Chaotic synchronization in lattice of partial-state coupled Lorenz equations

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Abstract

In this paper, we study chaotic synchronization in lattices of coupled Lorenz equations with Neumann or periodic boundary condition. Three different coupling configurations in the single x_i -, y_i - or z_i -component are considered. Synchronization is affected by coupling rules. We prove that synchronization occurs for either x_i - or y_i -component coupling provided the coupling coefficient is sufficiently large. Moreover, we determine the dependence of coupling coefficients on the lattice size. For the case of the z_i -component coupling, we demonstrate by numerical experience that the synchronization cannot occur. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Chaotic synchronization is a fundamental phenomenon in physical systems with dissipation. It was first observed in [19] for identical master–slave Lorenz equations. Later, this phenomenon was observed in many different fields—physics, electrical engineering, biology, laser systems, etc. Experimental observations show that chaotic subsystems in a lattice manifest synchronized chaotic behavior in time provided they are coupled with a dissipative coupling and a coefficient of this coupling is greater than some critical value.

Recently, synchronization of coupled chaotic circuits [4,9,12], coupled chaotic oscillators [1,2,6,7,10,11,16,18] and master–slave chaotic Lorenz equations [15,19] has been proved and well studied. A mathematical foundation of synchronization of these coupled systems was heavily dependent on the bounded dissipativeness, the coupling rule and the type of chaotic subsystems. Thus, the problem of coming up with a rigorous mathematical proof of chaotic synchronization for specified coupled systems appears to be attractive and important from both theoretical and practical point of view.

In this paper, we devote ourselves to studying the chaotic synchronization in lattice of identical Lorenz equations with partial-state couplings. Here are some motivations for the study of synchronization of Lorenz equations coupled with specified coupling rules: (a) Lorenz equation, a well known, simple model of convection rolls in atmosphere

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arising in meteorology, has chaotic behavior over a wide range of parameters [13]. (b) In contrast to the full state coupling of subsystems with dissipation (see, e.g. [14]), in many applications, one investigates if synchronization occurs on a lattice of subsystems for which only some partial states (single components) of subsystems are allowed to be coupled. For instance, in self-pulsating laser diode equations, only the photon density can be coupled with the electron density of active region. Numerical experiences show that chaotic synchronization of this coupled system can be observed [17]. Thus, three different coupling configurations in the x_i -, y_i - or z_i -component of Lorenz equations are considered. (c) The invariant manifold method and the analysis techniques proposed by Carralho and Rodrigues [5] and He and Vaidya [15], respectively, cannot be directly used to prove the synchronization of partial-state coupled Lorenz equations and to determine the dependence of coupling coefficients on the lattice size. Thus, a mathematical proof of synchronized behavior should be treated differently.

As mentioned above, Lorenz equations on a lattice are coupled in the x_i -, y_i - or z_i -component with dissipation. In fact, Lorenz equations can be modeled by identical or nonidentical forms. For coupled nonidentical Lorenz equations, perfect synchronization cannot occur, instead of synchronization, one can observe “asymptotic synchronization” (see [14] for definition) provided full (x_i, y_i, z_i) -state coupling is taken [8].

We now consider coupled identical Lorenz equations on an $n \times n$ squared lattice with different single component coupling:

(I) coupled Lorenz equations with x_i -component:

$$\dot{x}_i = \sigma(y_i - x_i) + d(\Delta x)_i, \quad \dot{y}_i = \gamma x_i - y_i - x_i z_i, \quad \dot{z}_i = -bz_i + x_i y_i, \quad (1.1)$$

(II) coupled Lorenz equations with y_i -component:

$$\dot{x}_i = \sigma(y_i - x_i), \quad \dot{y}_i = \gamma x_i - y_i - x_i z_i + d(\Delta y)_i, \quad \dot{z}_i = -bz_i + x_i y_i, \quad (1.2)$$

(III) coupled Lorenz equations with z_i -component:

$$\dot{x}_i = \sigma(y_i - x_i), \quad \dot{y}_i = \gamma x_i - y_i - x_i z_i, \quad \dot{z}_i = -bz_i + x_i y_i + d(\Delta z)_i, \quad (1.3)$$

where the index $i \equiv (i_1, i_2)$ for $1 \leq i_1, i_2 \leq n$, the constants $\sigma > 0, \gamma > 0, b > 0$ are suitable parameters of Lorenz equation, d is the coupling coefficient, $x = (x_i)$, $y = (y_i)$ and $z = (z_i)$ are vectors in $\mathbb{R}^{n^2}$, and Δ is the discretized Laplacian operator given by

$$(\Delta u)_i = u_{(i_1+1, i_2)} + u_{(i_1, i_2+1)} + u_{(i_1-1, i_2)} + u_{(i_1, i_2-1)} - 4u_{(i_1, i_2)}.$$

We now impose boundary conditions for systems (1.1)–(1.3), respectively. For Neumann condition (NC) and periodic condition (PC), the discretized Laplacian operators Δ are of the following forms:

(a) NC case:

$$u_{(0, i_2)} = u_{(1, i_2)}, \quad u_{(n+1, i_2)} = u_{(n, i_2)}, \quad u_{(i_1, 0)} = u_{(i_1, 1)}, \quad u_{(i_1, n+1)} = u_{(i_1, n)}$$

for $1 \leq i_1, i_2 \leq n$. The operator Δ with NC has the matrix form:

$$\Delta = I_n \otimes \Delta_N + \Delta_N \otimes I_n \quad \text{with} \quad \Delta_N = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}. \quad (1.4)$$

Here the symbol $\otimes$ denotes the Kronecker product of matrices [3].

(b) PC case:

$$u_{(0,i_2)} = u_{(n,i_2)}, \quad u_{(n+1,i_2)} = u_{(1,i_2)}, \quad u_{(i_1,0)} = u_{(i_1,n)}, \quad u_{(i_1,n+1)} = u_{(i_1,1)}$$

for $1 \leq i_1, i_2 \leq n$. The operator Δ with PC has the matrix form:

$$\Delta = I_n \otimes \Delta_p + \Delta_p \otimes I_n \quad \text{with} \quad \Delta_p = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}. \quad (1.5)$$

In both cases, the operator Δ has all negative eigenvalues except one simple zero associated with the eigenvector $e = [1, \dots, 1]^T \in \mathbb{R}^{n^2}$ (see, e.g. [2]).

For Dirichlet boundary condition, i.e., $u_{(0,i_2)} = u_{(i_1,0)} = u_{(i_1,n+1)} = u_{(n+1,i_2)} = 0$ for $1 \leq i_1, i_2 \leq n$, the operator Δ has only negative eigenvalues. Hence, one can apply the main theorem in Section 3 to show that (1.1) and (1.2) converge to the trivial equilibrium provided the coupling coefficient is sufficiently large. For this case, the behavior of solutions is not so interesting.

Synchronization of master–slave Lorenz equations has been observed by Pecora and Carroll [19]. The results in [19] indicated that synchronization occurs when either x_i - or y_i -component of the slave system is driven by the same component of the master system. However, the synchronization cannot occur for the model with z_i -component driving. Correspondingly, in this paper, we mainly give a rigorous mathematical proof for synchronization of systems (1.1) and (1.2), i.e., we show that the global attractors of (1.1) and (1.2), respectively, lie on the spatially homogeneous diagonal provided the coupling coefficient d is sufficiently large. Moreover, we determine the dependence of coupling coefficients for synchronization on the lattice size n . For the system (1.3) with z_i -component coupling, the proof technique for (1.1) and (1.2) cannot be directly used to show synchronization of (1.3). Numerical experiments in Section 4 indicate that synchronization of (1.3) cannot occur even though the coupling coefficient is chosen to be relatively large.

2. Bounded dissipativeness

In this section, we shall prove the bounded dissipativeness for the coupled system of (1.1)–(1.3) with NC and PC. The system of (1.1)–(1.3) is said to be bounded dissipative if there is a real-valued function $V : \mathbb{R}^{3n^2} \rightarrow \mathbb{R}_+$ and a positive number k_0 such that

$$V_{k_0} := \{(x, y, z) \in \mathbb{R}^{3n^2} \mid V(x, y, z) < k_0\}$$

is homeomorphic to an open ball in $\mathbb{R}^{3n^2}$ and $\dot{V} < 0$ along the solution $(x(t), y(t), z(t))$ of (1.1)–(1.3) with $(x(t), y(t), z(t)) \in \mathbb{R}^{3n^2} \setminus \bar{V}_{k_0}$. Thus, the vector field on the surface of $V(x, y, z) = k_0$ is oriented inside, i.e., the global attractor of systems lie strictly inside V_{k_0} .

Theorem 2.1. *The coupled systems (1.1)–(1.3) are bounded dissipative. Furthermore, the global attractors are contained in a $3n^2$ -dimensional ball with radius $O(n)$. Here the big “O” is a constant independent of the lattice size n .*

Proof. We only prove the bounded dissipativeness of the system (1.1) with x -component coupling. Similar arguments also hold for the systems (1.2) and (1.3) with single y - or z -component coupling.

Introduce a Lyapunov function given by

$$V(x, y, z) = \sum_i \left[\frac{1}{2} x_i^2 + \frac{\sigma}{2} y_i^2 + \frac{\sigma}{2} z_i^2 - \gamma \sigma z_i \right]. \quad (2.1)$$

The derivative of $V(x, y, z)$ along the trajectory of the coupled system (1.1) is

$$\begin{aligned} \dot{V}(x, y, z) &= \sum_i (-\sigma x_i^2 + \sigma x_i y_i - \sigma y_i^2 - \sigma b z_i^2 + \gamma \sigma b z_i) + \sum_i dx_i (\Delta x)_i \\ &= \sum_i \left[-\sigma \left(\left(x_i - \frac{1}{2} y_i \right)^2 + \frac{3}{4} y_i^2 \right) \right] + \sum_i \left[-\sigma b \left(z_i - \frac{\gamma}{2} \right)^2 + \frac{\sigma b \gamma^2}{4} \right] + \sum_i dx_i (\Delta x)_i. \end{aligned}$$

From (1.4), we see that the operator Δ is self-adjoint and has only nonpositive eigenvalues $\lambda_j \leq 0$, for $j \equiv (j_1, j_2)$, $1 \leq j_1, j_2 \leq n$. Let $\{h^j | j \equiv (j_1, j_2), 1 \leq j_1, j_2 \leq n\} \subseteq \mathbb{R}^{n^2}$ be the orthonormal basis consisted of eigenvectors of Δ . Then the vector $x = \{x_i | i = (i_1, i_2), 1 \leq i_1, i_2 \leq n\}$ can be represented by $x_i = \sum_j \alpha_j h_i^j$. This yields

$$\sum_i dx_i (\Delta x)_i = d \sum_j \lambda_j \alpha_j^2 \leq 0.$$

Therefore, we have

$$\begin{aligned} \dot{V}(x, y, z) &\leq \sum_i \left[-\sigma \left(\left(x_i - \frac{1}{2} y_i \right)^2 + \frac{3}{4} y_i^2 \right) - \sigma b \left(z_i - \frac{\gamma}{2} \right)^2 + \frac{\sigma b \gamma^2}{4} \right] \\ &= \sum_i \left[-\sigma \left(\left(x_i - \frac{1}{2} y_i \right)^2 + \frac{3}{4} y_i^2 \right) - \sigma b \left(z_i - \frac{\gamma}{2} \right)^2 \right] + \frac{\sigma b \gamma^2 n^2}{4}. \end{aligned}$$

Let

$$D = \left\{ (x, y, z) \in \mathbb{R}^{3n^2} \mid \sum_i \left(\sigma \left(x_i - \frac{1}{2} y_i \right)^2 + \frac{3}{4} y_i^2 + \sigma b \left(z_i - \frac{\gamma}{2} \right)^2 \right) < \frac{\sigma b n^2 \gamma^2}{4} \right\}. \quad (2.2)$$

Then D is a simply connected, bounded region and satisfies

$$\dot{V}(x, y, z) < 0 \quad \text{for all } (x, y, z) \notin \bar{D}.$$

Given any $k > 0$, we let

$$V_k = \{(x, y, z) \in \mathbb{R}^{3n^2} | V(x, y, z) < k\}.$$

It is easily seen that V_k is also a simply connected, bounded region satisfying

$$V_k \subset V_{k'} \quad \text{if } k < k'.$$

Choosing a sufficiently large number k_0 such that $V_{k_0} \supseteq D$, we conclude that every solution of the coupled system (1.1) will eventually enter the bounded region V_{k_0} and stay there. Moreover, from (2.1) and (2.2) k_0 can be chosen carefully so that V_{k_0} is contained in a $3n^2$ -dimensional ball with radius $O(n)$. This completes the proof of Theorem 2.1. $\square$

3. Main theorems

In this section, we prove the main result that the systems (1.1) and (1.2) with x - and y -components couplings, respectively, are synchronized. In order to produce chaotic behavior, the parameter b in (1.1) and (1.2) is chosen to be greater than 1.

For convenience, we designate a point $i \equiv (i_1, i_2)$ on the 2D lattice into a 1D ordering by $v \equiv v(i) = i_1 + n(i_2 - 1)$, for $1 \leq i_1, i_2 \leq n$. Any $n^2 \times 1$ vector of the form

$$u = [u_{(1,1)}, \dots, u_{(n,1)} | \dots | u_{(1,n)}, \dots, u_{(n,n)}]^T$$

can be identified by $u = [u_v | v = 1, 2, \dots, n^2]$ with $u_v = u_{v(i)} = u_{(i_1, i_2)}$.

Therefore, the system (1.1) can be regarded as a system of equations in 1D vector form:

$$\dot{x} = \sigma(y - x) + d(\Delta x), \quad \dot{y} = \gamma x - y - f(x, z), \quad \dot{z} = -bz + g(x, y), \quad (3.1)$$

where x, y and z are vectors in $\mathbb{R}^{n^2}$ with components $(x)_v = x_v, (y)_v = y_v$ and $(z)_v = z_v$, respectively, $f(x, z)$ and $g(x, y)$ are vector functions in $\mathbb{R}^{n^2}$ with components $(f(x, z))_v = x_v z_v$ and $(g(x, y))_v = x_v y_v$, respectively.

The matrix Δ as in (1.4) and (1.5) has the simple zero eigenvalue associated with the left eigenvector $e^T = [1, \dots, 1] \in \mathbb{R}^{1 \times n^2}$. Let $C = [e_1 - e_2, \dots, e_{n^2-1} - e_{n^2}]^T \in \mathbb{R}^{(n^2-1) \times n^2}$, where e_j is the unit vector in $\mathbb{R}^{n^2}$. Then

$$E = \begin{bmatrix} C \\ e^T \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}, \quad E^{-1} = \begin{bmatrix} C^T(CC^T)^{-1} \left| \frac{e^2}{n} \right. \end{bmatrix}$$

satisfying

$$E\Delta E^{-1} = \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.2)$$

where $\tilde{\Delta} \in \mathbb{R}^{(n^2-1) \times (n^2-1)}$ with $\max\{\lambda_j | \lambda_j \in \lambda(\tilde{\Delta})\} = \lambda_0 < 0$. Here $\lambda(\tilde{\Delta})$ denotes the spectrum of $\tilde{\Delta}$. From (1.4) and (1.5) we have

$$\lambda_0 = -8 \sin^2 \left(\frac{\pi}{2(n+1)} \right) \quad \text{for NC}, \quad \lambda_0 = -8 \sin^2 \left(\frac{\pi}{n} \right) \quad \text{for PC}. \quad (3.3)$$

We now introduce a new coordinate system with new variables ξ, ξ^*, η, η^* , and ζ, ζ^* such that

$$Ex = \begin{bmatrix} \xi \\ \xi^* \end{bmatrix}, \quad Ey = \begin{bmatrix} \eta \\ \eta^* \end{bmatrix}, \quad Ez = \begin{bmatrix} \zeta \\ \zeta^* \end{bmatrix}, \quad (3.4)$$

where ξ^*, η^* , and ζ^* are scalars satisfying

$$\xi^* = \sum_{v=1}^{n^2} x_v, \quad \eta^* = \sum_{v=1}^{n^2} y_v, \quad \zeta^* = \sum_{v=1}^{n^2} z_v,$$

and ξ, η, ζ are $(n^2 - 1) \times 1$ vectors satisfying

$$\xi_v = x_v - x_{v+1}, \quad \eta_v = y_v - y_{v+1}, \quad \zeta_v = z_v - z_{v+1} \quad (3.5)$$

for $v = 1, 2, \dots, n^2 - 1$. Substituting (3.4) into (3.1), we have

$$\dot{\xi} = \sigma(\eta - \xi) + d\tilde{\Delta}\xi, \quad \dot{\eta} = \gamma\xi - \eta - Cf(x, z), \quad \dot{\zeta} = -b\zeta + Cg(x, y). \quad (3.6)$$

Let $D_1(x) = \text{diag}(x_1, x_2, \dots, x_{n^2-1})$, $D_2(y) = \text{diag}(y_2, y_3, \dots, y_{n^2})$ and $D_3(z) = \text{diag}(z_2, z_3, \dots, z_{n^2})$.

Then we have

$$\begin{aligned} (Cf(x, z))_v &= (f(x, z))_v - (f(x, z))_{v+1} = x_v z_v - x_{v+1} z_{v+1} = x_v(z_v - z_{v+1}) + z_{v+1}(x_v - x_{v+1}) \\ &= (D_1(x))_v \zeta_v + (D_3(z))_v \xi_v \end{aligned}$$

and

$$(Cg(x, y))_v = (g(x, y))_v - (g(x, y))_{v+1} = x_v y_v - x_{v+1} y_{v+1} = (D_1(x))_v \eta_v + (D_2(y))_v \xi_v.$$

Hence, system (3.6) can be rewritten as

$$\dot{\xi} = \sigma(\eta - \xi) + d\tilde{\Delta}\xi, \quad \dot{\eta} = \gamma\xi - \eta - D_1(x)\zeta - D_3(z)\xi, \quad \dot{\zeta} = -b\zeta + D_1(x)\eta + D_2(y)\xi. \quad (3.7)$$

Because the new coordinate system as in (3.4) and (3.5) represents phase differences between different lattice sites, if we can show that

$$\|\xi(t)\|_2 \rightarrow 0, \quad \|\eta(t)\|_2 \rightarrow 0, \quad \|\zeta(t)\|_2 \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.8)$$

i.e., $|\xi_v(t)| \rightarrow 0$, $|\eta_v(t)| \rightarrow 0$ and $|\zeta_v(t)| \rightarrow 0$ as $t \rightarrow \infty$, then it follows from (3.5) that

$$|x_j(t) - x_\ell(t)| \leq \sum_{v=j}^{\ell-1} |\xi_v(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Similarly, it also holds that

$$|y_j(t) - y_\ell(t)| \rightarrow 0 \quad \text{and} \quad |z_j(t) - z_\ell(t)| \rightarrow 0$$

as $t \rightarrow \infty$. Hence, the synchronization for the system (1.1) holds. In the following, we mainly give a technique to show the assertion (3.8) holds.

Theorem 3.1. *Let $(x(t), y(t), z(t))$ be a solution of the coupled system (1.1) with NC or PC. Then there exists a positive constant d^* and $q > 2$ with*

$$d^* = O\left(\frac{n^q}{4 \sin^2(\pi/2(n-1)^2) |\lambda_0|}\right)$$

such that for any $d > d^*$, it holds

$$\lim_{t \rightarrow \infty} |x_j(t) - x_\ell(t)| = 0, \quad \lim_{t \rightarrow \infty} |y_j(t) - y_\ell(t)| = 0, \quad \lim_{t \rightarrow \infty} |z_j(t) - z_\ell(t)| = 0,$$

where $1 \leq j, \ell \leq n^2$ and λ_0 is given by (3.3).

Proof. By Theorem 2.1 and (3.4), there is a positive number $N = O(n)$ such that

$$\begin{aligned} \|x(t)\|_2 &\leq N, & \|y(t)\|_2 &\leq N, & \|z(t)\|_2 &\leq N, & \|\xi(t)\|_2 &\leq N, \\ \|\eta(t)\|_2 &\leq N, & \|\zeta(t)\|_2 &\leq N \end{aligned} \quad (3.9)$$

for t sufficiently large. Let M be a positive number such that

$$\frac{1}{3}M > \max\{N, \sigma, \gamma\}. \quad (3.10)$$

Define

$$L_{k+1} = K_k + q - 1 \quad \text{and} \quad K_{k+1} = K_k + \frac{1}{2}q - 1 \quad (3.11)$$

with $q > 2$, for $k = 0, 1, 2, \dots$. We shall show inductively on k that for each L_k and K_k there are times with $T_k < \bar{T}_k$ such that

$$\|\xi(t)\|_2 \leq \frac{1}{M^{L_k}} \quad \forall t \geq T_k, \quad (3.12)$$

and

$$\|\eta(t)\|_2 \leq \frac{1}{M^{K_k}}, \quad \|\zeta(t)\|_2 \leq \frac{1}{M^{K_k}} \quad \forall t \geq \bar{T}_k. \quad (3.13)$$

We now prove (3.12) and (3.13) by induction. For $k = 0$, let $K_0 = \frac{1}{2}q - 2$ and $L_0 = q - 2$. Applying the variation of constant formula to the first equation of (3.7), we get

$$\xi(t) = e^{t(-\sigma I + d\tilde{\Delta})}\xi(0) + \int_0^t e^{(t-s)(-\sigma I + d\tilde{\Delta})}\sigma\eta(s) ds.$$

Since $\tilde{\Delta} = (C \Delta C^T)(CC^T)^{-1}$ by (3.2), we then have

$$\|e^{t(-\sigma I + d\tilde{\Delta})}\|_2 \leq \Gamma e^{t(-\sigma I + d\lambda_0)}, \quad (3.14)$$

where $\lambda_0 = \max\{|\lambda_j| \mid \lambda_j \in \sigma(\tilde{\Delta})\}$ and $\Gamma = [4 \sin^2(\pi/2(n-1)^2)]^{-1/2} \approx O(n^2)$ (for n sufficiently large). From (3.9), (3.10) and (3.14) it follows that

$$\|\xi(t)\|_2 < \Gamma \|\xi(0)\|_2 e^{t(-\sigma + d\lambda_0)} + \frac{\Gamma M^2}{9} \int_0^t e^{(t-s)(-\sigma + d\lambda_0)} ds < \Gamma \|\xi(0)\|_2 e^{t(-\sigma + d\lambda_0)} + \frac{\Gamma M^2}{9(\sigma - d\lambda_0)}.$$

This implies that

$$\limsup_{t \rightarrow \infty} \|\xi(t)\|_2 \leq \frac{\Gamma M^2}{9(\sigma - d\lambda_0)}.$$

Choose $d^* > 0$ such that

$$(\sigma - d^*\lambda_0) \geq \Gamma M^q \quad \text{with } q > 2. \quad (3.15)$$

Then, for $d \geq d^*$, it follows that $\limsup_{t \rightarrow \infty} \|\xi(t)\|_2 < M^{2-q}$. This implies that there exists a time T_0 , so that

$$\|\xi(t)\|_2 \leq \frac{1}{M^{q-2}} \quad \forall t \geq T_0. \quad (3.16)$$

Now consider the v th scalar equation of the second and the third equations of (3.7):

$$\dot{\eta}_v = \gamma \xi_v - \eta_v - x_v \zeta_v - z_{v+1} \xi_v, \quad \dot{\zeta}_v = -b \zeta_v + x_v \eta_v + y_{v+1} \xi_v. \quad (3.17)$$

Construct the Lyapunov function

$$V(\eta, \zeta) = \frac{1}{2} \sum_v (\eta_v^2 + \zeta_v^2). \quad (3.18)$$

Then the derivative of $V(\eta, \zeta)$ along the trajectory of system (3.17) is

$$\dot{V}(\eta, \zeta) = \sum_v [-\eta_v^2 - b \zeta_v^2 + (\gamma \eta_v - z_{v+1} \eta_v + y_{v+1} \zeta_v) \xi_v]. \quad (3.19)$$

From (3.10) and (3.16) and Cauchy–Schwarz inequality we have

$$\sum_v |\gamma \eta_v - z_{v+1} \eta_v + y_{v+1} \zeta_v| |\xi_v| < \frac{1}{M^{q-4}} \quad \forall t \geq T_0.$$

Define the set A_0 by

$$A_0 = \left\{ (\eta, \zeta) \mid \|\eta\|_2 \geq \frac{1}{M^{(q-4)/2}} \text{ or } \|\zeta\|_2 \geq \frac{1}{\sqrt{b}M^{(q-4)/2}} \right\}.$$

From (3.19) we have $\dot{V}(\eta, \zeta) < 0$ for all $(\eta, \zeta) \in A_0$. Therefore, there exists a $\bar{T}_0 > T_0$ such that

$$\|\eta(t)\|_2 \leq \frac{1}{M^{(q-4)/2}} \quad \text{and} \quad \|\zeta(t)\|_2 \leq \frac{1}{M^{(q-4)/2}}$$

for all $t \geq \bar{T}_0$. We proved the case that $k = 0$.

Assume that L_k and K_k defined by (3.11) satisfy (3.12) and (3.13). As above by applying the variation of constant formula again, we have

$$\xi(t) = e^{t(-\sigma I + d\tilde{\Delta})}\xi(0) + \int_0^t e^{(t-s)(-\sigma I + d\tilde{\Delta})}\sigma\eta(s) ds.$$

From (3.10), (3.13) and (3.15), it follows that $\limsup_{t \rightarrow \infty} \|\xi(t)\|_2 < M^{1-q-K_k}$. This implies that there is a $T_{k+1} > \bar{T}_k$, such that

$$\|\xi(t)\|_2 \leq \frac{1}{M^{K_k+q-1}} \quad \text{for all } t \geq T_{k+1}. \quad (3.20)$$

By (3.10), (3.13) and (3.20) it holds

$$\sum_v |\gamma\eta_v - z_{v+1}\eta_v + y_{v+1}\zeta_v| |\xi_v| < \frac{1}{M^{2K_k+q-2}}.$$

Consider the Lyapunov function of (3.18) on the set

$$A_k = \left\{ (\eta, \zeta) \mid \|\eta\|_2 \geq \frac{1}{M^{K_k+(q/2)-1}} \text{ or } \|\zeta\|_2 \geq \frac{1}{\sqrt{b}M^{K_k+(q/2)-1}} \right\}.$$

We then have $\dot{V}(\eta, \zeta) < 0$ for all $(\eta, \zeta) \in A_k$. Therefore, there is a $\bar{T}_{k+1} > T_{k+1}$ such that

$$\|\eta(t)\|_2 \leq \frac{1}{M^{K_k+(q/2)-1}}, \quad \|\zeta(t)\|_2 \leq \frac{1}{M^{K_k+(q/2)-1}} \quad \text{for all } t \geq \bar{T}_{k+1}. \quad (3.21)$$

Thus, the inequalities of (3.12) and (3.13) hold for all $k = 0, 1, 2, \dots$

Clearly, $\lim_{k \rightarrow \infty} K_k = \infty$ and $\lim_{k \rightarrow \infty} L_k = \infty$. This implies that $\|\xi(t)\|_2 \rightarrow 0$, $\|\eta(t)\|_2 \rightarrow 0$, $\|\zeta(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$. Finally, from (3.10) and (3.15), the number d^* is chosen to be the positive: $d^* = O(n^q/[4 \sin^2(\pi/2(n-1)^2)|\lambda_0|])$ with $q > 2$. This completes the proof of the theorem. $\square$

As in the derivation of (3.7) the coupled system (1.2) with y-component coupling can also be transformed into the form by (3.4)

$$\dot{\xi} = \sigma(\eta - \xi), \quad \dot{\eta} = \gamma\xi - \eta - D_1(x)\zeta - D_3(z)\xi + d\tilde{\Delta}\eta, \quad \dot{\zeta} = -b\zeta + D_1(x)\eta + D_2(y)\xi. \quad (3.22)$$

In the following theorem, we prove that $\|\xi(t)\|_2 \rightarrow 0$, $\|\eta(t)\|_2 \rightarrow 0$ and $\|\zeta(t)\|_2 \rightarrow 0$. Consequently, the synchronization for (1.2) holds.

Theorem 3.2. *Let $(x(t), y(t), z(t))$ be a solution of the coupled system (1.2) with NC or PC. Then there exists a positive constant d^* and $q > 3$ with*

$$d^* = O\left(\frac{n^q}{4 \sin^2(\pi/2(n-1)^2)|\lambda_0|}\right) \quad (3.23)$$

such that for any $d > d^*$, it holds

$$\lim_{t \rightarrow \infty} |x_j(t) - x_\ell(t)| = 0, \quad \lim_{t \rightarrow \infty} |y_j(t) - y_\ell(t)| = 0, \quad \lim_{t \rightarrow \infty} |z_j(t) - z_\ell(t)| = 0,$$

where $1 \leq j, \ell \leq n^2$ and λ_0 is given by (3.3).

Proof. Let M be a positive number such that

$$\frac{1}{3}M > \max\{N, \gamma\}, \quad (3.24)$$

where $N = O(n)$ is given by (3.9). Then it holds

$$\|D_1(x)\|_2 \leq \frac{1}{3}M, \quad \|D_2(y)\|_2 \leq \frac{1}{3}M, \quad \|D_3(z)\|_2 \leq \frac{1}{3}M.$$

Define

$$L_{k+1} = L_k + q - 3 \quad (3.25)$$

with $q > 3$ for $k = 0, 1, 2, \dots$. We shall show inductively on k that for each L_k there are times with $\bar{T}_k > T_k$ such that

$$\|\xi(t)\|_2 \leq \frac{1}{M^{L_k}}, \quad \|\eta(t)\|_2 \leq \frac{1}{M^{L_k}} \quad \forall t \geq T_k, \quad (3.26)$$

and

$$\|\zeta(t)\|_2 \leq \frac{1}{M^{L_k-1}} \quad \forall t \geq \bar{T}_k. \quad (3.27)$$

For $k = 0$, let $L_0 = q - 2$. By applying the variation of constant formula to the second equation of system (3.22), we get

$$\eta(t) = e^{t(-I+d\tilde{\Delta})}\eta(0) + \int_0^t e^{(t-s)(-I+d\tilde{\Delta})}(\gamma\xi(s) - D_1(x(s))\zeta(s) - D_3(z(s))\xi(s))ds.$$

As in (3.14) there is a $\Gamma = [4 \sin^2(\pi/2(n-1)^2)]^{-1/2} \approx O(n^2)$ for n sufficiently large, such that

$$\|e^{t(-I+d\tilde{\Delta})}\|_2 \leq \Gamma e^{t(-1+d\lambda_0)},$$

where $\lambda_0 = \max\{\lambda_j | \lambda_j \in \sigma(\tilde{\Delta})\}$. By (3.9) and (3.24) it holds

$$\|\gamma\xi(s) - D_1(x(s))\zeta(s) - D_3(z(s))\xi(s)\|_2 < \frac{1}{3}M^2.$$

Thus,

$$\|\eta(t)\|_2 < \Gamma \|\eta(0)\|_2 e^{t(-1+d\lambda_0)} + \frac{\Gamma M^2}{3(1-d\lambda_0)}.$$

This implies that

$$\limsup_{t \rightarrow \infty} \|\eta(t)\|_2 < \frac{\Gamma M^2}{3(1-d\lambda_0)}.$$

Choose $d^* > 0$ such that $(1 - d^*\lambda_0) \geq \Gamma M^q$ with $q > 3$. Then, for $d \geq d^*$, we have $\limsup_{t \rightarrow \infty} \|\eta(t)\|_2 \leq \frac{1}{3}M^{2-q} < M^{2-q}$. There is a time τ_0 , such that

$$\|\eta(t)\|_2 < \frac{1}{M^{q-2}} \quad \forall t \geq \tau_0. \quad (3.28)$$

By applying the variation of constant formula again to the first equation of (3.22), we get

$$\xi(t) = e^{-\sigma t} \xi(0) + \int_0^t e^{-\sigma(t-s)} \sigma \eta(s) ds.$$

By (3.28) we have $\limsup_{t \rightarrow \infty} \|\xi(t)\|_2 < M^{2-q}$. This implies that there is a time $T_0 > \tau_0$ such that

$$\|\xi(t)\|_2 \leq \frac{1}{M^{q-2}} \quad \forall t \geq T_0.$$

Using the similar argument as above to the third equation of (3.22), we get $\limsup_{t \rightarrow \infty} \|\zeta(t)\|_2 < M^{3-q}$. Thus, there is a time $\bar{T}_0 > T_0$ such that

$$\|\zeta(t)\|_2 \leq \frac{1}{M^{q-3}} \quad \forall t \geq \bar{T}_0.$$

Assume now that L_k defined in (3.25) satisfies (3.26) and (3.27). Then by the variation of constant formula again, there is a time $T_{k+1} > \bar{T}_k$ such that

$$\|\eta(t)\|_2 \leq \frac{1}{M^{L_k+q-3}}, \quad \|\xi(t)\|_2 \leq \frac{1}{M^{L_k+q-3}} \quad \forall t \geq T_{k+1}.$$

Fig. 1. Synchronization for the coupled system (1.1): (a) $x_1 - x_2$ versus time for $0 \leq t \leq 100$; (b) $y_1 - y_2$ versus time for $0 \leq t \leq 100$; (c) $z_1 - z_2$ versus time for $0 \leq t \leq 100$.

Using the argument as above, there is a time $\bar{T}_{k+1} > T_{k+1}$ such that

$$\|\zeta(t)\|_2 \leq \frac{1}{M^{L_k+q-4}} \quad \forall t \geq \bar{T}_{k+1}.$$

Therefore, the inequalities of (3.26) and (3.27) hold for $k = 0, 1, 2, \dots$

Clearly, $\lim_{k \rightarrow \infty} L_k = \infty$. This implies that $\|\xi(t)\|_2 \rightarrow 0$, $\|\eta(t)\|_2 \rightarrow 0$, $\|\zeta(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$. The theorem follows by choosing $d^* = O(n^q / [4 \sin^2(\pi/2(n-1)^2) |\lambda_0|])$ with $q > 3$. $\square$

4. Numerical results

In this section, we present some numerical results to illustrate the synchronization of the coupled identical Lorenz systems (1.1) and (1.2) with x_i - and y_i -components couplings, respectively. We also perform numerical simulations for the study of dynamics of the coupled system (1.3) with z_i -component coupling. From numerical evidence, we argue that the synchronization of (1.3) with z_i -component coupling cannot occur.

Example 4.1. We consider the x_i -component coupled system (1.1) of two identical Lorenz equations with $\sigma = 10$, $\gamma = 28$, $b = \frac{8}{3}$ and $d = 15$. It is well known that Lorenz equation has chaotic behavior with this set of parameters

Fig. 2. Synchronization for the coupled system (1.2): (a) $x_1 - x_2$ versus time for $0 \leq t \leq 100$; (b) $y_1 - y_2$ versus time for $0 \leq t \leq 100$; (c) $z_1 - z_2$ versus time for $0 \leq t \leq 100$.

Fig. 3. Asynchronization for the coupled system (1.3) with $d = 15$: (a) $x_1 - x_2$ versus time for $0 \leq t \leq 100$; (b) $y_1 - y_2$ versus time for $0 \leq t \leq 100$; (c) $z_1 - z_2$ versus time for $0 \leq t \leq 100$.

[13]. Fig. 1 plots the graph of the time versus the phase differences of $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$, respectively. We see that the differences approach to zero as time becomes large. The synchronization for the coupled system (1.1) with x_i -component coupling can occur.

Example 4.2. We consider the y_i -component coupled system (1.2) of two identical Lorenz equations with same parameters and coupling coefficients as in Example 4.1. Fig. 2 shows the graph of the time versus the phase differences of $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$, respectively. The differences also approach to zero as time becomes large. The synchronization for the coupled system (1.2) with y_i -component coupling can also occur.

Example 4.3. We now consider the z_i -component coupled system (1.3) of two identical Lorenz equations with same parameters as in Example 4.1. Figs. 3 and 4 plot the graphs of the time versus the phase differences of $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$ with $d = 15$ and 100, respectively. We observe that synchronization for the coupled system (1.3) with z_i -component coupling cannot occur even when the time becomes sufficiently large.

We can also argue the above viewpoint by the observation of Pecora and Carroll [19]. Suppose the coupled system (1.3) is synchronized for all $d > d^*$. Then we take d sufficiently large so that $\limsup_{t \rightarrow \infty} |z_1(t) - z_2(t)| = 0$. For this case, the first two equations of (1.3) form a master–slave system of Lorenz equations. But the conditional Lyapunov exponent of this type of master–slave system was computed to be positive [19]. This contradicts synchronization.

Fig. 4. Asynchronization for the coupled system (1.3) with $d = 100$: (a) $x_1 - x_2$ versus time for $0 \leq t \leq 100$; (b) $y_1 - y_2$ versus time for $0 \leq t \leq 100$; (c) $z_1 - z_2$ versus time for $0 \leq t \leq 100$.

5. Conclusions

In this paper, we study the synchronization of three different component coupled systems of Lorenz equations on an $n \times n$ lattice with NC or PC. Coupled systems require that solutions stay in a bounded set for all time. Using the quadratic Lyapunov function (2.1) we show that there is a natural bounded dissipative mechanism for coupled systems (1.1)–(1.3). Theorems 3.1 and 3.2 prove that the synchronization of the x_i - or y_i -component coupled system occurs provided the coupling coefficient d is sufficiently large. A lower bound for d determined by the lattice size n is also given. Numerical experience shows that synchronized behavior of the coupled systems heavily depends on the coupling rules. Synchronization for the single z_i -component coupled system (1.3) cannot occur.

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