2} & 0 \end{bmatrix}.$$ Therefore, according to Example 2.3, we have
$$\|H_Z\|_{\infty} = \sqrt{\frac{1}{2}} \approx 0.7071067811865476 \times 10^{-1}.$$ Note that Example 2.3 also shows that $S_Z(\frac{1}{\sqrt{2}})$ is a singular pair.

For $r > \sqrt{\frac{1}{2}}$, the two vectors $[1, -1, -1, -1]^T$ and $[\alpha(r), \alpha(r), 1, 1]^T$ form a basis for the stable deflating subspace of $(S_{\infty_1}(r), S_{\infty_2}(r))$, where
$$\alpha(r) = \frac{4r^2 - 1 - \sqrt{20r^4 - 12r^2 + 1}}{-2r^2}.$$ However, $\begin{bmatrix} \alpha(r) & 1 \\ \alpha(r) & -1 \end{bmatrix}$ is invertible if and only if $r \neq 1$. So, when $r > \sqrt{\frac{1}{2}}$, $X(r) := \text{Ric}(S_{\infty}(r))$ exists except for $r = 1$. Define
$$\beta(r) = \frac{4r^2 - 1 + \sqrt{20r^4 - 12r^2 + 1}}{-2r^2}.$$
Direct computation shows that

\[
X(r) = \begin{cases} 
\frac{1}{2\alpha(r)} \begin{bmatrix} 1 + \alpha(r) & 1 - \alpha(r) \\ 1 - \alpha(r) & 1 + \alpha(r) \end{bmatrix} & \text{if } \sqrt{\frac{1}{2}} < r \neq 1, \\
\frac{1}{2\beta(r)} \begin{bmatrix} 1 + \beta(r) & 1 - \beta(r) \\ 1 - \beta(r) & 1 + \beta(r) \end{bmatrix} & \text{if } 0 < r < \sqrt{\frac{1}{10}}.
\end{cases}
\]

Since

\[
\alpha(r) \in \begin{cases} 
(0, 1) & \text{if } r > 1, \\
(-1, 0) & \text{if } \sqrt{\frac{1}{2}} < r < 1,
\end{cases}
\]

and \(\beta(r) > 1\) as \(0 < r < \sqrt{\frac{1}{10}}\), it can be shown that

\[
X(r) \sim \begin{cases} 
\frac{1}{2\alpha(r)} \begin{bmatrix} 4\alpha(r) & 0 \\ 0 & 1 + \alpha(r) \end{bmatrix} \begin{bmatrix} > 0 \\ \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} & \text{if } r > 1, \\
\frac{1}{2\beta(r)} \begin{bmatrix} 4\beta(r) & 0 \\ 0 & 1 + \beta(r) \end{bmatrix} > 0 & \text{if } 0 < r < \sqrt{\frac{1}{10}},
\end{cases}
\]

where \(\sim\) denotes the congruence relation.

On the other hand, observe the \((2,2)\) element of \(X(r) - \frac{r^2}{1-r^2}I\) when \(0 < r < \sqrt{\frac{1}{10}}\). It can be shown that

\[
\frac{1 + \beta(r)}{2\beta(r)} - \frac{r^2}{1-r^2} > 0 \quad \text{if } 0 < r < \sqrt{\frac{1}{10}}.
\]

Therefore \((1 - r^2)X(r) - r^2I \neq 0\) when \(0 < r < \sqrt{\frac{1}{10}}\). In addition, using \(X(r) > 0\) as \(r > 1\), it is easy to see that \((1 - r^2)X(r) - r^2I > 0\) when \(r > 1\). Thus, similar to Example 3.2, we conclude

\[
r_x = 1 > \|H_Z\|_\infty = \sqrt{\frac{1}{2}}
\]

for this example. Notice that \(r_x = r_s\) in this example, where \(r_s\) is defined in Theorem 3.2.

Below, we list the iterations generated by steps 1–4 of Algorithm 3.1. We use \(\|H_Z\|_\infty (= \sqrt{\frac{1}{2}})\) as the lower bound \(r_-\). As to the upper bound \(r_+\), we consider \(r_+ = 2\) for illustration (if we use the good bound (3.8), we immediately get the answer). The notation \(|\Delta \hat{r}|\) denotes the gap between the last iteration \(\hat{r}\) and the newly updated one.
4. Solution of the main problem. Now we return to the discussion of the main problem. We want to find the infimum $r^*$ of $r$ such that conditions (C1)–(C3) of section 1 hold simultaneously. Because we have established algorithms for solving (C1) (dually, (C2)), it remains to consider (C3). The remaining part of this work follows [16, section 3]. Below, for completeness of the paper, we excerpt only the main points from [16].

Let $r_x$ (respectively, $r_y$) be the computed infimum of $r$ that satisfies (C1) (respectively, (C2)). And let $r_m = \max\{r_x, r_y\}$. Then $r_m$ is a lower bound of $r^*$. For every $r \geq r_m$, define $\rho(r) = \rho(X(r)Y(r))$. Using the techniques of [6] and [11], it can be shown that $\rho(r)$ is a decreasing function of $r$ on the interval $[r_m, \infty)$. The graph of $\rho$ is shown in Figure 6, where we also depict the graph of $y(r) = r^2$. Our goal is to find $r^*$, at which the graph of $y$ intersects the graph of $\rho$. So, if we know in advance that $\rho(r_m) \leq r_m^2$ (which is not the case shown in Figure 6), we conclude $r^* = r_m$; otherwise, as shown in Figure 6, we get an upper bound $\sqrt{\rho(r_m)}$ and a lower bound $r_m$ of $r^*$. In the latter case, we can further apply the bisection or the secant method to find the zero of the function $\rho(r) - r^2$ on the interval $[r_m, \sqrt{\rho(r_m)}]$ to get $r^*$.

The major part of the numerical errors arising from the use of the above method mainly comes out with the evaluation of $\rho(X(r)Y(r))$. To avoid the explicit computation of $X(r)$ and $Y(r)$, consider the deflating subspaces $\mathcal{X}_-(S_{\infty}(r))$ and $\mathcal{X}_-(T_{\infty}(r))$, where $S_{\infty}(r)$ and $T_{\infty}(r)$ are defined in (1.1) and (1.2), respectively. We first find
$X_1(r), X_2(r), Y_1(r), Y_2(r) \in \mathbb{R}^{n \times n}$ such that

$$\text{Im} \left( \begin{bmatrix} X_1(r) \\ X_2(r) \end{bmatrix} \right) = \mathcal{X}_- (S_\infty(r)) \quad \text{and} \quad \text{Im} \left( \begin{bmatrix} Y_1(r) \\ Y_2(r) \end{bmatrix} \right) = \mathcal{X}_- (T_\infty(r)).$$

Then apply the periodic QZ-algorithm [2], [10] to find orthogonal matrices $U_1, U_2, V_1,$ and $V_2$ such that

$$U_1X_2(r)V_2 = \tilde{X}_2 = \begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix}, \quad U_2Y_2(r)V_1 = \tilde{Y}_2 = \begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix},$$

$$U_2X_1(r)V_2 = \tilde{X}_1 = \begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix}, \quad \text{and} \quad U_1Y_1(r)V_1 = \tilde{Y}_1 = \begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix},$$

are all upper triangular. Using this, we see that

$$X(r)Y(r) = X_2(r)X_1^{-1}(r)Y_2(r)Y_1^{-1}(r)$$

is similar to $\tilde{X}_2 \tilde{X}_1^{-1} \tilde{Y}_2 \tilde{Y}_1^{-1}$. Thus, one can easily get $\rho(X(r)Y(r))$ from the diagonal elements of the upper triangular matrices $\tilde{X}_1, \tilde{X}_2, \tilde{Y}_1,$ and $\tilde{Y}_2$. This avoids the instability of matrix inversion and the computation of eigenvalues.

Let $X_1, X_2, Y_1,$ and $Y_2$ be the computed $\tilde{X}_1, \tilde{X}_2, \tilde{Y}_1,$ and $\tilde{Y}_2$, respectively. Using [10] we have

$$\hat{X}_k = \tilde{X}_k + \Delta X_k \quad \text{and} \quad \hat{Y}_k = \tilde{Y}_k + \Delta Y_k,$$

where

$$\|\Delta X_k\|_2 \leq c_{X_k}(n) \|X_k(r)\|_2 \cdot \epsilon \quad \text{and} \quad \|\Delta Y_k\|_2 \leq c_{Y_k}(n) \|Y_k(r)\|_2 \cdot \epsilon$$

for $k = 1, 2$, and $c_{X_k}(n)$'s, $c_{Y_k}(n)$'s are low-degree polynomials in $n$. Thus the computed spectral radius $\hat{\rho}(X(r)Y(r))$ of $X(r)Y(r)$ satisfies

$$|\hat{\rho}(X(r)Y(r)) - \rho(X(r)Y(r))| \leq c_\rho(n) \frac{\|X(r)Y(r)\|_2}{\|v(r)^T u(r)\|} \cdot \epsilon,$$

where $u(r), v(r)$ are, respectively, the right and left eigenvectors of $X(r)Y(r)$ that correspond to the eigenvalue $\lambda = \rho(X(r)Y(r))$, and $c_\rho(n)$ is a low-degree polynomial in $n$. Based on (4.1), below we give a stop criterion for using the bisection or the secant method to find the zero of $\rho(r) - r^2$.

Let $r^*$ denote the root (if it exists) of $\rho(r) - r^2$. Assume that $|\hat{r} - r^*| < r^* \cdot \epsilon$; here $\hat{r}$ denotes the iteration. Let

$$\Phi_X = \left( I + \left( B_2 B_2^T - \frac{1}{\hat{r}^2} B_1 B_1^T \right) X_\hat{(r)} \right)^{-1} A,$$

$$\Phi_Y = \left( I + \left( C_2 C_2^T - \frac{1}{\hat{r}^2} C_1 C_1^T \right) Y_\hat{(r)} \right)^{-1} A^T,$$

$$K_X = X_\hat{(r)} \Phi_X, \quad K_Y = Y_\hat{(r)} \Phi_Y,$$

$$T_X = I_{n^2} - \Phi_X^T \otimes \Phi_X \quad \text{and} \quad T_Y = I_{n^2} - \Phi_Y^T \otimes \Phi_Y,$$

$$q_X = \|T_X^{-1} (K_X^T \otimes K_X)\|_2 \quad \text{and} \quad q_Y = \|T_Y^{-1} (K_Y^T \otimes K_Y)\|_2.$$ 

Then from [22] we have

$$\|X(r) - X(r^*)\|_2 \leq \frac{2}{\hat{r}^2} \cdot q_X \|B_1\|^2_2 \cdot \epsilon$$

in (4.2).
and
\[ (4.3) \quad \|Y(\hat{r}) - Y(r^*)\|_2 \leq \frac{2}{\sqrt{r}} \cdot q_Y \|C_1\|_2^2 \cdot \epsilon. \]

Therefore, by [8, Theorem 7.2.3], (4.2), and (4.3),
\[ |\rho(X(\hat{r})Y(\hat{r})) - \rho(X(r^*)Y(r^*))| \leq \kappa \cdot \|X(\hat{r})Y(\hat{r}) - X(r^*)Y(r^*)\|_2 \]
\[ \leq \frac{2\kappa}{\sqrt{r}} \left( q_x \|B_1\|_2^2 \|Y(\hat{r})\|_2 + q_Y \|C_1\|_2^2 \|X(\hat{r})\|_2 \right) \epsilon, \]

where \( \kappa \) is some constant. Thus, using (4.1), (4.4), and the fact \( (r^*)^2 = \rho(X(r^*)Y(r^*)) \), we immediately get
\[ \frac{2\kappa}{\sqrt{r}} \left( q_x \|B_1\|_2^2 \|Y(\hat{r})\|_2 + q_Y \|C_1\|_2^2 \|X(\hat{r})\|_2 \right) \epsilon. \]

So we may use the right-most side of inequality (4.5) as our stop criterion. In practical implementation, we may choose \( c_p(n) = n, \kappa = n^2 \), and \( |v(\hat{r})^T u(\hat{r})| = 1 \).

Now we continue the analysis of the three examples given in section 3.3 and show their respective numerical results. All examples are tested using PC-MATLAB. The notations \( r_x, r_y, r_m \), and \( r^* \) are defined at the very beginning of the section. \( X(r) \) and \( Y(r) \) are defined as \( X(r) = Ric(S_\infty(r)) \), \( Y(r) = Ric(T_\infty(r)) \), where \( S_\infty(r) \) and \( T_\infty(r) \) are given in (1.1) and (1.2), respectively. As before, we manually assign the lower bound \( r_m \) (to save the listing of iterations for obtaining \( r_y \) for each example. We switch the approximation method from bisection to secant method when the relative error between two consecutive iterations is less than \( 10^{-3} \). Note that in general we could not determine the radius of convergence for the secant method. Therefore the threshold check mentioned in section 3 is needed for general problem. Here we use \( 10^{-3} \) as the threshold and list only the secant iterations.

**Example 4.1.** We first look at the control system \( G(z) \) given by Example 3.1. Since \( S_\infty(r) = T_\infty(r) \) for this example, it follows from Example 3.1 that
\[ r_m = r_x = r_y = \sqrt{\frac{4}{5}} \approx 0.894427190999916 \times 10^{-1}. \]

Furthermore,
\[ X(r) = Y(r) = \frac{8}{-1 + \frac{4}{r^2} + \sqrt{(9 - \frac{4}{r^2})^2 - 16}} > 0 \quad \text{for} \quad r > r_m. \]

Thus, when \( r > r_m \), \( \rho(X(r)Y(r)) < r^2 \) if and only if
\[ \frac{8}{-1 + \frac{4}{r^2} + \sqrt{(9 - \frac{4}{r^2})^2 - 16}} < r, \]
which holds if and only if

\[
 r > \frac{1}{12} \left( \sqrt[3]{-719 + 24 \sqrt{687i}} + \frac{97}{\sqrt[3]{-719 + 24 \sqrt{687i}}} + 1 \right) 
n \approx 1.21792178858753.
\]

Therefore we have \( r^* \approx 1.21792178858753 \). Below, we list the secant iterations with lower and upper bounds given by

1.21616614072869 and 1.21724580163382

(which are exhibited when the bisection method stops with a relative error less than 1.0e-3), respectively. The secant method stops with \(|\rho(X(\tilde{r})Y(\tilde{r})) - \tilde{r}^2| < 1.0e-15\). \(|\Delta \tilde{r}|\) denotes the gap between the last iteration \( \tilde{r} \) and the newly updated one.

<table>
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<tr>
<td>Iteration</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>3</td>
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**Example 4.2.** Now we consider the control system \( G(z) \) of Example 3.2. Although \( S_\infty(r) \) and \( T_\infty(r) \) are not identical for this example, it can be shown that \( X(r) = Y(r) \) whenever they exist. So by Example 3.2 we have

\[
r_m = r_x = r_y = 1
\]

and

\[
 X(r) = Y(r) = \frac{2r^2}{1 - r^2 + \sqrt{5r^4 - 6r^2 + 1}} \cdot J > 0 \text{ for } r > r_m.
\]

Similar to Example 4.1, it can be shown that

\[
r^* = \frac{1}{6} \left( \sqrt[3]{-28 + 84 \sqrt{3i}} + \frac{28}{\sqrt[3]{-28 + 84 \sqrt{3i}}} + 2 \right) 
n \approx 1.801937735804838
\]

in this example. Below, we list the secant iterations with lower and upper bounds given by

1.800756480202620 and 1.801944546196986

(which are exhibited when the bisection method stops with a relative error less than 1.0e - 3), respectively. The secant method stops with \(|\Delta \tilde{r}| < 1.0e - 15\), where \(|\Delta \tilde{r}|\) denotes the gap between the last iteration \( \tilde{r} \) and the newly updated one.

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<td>3</td>
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</table>
Example 4.3. Finally, we consider the control system $G(z)$ of Example 3.3. Since $S_{\infty}(r) = T_{\infty}(r)$ for this example, it follows from Example 3.3 that

$$r_m = r_x = r_y = 1$$

and that

$$X(r) = Y(r) = \frac{1}{2\alpha(r)} \begin{bmatrix} 1 + \alpha(r) & 1 - \alpha(r) \\ 1 - \alpha(r) & 1 + \alpha(r) \end{bmatrix} > 0 \quad \text{for} \quad r > r_m,$$

where

$$\alpha(r) = \left(\frac{-1}{2r^2}\right) \left(4r^2 - 1 - \sqrt{20r^4 - 12r^2 + 1}\right) \in (0, 1) \quad \text{when} \quad r > r_m.$$  

Therefore, when $r > r_m$, $\rho(X(r)Y(r)) < r^2$ if and only if

$$\begin{bmatrix} 1 + (1 - 2r)\alpha(r) & 1 - \alpha(r) \\ 1 - \alpha(r) & 1 + (1 - 2r)\alpha(r) \end{bmatrix} < 0,$$

which holds if and only if

$$1 + (1 - 2r)\alpha(r) < 0$$

and

$$(1 - (1 - 2r)\alpha(r))^2 - (1 - \alpha(r))^2 > 0$$

hold simultaneously. It can be shown that the above two inequalities hold simultaneously if and only if

$$r > \frac{1}{6} \left( \sqrt{692 + 12\sqrt{1407i}} + \frac{88}{\sqrt{692 + 12\sqrt{1407i}}} + 8 \right)$$

$$\approx 4.402678829521188.$$  

Therefore we have $r^* \approx 4.402678829521188$. Below, we list the secant iterations with lower and upper bounds given by

$$4.405386183738079 \quad \text{and} \quad 4.402208659056787$$

(which are exhibited when the bisection method stops with a relative error less than $1.0e - 3$), respectively. The secant method stops with $|\rho(\tilde{r})Y(\tilde{r})) - \tilde{r}^2| < 1.0e - 15$. $|\Delta \tilde{r}|$ denotes the gap between the last iteration $\tilde{r}$ and the newly updated one.

**The secant method**

| Iteration | $\tilde{r}$ | $|\Delta \tilde{r}|$ |
|-----------|------------|-----------------|
| 1         | 4.402678730798692 | -               |
| 2         | 4.402678829500458 | 9.87e - 8       |
| 3         | 4.402678829521188 | 2.07e - 11      |
5. Concluding remarks. In this paper, we first review and give deeper insight into the computational work of the $H_\infty$ norm of the transfer function matrix in the discrete-time case. We pay attention to the singular case and construct four typical examples in section 2 to illustrate the geometrical relation between the $H_\infty$ norm of the transfer function and the eigenvalue curves of the associated symplectic pair. We use the results of section 2 as the basic tools for obtaining a preliminary and tight lower bound of our main problem. Next, we characterize the solution of the main problem directly in terms of the basis vectors of the stable deflating subspace of $S_\infty(r)$ and $T_\infty(r)$. Using the characterization, we devise multiple secant methods in sections 3 and 4 to solve the problem. In addition, we also construct three algebraic verifiable examples to compare with the numerical results. Numerical aspects are addressed throughout.

REFERENCES


