PERTURBATION ANALYSIS OF THE PERIODIC DISCRETE-TIME
ALGEBRAIC RICCATI EQUATION*

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Abstract. This paper is devoted to the perturbation analysis for the periodic discrete-time algebraic Riccati equations (P-DAREs). Perturbation bounds and condition numbers of the Hermitian positive semidefinite solution set to the P-DAREs are obtained. The results are illustrated by numerical examples.

Key words. periodic Riccati equation, periodic Hermitian positive semidefinite solution set, perturbation bound, condition number

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1. Introduction. We consider the periodic discrete-time algebraic Riccati equation (P-DARE) with period \( p \geq 1 \),

\[
X_{j-1} = A_j^H X_j A_j - A_j^H X_j B_j (R_j + B_j^H X_j B_j)^{-1} B_j^H X_j A_j + H_j
\]

\[
= A_j^H X_j (I + G_j X_j)^{-1} A_j + H_j,
\]

where, for all \( j \), \( A_j = A_{j+p} \), \( H_j = H_{j+p} \), and \( X_j = X_{j+p} \) are \( n \times n \) matrices, \( B_j = B_{j+p} \) are \( m \times m \) matrices, and \( R_j = R_{j+p} \) are full column rank, \( R_j \) is Hermitian positive definite \( (R_j > 0) \), \( G_j \equiv B_j R_j^{-1} B_j^H = G_{j+p} \), and \( H_j \) is Hermitian positive semidefinite (p.s.d.) with \( H_j = C_j^H C_j \geq 0 \), a full rank decomposition (f.r.d.). Note that the second equation of (1.1) is obtained by the Sherman–Morrison–Woodbury formula (see, e.g., [9, p. 50]) provided that \( (I + G_j X_j)^{-1} \) exists. In this paper, the indices \( j \) for all periodic coefficient matrices are chosen in \( \{1, \ldots, p\} \) modulo \( p \) without ambiguity.

Appropriate assumptions on the periodic coefficient matrices will be made in the following to guarantee the existence and uniqueness of the Hermitian p.s.d. solution set \( \{X_j\}_{j=1}^p \) to the P-DARE (1.1). The equation (1.1) arises frequently in solving the periodic discrete-time linear optimal control problem

\[
\begin{align*}
\text{Minimize} \quad J &= \frac{1}{2} \sum_{j=1}^\infty \|x_j^H H_j x_j + u_j^H R_j u_j\|, \\
\text{subject to} \quad x_{j+1} &= A_j x_j + B_j u_j.
\end{align*}
\]

The periodic optimal feedback vector \( u_j^* \) for (1.2) is given by [2]

\[
u_j^* = -(R_j + B_j^H X_j B_j)^{-1} B_j^H X_j A_j x_j
\]

for \( j = 1, \ldots, p \), where \( \{X_j\}_{j=1}^p \) is the Hermitian p.s.d. solution set to (1.1). The real case, i.e., to find the real symmetric p.s.d. solution set \( \{X_j\}_{j=1}^p \) to the P-DARE (1.1)
when all of the periodic coefficient matrices are real, is essentially important in many applications. We consider here the real case as well as the general, that is, complex, case.

The P-DARE can be regarded as an extension of the time-invariant case. For \( p = 1 \), the P-DARE becomes the usual discrete-time algebraic Riccati equation (DARE) by setting \( X_j = X_{j-1} \) in (1.1). There are many contributions in the literature on the perturbation theory and numerical methods of the DARE (see, e.g., [13], [20], [21], [11], [22], [17]). In the case of \( p > 1 \), many research efforts have been devoted to the existence of different types of solution sets to the P-DARE under variant assumptions [1], [2], [5], [7], [12], [15], [18], [23]. In this paper, we study the perturbation theory for the P-DARE. This work, as a generalization of the results given by [20] and [21], derives perturbation bounds and condition numbers of the Hermitian p.s.d. solution set \( \{X_j\}_{j=1}^p \) to the P-DARE (1.1). The interest in this topic is motivated by the fact that the P-DARE is usually subject to perturbation in the coefficient matrices, reflecting various errors in the formulation of the problem and in its solution by a computer. (See, e.g., [3], [6] for numerical methods for solving the P-DARE.)

Throughout this paper, we denote by \( \mathcal{H}_n(S_n) \) and \( \mathbb{C}_n(\mathbb{R}_n) \) the sets of \( n \times n \) Hermitian (real symmetric) and \( n \times n \) complex (real) matrices, respectively, and we denote by \( \mathcal{H}_{p}^n \) and \( \mathbb{C}_{p}^n \) the \( p \)-tuple product spaces \( \mathcal{H}_n \times \cdots \times \mathcal{H}_n \) and \( \mathbb{C}_n \times \cdots \times \mathbb{C}_n \), respectively. \( \overline{A} \) denotes the conjugate of a matrix \( A \), and \( A^\top \) denotes the transpose of \( A \), and \( I_n \) is the identity matrix of order \( n \), and \( 0 \) is the null matrix. The set of all eigenvalues of \( A \in \mathbb{C}_n \) is denoted by \( \lambda(A) \). The spectral radius \( \rho(A) \) is defined by \( \rho(A) = \max\{|\lambda_i| : \lambda_i \in \lambda(A)\} \). The symbol \( || \cdot ||_F \) is the Frobenius norm, and \( || \cdot ||_2 \) is the spectral norm and the Euclidean vector norm. For \( A = (a_{ij}) \in \mathbb{C}_n \) and a matrix \( B, A \otimes B = (a_{ij}B) \) is a Kronecker product, and \( \text{vec}(A) \) is a vector defined by \( \text{vec}(A) = (a_{11}, \ldots, a_{nn})^\top \). An \( n \times n \) matrix \( \Phi \) is said to be d-stable if \( \lambda(\Phi) \subset \mathcal{D} \), where \( \mathcal{D} = \{z \in \mathbb{C} : |z| < 1\} \). In order to save the space of the matrix representation, we also use the following notation:

\[
\text{diag}\{N_j\}_{j=1}^p = \begin{bmatrix} N_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N_p \end{bmatrix}, \quad \text{cyc}\{N_j\}_{j=1}^p = \begin{bmatrix} 0 & \cdots & 0 & N_1 \\ N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & N_p & 0 \end{bmatrix}.
\]

**Definition 1.1.** Let \( \Phi_1, \ldots, \Phi_p \in \mathbb{C}_n \). If there are complex numbers \( \alpha_1, \ldots, \alpha_p \) such that

\[
\det \left[ \text{diag}\{\alpha_j I\}_{j=1}^p - \text{cyc}\{\Phi_j\}_{j=1}^p \right] = 0,
\]

then \( \alpha_1 \alpha_2 \cdots \alpha_p \) is an eigenvalue of the periodic matrix set \( \{\Phi_j\}_{j=1}^p \).

The set of all eigenvalues of \( \{\Phi_j\}_{j=1}^p \) is denoted by \( \lambda(\{\Phi_j\}_{j=1}^p) \). Note that it is easily seen that \( \lambda(\{\Phi_j\}_{j=1}^p) = \lambda(\Phi_p \Phi_{p-1} \cdots \Phi_1) \), and so \( \rho(\{\Phi_j\}_{j=1}^p) = \rho(\Phi_p \Phi_{p-1} \cdots \Phi_1) \).

**Definition 1.2.** Let \( \Phi_1, \ldots, \Phi_p \in \mathbb{C}_n \). The periodic matrix set \( \{\Phi_j\}_{j=1}^p \) is said to be pd-stable if the matrix \( \Phi_p \Phi_{p-1} \cdots \Phi_1 \) is d-stable, i.e., \( \lambda(\Phi_p \Phi_{p-1} \cdots \Phi_1) \subset \mathcal{D} \).

From Definition 1.2, we see that, if \( \{\Phi_j\}_{j=1}^p \) is pd-stable, then \( \lambda(\Phi_p \cdots \Phi_1) \), \( \lambda(\Phi_1) \), \( \lambda(\Phi_p \cdots \Phi_2) \) \( \subset \mathcal{D} \).

**Definition 1.3 (see [2]).** The periodic matrix pair sets \( \{(A_j, B_j)\}_{j=1}^p \) and \( \{(A_j, C_j)\}_{j=1}^p \) are said to be pd-stabilizable and pd-detectable, respectively, if the pairs \( (A_j, B_j) \) and \( (A_j, C_j) \) are d-stabilizable and d-detectable, respectively, for \( j = 1, \ldots, p \),
where

\[ A_j = A_{\pi_j(p)} \cdots A_{\pi_j(1)}; \]
\[ B_j = [A_{\pi_j(p)} \cdots A_{\pi_j(2)} B_{\pi_j(1)} A_{\pi_j(p)} \cdots A_{\pi_j(3)} B_{\pi_j(2)}] \cdots [A_{\pi_j(p)} B_{\pi_j(p-1)} B_{\pi_j(p)}], \]
\[ C_j = [C_{\pi_j(1)} A_{\pi_j(1)} C_{\pi_j(2)} A_{\pi_j(1)} A_{\pi_j(2)} C_{\pi_j(2)}] \cdots [A_{\pi_j(1)} C_{\pi_j(p-1)} A_{\pi_j(p-1)} C_{\pi_j(p)}] \top, \]

and \( \pi_j(k) \) is a permutation defined by

\[ \pi_j(k) = \begin{cases} k - j + 1 + p & \text{for } k = 1, \ldots, j - 1, \\ k - j + 1 & \text{for } k = j, \ldots, p. \end{cases} \]

Note that the pair \((A, B)\) is d-stabilizable if \( w^H B = 0, w^H A = \lambda w^H \) for some constant \( \lambda \) implies \( |\lambda| < 1 \) or \( w = 0 \), and the pair \((A, C)\) is d-detectable if \((A^H, C^H)\) is d-stabilizable.

Throughout this paper, the periodic matrix pair sets \( \{ (A_j, B_j) \}_{j=1}^p \) and \( \{ (A_j, C_j) \}_{j=1}^p \) of (1.1) are assumed to be pd-stabilizable and pd-detectable, respectively. The existence and uniqueness of the Hermitian p.s.d. solution set to the P-DARE (1.1) are studied in [1] and [2].

**Theorem 1.1** (see [1],[2]). For the P-DARE (1.1), if \( \{ (A_j, B_j) \}_{j=1}^p \) and \( \{ (A_j, C_j) \}_{j=1}^p \) are pd-stabilizable and pd-detectable, respectively, then there is a unique Hermitian p.s.d. solution set \( \{ X_j \}_{j=1}^p \) to the P-DARE (1.1). Moreover, the periodic matrix set \( \{ (I + G_j X_j)^{-1} A_j \}_{j=1}^p \) is pd-stable.

Let

\[ \bar{X}_{j-1} = \bar{A}_j^H \bar{X}_j (I + \bar{G}_j \bar{X}_j)^{-1} \bar{A}_j + \bar{B}_j \]

for \( j = 1, \ldots, p \) be a perturbed P-DARE of (1.1). Based on the technique described in [20], we shall construct an easily treated system of periodic equations of \( \Delta X_j \equiv \bar{X}_j - X_j \) for deriving sharp upper bounds for \( \| \bar{X}_j - X_j \|_F (j = 1, \ldots, p) \) and find some reasonable restrictions on the perturbations in the periodic coefficient matrices of the P-DARE (1.1) such that the perturbed P-DARE (1.4) has a unique Hermitian p.s.d. solution set \( \{ \bar{X}_j \}_{j=1}^p \). Moreover, applying the theory of condition developed by Rice [19], we define a condition number of the Hermitian p.s.d. solution set to the P-DARE (1.1), and, by using the techniques described in [4] and [14], we derive explicit expressions of the condition number.

This paper is organized as follows. In section 2, we prove some lemmas. In section 3, we first construct a perturbation equation for the P-DARE and then derive perturbation bounds for the Hermitian p.s.d. solution set. In section 4, we define a condition number of the Hermitian p.s.d. solution set and derive explicit expressions of the condition number. In section 5, we use a numerical example to illustrate our results.

2. **Lemmas.** In this section, we prove some preliminary lemmas which are used in sections 3 and 4.

Let \( \Phi_j \in \mathbb{C}_n, j = 1, \ldots, p \). Define the linear operator \( L : \mathcal{H}_n^p \rightarrow \mathcal{H}_n^p \) by

\[ (2.1) \quad L(W_1, \ldots, W_p) = (W_1 - \Phi_2^H W_2 \Phi_2, \ldots, W_{p-1} - \Phi_p^H W_p \Phi_p, W_p - \Phi_1^H W_1 \Phi_1), \]

where \( W_j \in \mathcal{H}_n \) for \( j = 1, \ldots, p \).

**Lemma 2.1.** The linear operator \( L \) defined by (2.1) is singular provided that there is an eigenvalue \( \lambda \in \lambda(\{ \Phi_j \}_{j=1}^p) \) with \( |\lambda| = 1 \).
Proof. By the periodic Schur theorem [3], there is a unitary matrix set \( \{U_j\}_{j=1}^p \) such that

\[
U_j^T \Phi_j U_{j-1} = \begin{bmatrix} \phi_j & 0 \\ * & * \end{bmatrix}
\]

for \( j = 1, \ldots, p \), and \( \hat{\lambda} = \phi_p \phi_{p-1} \cdots \phi_1 \). Without loss of generality, we may assume first that \( \Phi_j \) has the lower triangular form as in (2.2) for \( j = 1, \ldots, p \). Taking

\[
W_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_j = \begin{bmatrix} |\phi_{j+1}|^2 \cdots |\phi_p|^2 \\ 0 \\ 0 \end{bmatrix}, \quad j = p-1, \ldots, 1,
\]

and substituting (2.3) into (2.1), we have

\[
W_j - \Phi_j^T W_j \Phi_j = 0, \quad j = 1, \ldots, p,
\]

and, by assumption, \( |\phi_1|^2 \cdots |\phi_p|^2 = 1 \). Setting \( W_j^* = U_j^T W_j U_j \) for \( j = 1, \ldots, p \), there is a nonzero element \( (W_1^*, \ldots, W_p^*) \in \mathcal{H}_n^p \) such that \( L(W_1^*, \ldots, W_p^*) = (0, \ldots, 0) \), which implies the assertion. \( \square \)

**Lemma 2.2.** Let \( \Phi = \text{cyc}\{\Phi_j\}_{j=1}^p \), where \( \Phi_j \in \mathbb{C}_n, j = 1, \ldots, p \). If \( \{\Phi_j\}_{j=1}^p \) is pd-stable, then \( \Phi \) is d-stable.

**Proof.** Suppose that \( \lambda \in \lambda(\Phi) \). Then there are \( n \times 1 \) vectors \( x_1, \ldots, x_p \) with \( (x_1^T, \ldots, x_p^T)^T \neq 0 \) such that

\[
\text{cyc}\{\Phi_j\}_{j=1}^p \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}.
\]

Suppose that \( x_j \neq 0 \) for some \( j \). Comparing the two sides of (2.5), we have

\[
(\Phi_j \cdots \Phi_1 \Phi_p \cdots \Phi_{j+1}) x_j = \lambda^p x_j.
\]

By the assumption of the pd-stability for \( \{\Phi_j\}_{j=1}^p \) (see Definition 1.2), we have \( |\lambda| < 1 \). \( \square \)

Let

\[
L = I_{pn^2} - \begin{bmatrix} 0 & \Phi_2^T \otimes \Phi_2^H & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Phi_p^T \otimes \Phi_p^H \\ \Phi_1^T \otimes \Phi_1^H & \cdots & \cdots & 0 \end{bmatrix}.
\]

Then \( L \) is a matrix representation of \( L \) on

\[
\mathcal{H}^{pn^2} \equiv \{[w_1^T, \ldots, w_p^T]^T | w_j = \text{vec}(W_j), W_j \in \mathcal{H}_n, j = 1, \ldots, p \}.
\]

Assume that \( \{\Phi_j\}_{j=1}^p \) is pd-stable. By Lemma 2.2, the matrix \( L \) defined by (2.6) is nonsingular, and thus \( L^{-1} \) exists. In such a case, we define the quantity \( \ell \) by

\[
\ell = \|L^{-1}\|^{-1},
\]

where the operator norm \( \|\cdot\| \) for \( L^{-1} \) is induced by the Frobenius norm \( \|\cdot\|_F \) on \( \mathbb{C}_n^{pn^2} \).

In Appendix B, we shall prove that the quantity \( \ell \) can be expressed by \( \ell = \|L^{-1}\|_2^{-1} \).
For the pd-stable periodic matrix set \( \{ \Phi_j \}_{j=1}^p \), we define the quantity \( s \) by

\[
(2.8) \quad s = \min \left\{ \max_{1 \leq j \leq p} \| E_j \|_2 : \rho(\{ \Phi_j + E_j \}_{j=1}^p) = 1, \ E_j \in \mathbb{C}_n \right\}.
\]

The quantity \( s \) measures the smallest \( \max_{1 \leq j \leq p} \| E_j \|_2 \) such that \( \{ \Phi_j + E_j \}_{j=1}^p \) has an eigenvalue on the unit circle.

**Lemma 2.3.** Let \( \{ \Phi_j \}_{j=1}^p \) be pd-stable, and let \( L \) be the linear operator defined by (2.1) with \( L \) of (2.6) as its matrix representation. Let \( \varphi = \max_{1 \leq j \leq p} \| \Phi_j \|_2 \), \( \ell = \| L^{-1} \|_2^{-1} \), and \( s \) be given by (2.8). Then

\[
(2.9) \quad \frac{\ell}{\varphi + \sqrt{\varphi^2 + \ell}} \leq s.
\]

**Proof.** Let \( E_j^* \in \mathbb{C}_n \) \( (j = 1, \ldots, p) \) satisfy

\[
(2.10) \quad s = \max_{1 \leq j \leq p} \| E_j^* \|_2 \text{ with } \rho(\{ \Phi_j + E_j^* \}_{j=1}^p) = 1.
\]

By Lemma 2.1, the transformation

\[
(2.11) \quad \begin{bmatrix} W_1 \\ \vdots \\ W_p \end{bmatrix} \mapsto \begin{bmatrix} W_1 - (\Phi_2 + E_2^*)^H W_2 (\Phi_2 + E_2^*) \\ \vdots \\ W_{p-1} - (\Phi_p + E_p^*)^H W_p (\Phi_p + E_p^*) \\ W_p - (\Phi_1 + E_1^*)^H W_1 (\Phi_1 + E_1^*) \end{bmatrix}
\]

is singular, where \( W_j \in \mathcal{H}_n, j = 1, \ldots, p \). Thus there are Hermitian matrices \( W_1^*, \ldots, W_p^* \) with \( W_j^* \neq 0 \) for some \( j \in \{1, \ldots, p\} \) such that

\[
(2.12) \quad L \begin{bmatrix} W_1^* \\ W_2^* \\ \vdots \\ W_p^* \end{bmatrix} = \begin{bmatrix} \Phi_2^H W_2^* E_2^* + E_2^* W_2^* \Phi_2 + E_2^* W_2^* E_2^* \\ \vdots \\ \Phi_p^H W_p^* E_p^* + E_p^* W_p^* \Phi_p + E_p^* W_p^* E_p^* \\ \Phi_1^H W_1^* E_1^* + E_1^* W_1^* \Phi_1 + E_1^* W_1^* E_1^* \end{bmatrix};
\]

or, equivalently, by letting \( \text{vec}(W_j^*) = w_j^* \), we have

\[
(2.13) \quad L \begin{bmatrix} w_1^* \\ \vdots \\ w_p^* \end{bmatrix} = (\text{cyc}(\Omega_j^p))^{\top} \begin{bmatrix} w_1^* \\ \vdots \\ w_p^* \end{bmatrix},
\]

where \( \Omega_j \equiv E_j^\top \odot \Phi_j^H + \Phi_j^\top \odot E_j^H + E_j^\top \odot E_j^H \) for \( j = 1, \ldots, p \). Inverting \( L \) and taking the 2-norm of (2.13), we get \( s^2 + 2 \varphi s - \ell \geq 0 \), which implies the inequality (2.9).

The following lemma is an immediate corollary of Lemma 2.3.

**Lemma 2.4.** Let \( \{ \Phi_j \}_{j=1}^p \) be pd-stable, and let \( L \) be the linear operator defined by (2.1) with \( L \) in (2.6) as its matrix representation. Let \( \varphi = \max_{1 \leq j \leq p} \| \Phi_j \|_2 \) and \( \ell = \| L^{-1} \|_2^{-1} \). If \( E_j \in \mathbb{C}_n (j = 1, \ldots, p) \) satisfy

\[
\max_{1 \leq j \leq p} \| E_j \|_2 \leq \frac{\ell}{\varphi + \sqrt{\varphi^2 + \ell}},
\]

then \( \{ (\Phi_j + E_j) \}_{j=1}^p \) is pd-stable. □
3. Perturbation results for the P-DARE. In this section, we present perturbation bounds for the Hermitian p.s.d. solution set to the P-DARE (1.1).

Consider the P-DARE (1.1),

\[ X_{j-1} = A_j^H X_j (I + G_j X_j)^{-1} A_j + H_j, \quad j = 1, \ldots, p, \]

and a perturbed P-DARE (1.4),

\[ \tilde{X}_{j-1} = \tilde{A}_j^H \tilde{X}_j (I + \tilde{G}_j \tilde{X}_j)^{-1} \tilde{A}_j + \tilde{H}_j, \quad j = 1, \ldots, p. \]

For simplicity, we now consider the case of \( p = 3 \). Define

\[
F = (I + G_j X_j)^{-1}, \quad \Phi_j = (I + G_j X_j)^{-1} A_j,
\]

and define

\[
\Delta X_j = \tilde{X}_j - X_j, \quad \Delta A_j = \tilde{A}_j - A_j, \quad \Delta G_j = \tilde{G}_j - G_j, \quad \Delta H_j = \tilde{H}_j - H_j
\]

for \( j = 1, 2, 3 \).

Recall the linear operator \( L : \mathcal{H}_n^3 \rightarrow \mathcal{H}_n^3 \) defined by (2.1), that is,

\[
L(W_1, W_2, W_3) = (W_1, W_2, W_3) - (\Phi_2^H W_2 \Phi_2, \Phi_3^H W_3 \Phi_3, \Phi_1^H W_1 \Phi_1),
\]

where \( W_j \in \mathcal{H}_n \) for \( j = 1, 2, 3 \), and recall its matrix representation \( L \) given by (2.6).

It is easily seen that

\[
\lambda(L) = \{1 - \lambda \mid \lambda^3 \in \lambda((\Phi_3 \Phi_2 \Phi_1)^T \otimes (\Phi_3 \Phi_2 \Phi_1)^H)\}.
\]

From (3.1) and Theorem 1.1, it follows that \( |\lambda| < 1, k = 1, \ldots, n \). Hence \( L \), and thus \( L \), are invertible.

Further, we define the operator \( P : \mathbb{C}_n^3 \rightarrow \mathcal{H}_n^3 \) and the linear operator \( Q : \mathcal{H}_n^3 \rightarrow \mathcal{H}_n^3 \) by

\[
P(N_1, N_2, N_3) = L^{-1}(K_2^H N_2 + N_2^H K_2, K_3^H N_3 + N_3^H K_3, K_1^H N_1 + N_1^H K_1)
\]

and

\[
Q(M_1, M_2, M_3) = L^{-1}(K_2^H M_2 K_2, K_3^H M_3 K_3, K_1^H M_1 K_1),
\]

respectively, where \( N_1, N_2, N_3 \in \mathbb{C}_n \) and \( M_1, M_2, M_3 \in \mathcal{H}_n \).

The main result of this section is the following theorem.

**Theorem 3.1.** Let \( \{X_j\}_{j=1}^p \) be the unique Hermitian p.s.d. solution set to the P-DARE (1.1), and let \( \tilde{A}_j = A_j + \Delta A_j, \tilde{G}_j = G_j + \Delta G_j, \tilde{H}_j = H_j + \Delta H_j \) \( (j = 1, \ldots, p) \) be the coefficient matrices of the perturbed P-DARE (1.4). Define the operators \( L, P, \) and \( Q \) by (3.4), (3.6), and (3.7), respectively. Let

\[
\ell = \|L^{-1}\|^{-1}, \quad p_d = \|P\|, \quad q_d = \|Q\|
\]

where \( \|\cdot\| \) denotes the operator norm induced by \( \|\cdot\|_F \). Moreover, let

\[
\varphi = \max_{1 \leq j \leq p} \|\Phi_j\|_2
\]
(3.10) \[ \alpha = \max_{1 \leq j \leq p} \left\{ \frac{\| F_j \|_2 (\| A_j \|_2 + \| \Delta A_j \|_2)}{1 - \| \Psi_j \|_2 \| \Delta G_j \|_2} \right\}, \]

(3.11) \[ \gamma = \max_{1 \leq j \leq p} \left\{ \frac{\| F_j \|_2 (\| G_j \|_2 + \| \Delta G_j \|_2)}{1 - \| \Psi_j \|_2 \| \Delta G_j \|_2} \right\}, \]

(3.12) \[ \delta_j = \frac{\| \Delta A_j \|_2 + \| K_j \|_2 \| \Delta G_j \|_2}{1 - \| \Psi_j \|_2 \| \Delta G_j \|_2}, \quad j = 1, \ldots, p, \]

(3.13) \[ \zeta = \max_{1 \leq j \leq p} \{ |F_j|_2 \delta_j (2\varphi + |F_j|_2 \delta_j)\}, \]

\[ \epsilon_1 = \frac{1}{\ell} \left( \| \Delta H_1, \ldots, \Delta H_p \|_F + \sum_{j=1}^{p} (|\Delta A_j|_F + \| K_j \|_2 \| \Delta G_j \|_F)^2 \right)^{\frac{1}{2}}, \]

and

(3.16) \[ \xi_* = \frac{2\epsilon}{\ell - \zeta + \ell \gamma \epsilon + \sqrt{\ell - \zeta + \ell \gamma \epsilon} \epsilon - 4 \ell \gamma (\ell - \zeta + \alpha^2) \epsilon}. \]

If \( \tilde{G}_j, \tilde{H}_j \geq 0 \) \( (j = 1, \ldots, p) \), and if

(3.17) \[ 1 - \| \Psi_j \|_2 \| \Delta G_j \|_2 > 0 \quad (j = 1, \ldots, p), \]

(3.18) \[ 1 - \gamma \xi_* > 0, \]

(3.19) \[ \max_{1 \leq j \leq p} \left\{ \frac{\| F_j \|_2 \delta_j + \| \Phi_j \|_2 \gamma \xi_*}{1 - \gamma \xi_*} \right\} < \frac{\ell}{\varphi + \sqrt{\varphi^2 + \ell}}, \]

and

(3.20) \[ \epsilon \leq \frac{(\ell - \zeta)^2}{\ell \gamma (\ell - \zeta + 2\alpha + \sqrt{\ell - \zeta + 2\alpha})^2 - (\ell - \zeta)^2}, \]

then the perturbed P-DARE (1.4) has a unique Hermitian p.s.d. solution set \( \{ \tilde{X}_j \}_{j=1}^{p} \), and

(3.21) \[ \| (\tilde{X}_1 - X_1, \ldots, \tilde{X}_p - X_p) \|_F \leq \xi*. \]

See Appendix A for a proof of Theorem 3.1.

Let

\[ \delta_{A,G,H} = \sqrt{\sum_{j=1}^{p} (\| \Delta A_j, \Delta G_j, \Delta H_j \|_F^2).} \]

According to the definitions of \( \xi_*, \delta_j \) \( (j = 1, \ldots, p) \), and \( \epsilon \), the conditions (3.17)–(3.20) simply mean that the quantity \( \delta_{A,G,H} \) should be sufficiently small. Theorem 3.1 concludes that, if \( \delta_{A,G,H} \) is so small that the conditions (3.17)–(3.20) are satisfied and \( \tilde{G}_j, \tilde{H}_j \geq 0 \) \( (j = 1, \ldots, p) \), then the perturbed P-DARE (1.4) has a unique Hermitian p.s.d. solution set \( \{ \tilde{X}_j \}_{j=1}^{p} \) with the estimate (3.21).
Remark 3.1. First order estimates. Obviously, the estimate (3.21) can be expressed by
\[ \|(\Delta X_1, \ldots, \Delta X_p)\|_F = O(\delta_{A,G,H}), \quad \text{as} \quad \delta_{A,G,H} \to 0, \]
where \( \Delta X_j = \tilde{X}_j - X_j \) for \( j = 1, \ldots, p \). Moreover, by the proof of Theorem 3.1 (see Appendix A), we have the first order perturbation expansion of \((\tilde{X}_1, \ldots, \tilde{X}_p)\) at \((X_1, \ldots, X_p)\),
\[ (\tilde{X}_1, \ldots, \tilde{X}_p) = (X_1, \ldots, X_p) + L^{-1}(\Delta H_2, \ldots, \Delta H_p, \Delta H_1) + P(\Delta A_2, \ldots, \Delta A_p, \Delta A_1) \]
\[ - Q(\Delta G_2, \ldots, \Delta G_p, \Delta G_1) + O(\delta_{A,G,H}^2), \]
(3.22)
as \( \delta_{A,G,H}^2 \to 0 \), and thus we get the first order perturbation bound for the solution set \( \{X_j\}_{j=1}^p \) :
\[ \|(\tilde{X}_1 - X_1, \ldots, \tilde{X}_p - X_p)\|_F \leq \frac{1}{\ell} \|(\Delta H_1, \ldots, \Delta H_p)\|_F + p_d \|(\Delta A_1, \ldots, \Delta A_p)\|_F + q_d \|(\Delta G_1, \ldots, \Delta G_p)\|_F \]
\[ = \epsilon_1 \quad \text{as} \quad \delta_{A,G,H} \to 0. \]
(3.23)

Remark 3.2. Outline of the proof of Theorem 3.1. We prove Theorem 3.1 by three steps (see Appendix A for the details).

Step 1. From the P-DARE (1.1) and the perturbed P-DARE (1.4), we get an equation for \((\Delta X_1, \ldots, \Delta X_p)\), i.e., a perturbation equation.

Step 2. According to the perturbation equation, we define a continuous mapping \( M : \mathcal{H}_n^p \to \mathcal{H}_n^p \) so that any fixed point of \( M \) is a solution of the equation. Thus the problem of finding a perturbation bound for the Hermitian p.s.d. solution set \( \{X_j\}_{j=1}^p \) to the P-DARE (1.1) reduces to the problem of showing the existence of a fixed point \((\Delta X_1^*, \ldots, \Delta X_p^*)\) of \( M \) and determining a bound on its size. This can be done by applying the Schauder fixed-point theorem under certain assumptions on the perturbations in \( A_j, G_j, \) and \( H_j \) for \( j = 1, \ldots, p \).

Step 3. We prove that \((X_1 + \Delta X_1^*, \ldots, X_p + \Delta X_p^*)\) is the unique Hermitian p.s.d. solution set to the perturbed P-DARE (1.4).

Remark 3.3. The nonperiodic case \((p = 1)\). The DARE is in the form
\[ X = A^H X (I + GX)^{-1} A + H, \]
where \( A \in \mathbb{C}_n, G, H \in \mathcal{H}_n, \) and \( G, H \geq 0 \). Appropriate assumptions on the coefficient matrices guarantee the existence and uniqueness of a Hermitian p.s.d. solution \( X \). Set \( p = 1 \) in Theorem 3.1; then we obtain a perturbation result for the DARE which just coincides with [20, Theorem 4.1], but the operator \( L \) is defined by
\[ LW = W - \Phi^H W \Phi, \quad W \in \mathcal{H}_n, \]
where \( \Phi = (I + GX)^{-1} A \) is d-stable, and the operators \( P \) and \( Q \) are defined by
\[ PN = L^{-1} (K^H N + N^H K), \quad N \in \mathbb{C}_n, \]
and
\[ QM = L^{-1} (K^H MK), \quad M \in \mathcal{H}_n, \]
respectively, in which $K = X(I + GX)^{-1}A$.

Remark 3.4. Expression of $\ell$, $p_d$, and $q_d$. Let $L, P$, and $Q$ be the operators defined by (2.1), (3.6), and (3.7), respectively, and let $\ell, p_d$, and $q_d$ be the quantities defined by (3.8). Let

$$L = I_{pn^2} - [\text{cyc}\{\Phi_j \otimes \bar{\Phi}_j\}_{j=1}^p]^\top,$$

as in (2.6), where $\{\Phi_j\}_{j=1}^p$ is pd-stable. Let

$$L^{-1}[\text{cyc}\{I \otimes \bar{K}_j\}_{j=1}^p]^\top = \Omega_1 + i\Omega_2,$$

$$L^{-1}[\text{cyc}\{\Pi^\top(K_j \otimes I)\}_{j=1}^p]^\top = \Theta_1 + i\Theta_2,$$

where $\Omega_1, \Omega_2, \Theta_1$, and $\Theta_2$ are real matrices, $\Pi$ is the vec-permutation matrix [10, pp. 32–34], and $K_j = X_j(I + G_jX_j)^{-1}A_j$ for $j = 1, \ldots, p$, as in (3.5). Then

$$(3.24) \quad \ell = \|L^{-1}\|_2^{-1},$$

$$(3.25) \quad p_d = \left\| \begin{bmatrix} \Omega_1 + \Theta_1 & \Theta_2 - \Omega_2 \\ \Omega_2 + \Theta_2 & \Omega_1 - \Theta_1 \end{bmatrix} \right\|_2$$

for the real case, and, especially,

$$(3.26) \quad p_d = \|L^{-1}[\text{cyc}\{I \otimes K_j + \Pi^\top(K_j \otimes I)\}_{j=1}^p]^\top\|_2$$

and

$$(3.27) \quad q_d = \|L^{-1}[\text{cyc}\{K_j \otimes \bar{K}_j\}_{j=1}^p]^\top\|_2.$$

See Appendix B for a proof of the formulae (3.24)–(3.27).

4. Condition number of $\{X_j\}_{j=1}^p$. In this section, we define a condition number of the Hermitian p.s.d. solution set $\{X_j\}_{j=1}^p$ to the P-DARE (1.1) and derive explicit expressions of the condition number.

For simplicity, we first consider $p = 3$. From Theorem 3.1 and (3.22), we see that, if $G_j + \Delta G_j \succeq 0$ and $H_j + \Delta H_j \succeq 0$ for all $j$, then

$$(\Delta X_1, \Delta X_2, \Delta X_3) = L^{-1}(\Delta H_2, \Delta H_3, \Delta H_1) + P(\Delta A_2, \Delta A_3, \Delta A_1)$$

$$- Q(\Delta G_2, \Delta G_3, \Delta G_1) + O(\delta_{A,G,H}^2)$$

$$= L^{-1}(\Delta H_2, \Delta H_3, \Delta H_1) + (K^H_2 \Delta A_2 + \Delta A_2^H K_2, K^H_3 \Delta A_3 + \Delta A_3^H K_3, K^H_1 \Delta A_1 + \Delta A_1^H K_1) - (K^H_2 \Delta G_2 K_2, K^H_3 \Delta G_3 K_3, K^H_1 \Delta G_1 K_1) + O(\delta_{A,G,H}^2),$$

as $\delta_{A,G,H} \to 0$,

$$(4.1)$$

where $\delta_{A,G,H} = \sqrt{\sum_{j=1}^3 \|\Delta H_j, \Delta A_j, \Delta G_j\|^2}$, $\Delta H_j, \Delta G_j \in \mathcal{H}_n, \Delta A_j \in \mathbb{C}_n$ for $j = 1, 2, 3$. Let

$$(4.2) \quad \rho = \left\| \begin{bmatrix} \Delta H_1 \eta_1 & \Delta H_2 \eta_2 & \Delta H_3 \eta_3 \\ \Delta A_1 \alpha_1 & \Delta A_2 \alpha_2 & \Delta A_3 \alpha_3 \\ \Delta G_1 \gamma_1 & \Delta G_2 \gamma_2 & \Delta G_3 \gamma_3 \end{bmatrix} \right\|_F,$$
where ηj, αj, γj are positive parameters. By the theory of condition developed by Rice [19], we define the condition number c(X1, X2, X3) by

\[
\text{(4.3a)} \quad c(X_1, X_2, X_3) = \lim_{\delta \to 0} \sup_{F \geq 0 \atop \delta H_j + \Delta H_j \geq 0} \frac{\| (\Delta x_1, \Delta x_2, \Delta x_3) \|_F}{\delta},
\]

where ξ1, ξ2, ξ3 are positive parameters.

By using the technique described by [21], we need only to derive an explicit expansion of c(X1, X2, X3) in the case of Gj + ΔGj > 0 and Hj + ΔHj > 0 for all j; and in such a case, the definition (4.3a) can be written

\[
\text{(4.3b)} \quad c(X_1, X_2, X_3) = \lim_{\delta \to 0} \sup_{\rho \leq \delta} \frac{\| (\Delta x_1, \Delta x_2, \Delta x_3) \|_F}{\rho},
\]

Define the operator \( \mathbf{V} : \mathcal{H}_n^3 \times \mathbb{C}_n^3 \times \mathcal{H}_n^3 \to \mathcal{H}_n^3 \) by

\[
\mathbf{V}(M_1, M_2, M_3, D_1, D_2, D_3, Q_1, Q_2, Q_3)
= L^{-1}[(M_2, M_3, M_1)H^{(2)} + (K_2^H D_2 + D_2^H K_2, K_3^H D_3 + D_3^H K_3, K_1^H D_1 + D_1^H K_1)A^{(2)} - (K_2^H Q_2 K_2, K_3^H Q_3 K_3, K_1^H Q_1 K_1)\Gamma^{(2)}]^{1/2},
\]

where \( M_j, Q_j \in \mathcal{H}_n, D_j \in \mathbb{C}_n \) for \( j = 1, 2, 3 \), and

\[
H^{(2)} = \text{diag}(\eta_2 I_n, \eta_3 I_n, \eta_1 I_n), \quad A^{(2)} = \text{diag}(\alpha_2 I_n, \alpha_3 I_n, \alpha_1 I_n), \quad \Gamma^{(2)} = \text{diag}(\gamma_2 I_n, \gamma_3 I_n, \gamma_1 I_n), \quad \Xi = \text{diag}(\xi_1 I_n, \xi_2 I_n, \xi_3 I_n).
\]

Substituting (4.1) into (4.3b) gives

\[
\text{(4.6)} \quad c(X_1, X_2, X_3) = \sup_{(M_1, \ldots, D_1, \ldots, Q_1, \ldots) \neq 0 \atop M_j, Q_j \in \mathcal{H}_n, D_j \in \mathbb{C}_n} \frac{\| \mathbf{V}(M_1, M_2, M_3, D_1, D_2, D_3, Q_1, Q_2, Q_3) \|_F}{\| (M_1, M_2, M_3, D_1, D_2, D_3, Q_1, Q_2, Q_3) \|_F}.
\]

Further, we define the operator \( \tilde{\mathbf{V}} : \mathbb{C}_n^3 \times \mathbb{C}_n^3 \times \mathbb{C}_n^3 \to \mathbb{C}_n^3 \) by

\[
\tilde{\mathbf{V}}(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3)
= \tilde{L}^{-1}[(N_2, N_3, N_1)H^{(2)} + (K_2^H E_2 + E_2^H K_2, K_3^H E_3 + E_3^H K_3, K_1^H E_1 + E_1^H K_1)A^{(2)} - (K_2^H R_2 K_2, K_3^H R_3 K_3, K_1^H R_1 K_1)\Gamma^{(2)}]^{1/2},
\]

where \( N_j, E_j, R_j \in \mathbb{C}_n \) for \( j = 1, 2, 3 \), and \( \tilde{L} \) is a natural extension of \( L \) on \( \mathbb{C}_n^3 \). From the definitions (4.4) and (4.7), we know that

\[
\text{(4.8a)} \quad \sup_{(M_1, \ldots, D_1, \ldots, Q_1, \ldots) \neq \mathbb{C}_n^3} \frac{\| \mathbf{V}(M_1, M_2, M_3, D_1, D_2, D_3, Q_1, Q_2, Q_3) \|_F}{\| (M_1, M_2, M_3, D_1, D_2, D_3, Q_1, Q_2, Q_3) \|_F}.
\]

\[
\text{(4.8b)} \quad \sup_{(N_1, \ldots, E_1, \ldots, R_1, \ldots) \neq \mathbb{C}_n^3} \frac{\| \tilde{\mathbf{V}}(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3) \|_F}{\| (N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3) \|_F}.
\]
We now prove that the equality in (4.8) holds. Let \((N_1^*, N_2^*, N_3^*, E_1^*, E_2^*, E_3^*, R_1^*, R_2^*, R_3^*)\)
be the singular “vector” of \(\hat{V}\) corresponding to the maximal singular value; that is, the right-hand side of (4.8b) equals
\[
\frac{\|\hat{V}(N_1^*, N_2^*, N_3^*, E_1^*, E_2^*, E_3^*, R_1^*, R_2^*, R_3^*)\|_F}{\|(N_1^*, N_2^*, N_3^*, E_1^*, E_2^*, E_3^*, R_1^*, R_2^*, R_3^*)\|_F}
\]
(4.9)
Let

\[
(Z_1^*, Z_2^*, Z_3^*) = \hat{V}(N_1^*, N_2^*, N_3^*, E_1^*, E_2^*, E_3^*, R_1^*, R_2^*, R_3^*) \in \mathbb{C}^3.
\]

(4.10)
Then, by the definition (4.7) and the definition of \(\hat{L}\), we have
\[
(Z_1^*, Z_2^*, Z_3^*) = (\Phi_1^H Z_2^* \Phi_2^*, \Phi_3^H Z_3^* \Phi_3^*, \Phi_1^H Z_1^* \Phi_1^*) = \hat{L}(Z_1^*, Z_2^*, Z_3^*)
\]
(4.11)
Further, from (4.11),
\[
(Z_1^{**}, Z_2^{**}, Z_3^{**}) = \hat{L}^{-1}[(N_2^{**}, N_3^{**}, N_1^{**}, H^{(2)} + (K_2^H E_2^* + E_2^H K_2^* K_2^* E_2^* + E_3^H K_3^* K_1^H E_1^* + E_1^H K_1^*) A^{(2)} - (K_2^H R_2^* K_2^* K_3^* R_3^* K_3^* K_1^H R_1^* K_1^*) \Gamma^{(2)}].
\]
(4.12)
Since \(\|(Z_1^{**}, Z_2^{**}, Z_3^{**})\|_F = \|(Z_1^*, Z_2^*, Z_3^*)\|_F\), from (4.9), (4.10), and (4.12), it follows that the right-hand side of (4.8b) equals
\[
\frac{\|\hat{V}(N_1^{**}, N_2^{**}, N_3^{**}, E_1^{**}, E_2^{**}, E_3^{**}, R_1^{**}, R_2^{**}, R_3^{**})\|_F}{\|(N_1^{**}, N_2^{**}, N_3^{**}, E_1^{**}, E_2^{**}, E_3^{**}, R_1^{**}, R_2^{**}, R_3^{**})\|_F};
\]
(4.13)
that is, \((N_1^{**}, N_2^{**}, N_3^{**}, E_1^{**}, E_2^{**}, E_3^{**}, R_1^{**}, R_2^{**}, R_3^{**})\) is also a singular “vector” of \(\hat{V}\) corresponding to the maximal singular value.

Let
\[
M_j^* = N_j^* + N_j^{**} \in \mathcal{H}_n, \quad D_j^* = 2E_j^* \in \mathbb{C}_n, \quad Q_j^* = R_j^* + R_j^{**} \in \mathcal{H}_n
\]
(4.14)
for \(j = 1, 2, 3\). If \((M_1^*, M_2^*, M_3^*, D_1^*, D_2^*, D_3^*, Q_1^*, Q_2^*, Q_3^*) \neq 0\), then it is also a singular “vector” of \(\hat{V}\) corresponding to the maximal singular value. By (4.4), (4.7), and the pd-stability of \(\{\Phi_j\}_{j=1}^3\), the right-hand side of (4.8b) equals
\[
\frac{\|\hat{V}(M_1^*, M_2^*, M_3^*, D_1^*, D_2^*, D_3^*, Q_1^*, Q_2^*, Q_3^*)\|_F}{\|(M_1^*, M_2^*, M_3^*, D_1^*, D_2^*, D_3^*, Q_1^*, Q_2^*, Q_3^*)\|_F}
\]
(4.15)
If \((M_1^*, M_2^*, M_3^*, D_1^*, D_2^*, D_3^*, Q_1^*, Q_2^*, Q_3^*) = 0\), then we set
\[
M_j^o = iN_j^o \in \mathcal{H}_n, \quad D_j^o = 0 \in \mathbb{C}_n, \quad Q_j^o = iR_j^o \in \mathcal{H}_n
\]
(4.16)
for $j = 1, 2, 3$. In such a case, $(M_1^0, M_2^0, M_3^0, D_1^0, D_2^0, D_3^0, Q_1^0, Q_2^0, Q_3^0)$ is also a singular “vector” of $\hat{V}$ corresponding to the maximal singular value. Hence the right-hand side of (4.8b) equals

$$
\frac{\|\hat{V}(M_1^0, M_2^0, M_3^0, D_1^0, D_2^0, D_3^0, Q_1^0, Q_2^0, Q_3^0)\|_F}{\|(M_1^0, M_2^0, M_3^0, D_1^0, D_2^0, D_3^0, Q_1^0, Q_2^0, Q_3^0)\|_F} = \frac{\|V(M_1^0, M_2^0, M_3^0, D_1^0, D_2^0, D_3^0, Q_1^0, Q_2^0, Q_3^0)\|_F}{\|(M_1^0, M_2^0, M_3^0, D_1^0, D_2^0, D_3^0, Q_1^0, Q_2^0, Q_3^0)\|_F}.
$$

(4.17)

Therefore, from (4.15) and (4.17), it follows that the equality in (4.8) holds. Combining this result with (4.4), (4.6), and (4.7), we obtain

$$
c(X_1, X_2, X_3) = \sup_{(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3) \neq 0} \frac{\|C(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3)\|_F}{\|(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3)\|_F},
$$

(4.18)

where

$$
C(N_1, N_2, N_3, E_1, E_2, E_3, R_1, R_2, R_3)
= \tilde{L}^{-1}((N_2, N_3, N_1)H(2) + (K_2H)E_2 + (K_2H)K_2) E_3 + (K_3H)K_3 + (K_1H)E_1 + (K_1H)K_1)A(2)
$$

(4.19) 

$- (K_2H)R_2K_2 + (K_3H)R_3K_3 + (K_1H)R_1K_1)\Omega(2)\Xi^{-1}.$

For the general case of an arbitrary $p \geq 2$, we have a similar formula to (4.18) and (4.19). Let $z_j = \text{vec}(N_j), w_j = \text{vec}(E_j), c_j = \text{vec}(R_j)$ for $j = 1, \ldots, p$. Then, from (4.18) and (4.19), we see that $c(X_1, \ldots, X_p)$ can be written as

$$
c(X_1, \ldots, X_p) = \sup_{[\mathbf{z}_1^T, \ldots, \mathbf{z}_p^T, \mathbf{w}_1^T, \ldots, \mathbf{w}_p^T, \mathbf{c}_1^T, \ldots, \mathbf{c}_p^T]^T \neq 0} \left\| \tilde{\Xi}^{-1} \left( L^{-1} \hat{H}(2) \begin{bmatrix} \mathbf{z}_1^T \\ \vdots \\ \mathbf{z}_p^T \end{bmatrix} + L^{-1} \hat{A}(2) \begin{bmatrix} (I \otimes K_1H)w_2 + (K_2kH)w_2 \\ \vdots \\ (I \otimes K_1H)w_p + (K_2kH)w_p \end{bmatrix} - L^{-1} \hat{F}(2) \begin{bmatrix} (K_1H)k \mathbf{c}_2 \\ \vdots \\ (K_1H)k \mathbf{c}_p \end{bmatrix} \right) \right\|_F
$$

$$
= \sup_{[\mathbf{z}_1^T, \ldots, \mathbf{z}_p^T, \mathbf{w}_1^T, \ldots, \mathbf{w}_p^T, \mathbf{c}_1^T, \ldots, \mathbf{c}_p^T]^T \neq 0} \left\| \tilde{\Xi}^{-1} \left( L^{-1} \hat{H}(2) \begin{bmatrix} \mathbf{z}_1^T \\ \vdots \\ \mathbf{z}_p^T \end{bmatrix} + P_1 \begin{bmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_p^T \end{bmatrix} + P_2 \begin{bmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_p^T \end{bmatrix} - Q \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_p^T \end{bmatrix} \right) \right\|_F
$$

$$
= \sup_{[\mathbf{z}_1^T, \ldots, \mathbf{z}_p^T, \mathbf{w}_1^T, \ldots, \mathbf{w}_p^T, \mathbf{c}_1^T, \ldots, \mathbf{c}_p^T]^T \neq 0} \sqrt{\sum_{j=1}^p (\|z_j\|_2^2 + \|w_j\|_2^2 + \|c_j\|_2^2)}
$$

(4.20)

where

$$
\hat{H}(2) = \text{diag}(\eta_1 I_n^2, \ldots, \eta_p I_n^2, \eta_1 I_n^2), \quad \hat{A}(2) = \text{diag}(\alpha_2 I_n^2, \ldots, \alpha_p I_n^2, \alpha_1 I_n^2),
$$

$$
\hat{F}(2) = \text{diag}(\gamma_2 I_n^2, \ldots, \gamma_p I_n^2, \gamma_1 I_n^2), \quad \hat{\Xi} = \text{diag}(\xi_1 I_n^2, \ldots, \xi_p I_n^2)\]
and
\[ P_1 = L^{-1}[\text{cyc}\{\alpha_j I \otimes \mathbf{K}_j\}_{j=1}^p]^\top, \quad P_2 = L^{-1}[\text{cyc}\{\alpha_j \mathbf{K}_j \otimes I\}_{j=1}^p]^\top, \quad Q = L^{-1}[\text{cyc}\{\gamma_j \mathbf{K}_j \otimes I\}_{j=1}^p]^\top. \]

Denote
\[ z_j = x_j + iy_j, \quad w_j = u_j + iv_j, \quad c_j = a_j + ib_j, \]
where \( x_j, y_j, u_j, v_j, a_j, b_j \in \mathbb{R}^n \) for \( j = 1, \ldots, p \), and
\[ x = [x_1^\top, \ldots, x_p^\top]^\top, \quad y = [y_1^\top, \ldots, y_p^\top]^\top, \quad u = [u_1^\top, \ldots, u_p^\top]^\top, \quad v = [v_1^\top, \ldots, v_p^\top]^\top, \quad a = [a_1^\top, \ldots, a_p^\top]^\top, \quad b = [b_1^\top, \ldots, b_p^\top]^\top \]
as well as
\[ L^{-1} \hat{H}^{(2)} = \Omega_1 + i\Omega_2, \quad P_1 = U_1 + iU_2, \quad P_2 = V_1 + iV_2, \quad Q = \Theta + i\Theta_2, \]
where \( \Omega_k, U_k, V_k, \Theta_k \) are real \( pn^2 \times pn^2 \)-matrices \((k = 1, 2)\). By a technique given by [14], substituting (4.22) into (4.21), we get
\[
c(X_1, \ldots, X_p) = \sup_{[x^\top, y^\top, u^\top, v^\top, a^\top, b^\top]^\top \neq 0} \frac{\| \hat{\Xi}^{-1} \left[ \begin{array}{cccc} 0 & \Omega_1 & -\Omega_2 & U_1 + V_1 \\ \Omega_2 & -\Omega_1 & U_2 + V_2 & V_2 - U_2 \\ U_1 - V_1 & U_2 - V_2 & -\Theta_1 & -\Theta_2 \\ -\Theta_1 & -\Theta_2 & -\Theta_1 1 & 1 & 1 & 1 & 1 & 1 \\ \end{array} \right] \left[ \begin{array}{c} x \\ y \\ u \\ v \\ a \\ b \end{array} \right] \|_2}{\sqrt{\|x\|^2 + \|y\|^2 + \|u\|^2 + \|v\|^2 + \|a\|^2 + \|b\|^2}}.
\]
(4.23)

Taking
\[ \xi_j = \eta_j = \alpha_j = \gamma_j = 1 \quad (j = 1, \ldots, p), \]
we get the absolute condition number \( c_{\text{abs}}(X_1, \ldots, X_p) \); and taking
\[ \xi_j = \|X_j\|_F, \quad \eta_j = \|H_j\|_F, \quad \alpha_j = \|A_j\|_F, \quad \gamma_j = \|G_j\|_F \quad (j = 1, \ldots, p), \]
we get the relative condition number \( c_{\text{rel}}(X_1, \ldots, X_p) \).

For the real case, we can prove that the equality in (4.9) also holds. Consequently, from (4.21), the condition number \( c(X_1, \ldots, X_p) \) can be explicitly expressed as follows:
\[
c(X_1, \ldots, X_p) = \sup_{[z^\top, w^\top, c^\top]^\top \neq 0} \frac{\|L^{-1} \hat{H}^{(2)}z + P_1w + P_2w - Qc\|_2}{\sqrt{\|z\|^2 + \|w\|^2 + \|c\|^2}}
= \|\hat{\Xi}^{-1}L^{-1}[\hat{H}^{(2)}; \text{cyc}\{\alpha_j I \otimes K_j + \mathbf{K}_j \otimes I\}_{j=1}^p]^\top, \text{cyc}\{\gamma_j \mathbf{K}_j \otimes I\}_{j=1}^p]^\top\|_2.
\]
(4.24)
The absolute condition number $c_{\text{abs}}(X_1, \ldots, X_p)$ and the relative condition number $c_{\text{rel}}(X_1, \ldots, X_p)$ for the real case can be obtained by evaluating $\xi_j, \eta_j, \alpha_j$, and $\gamma_j$ as above.

5. A numerical example. In this section, we use numerical examples to illustrate our perturbation results given in sections 3 and 4. All computations were performed using MATLAB version 5.3 on a Compaq/DS20 workstation. The machine precision is $2.22 \times 10^{-16}$.

Example 5.1 (see [13] for $p = 1$). Consider the P-DARE (1.1) with $n = 3$ and $p = 3$. Let

$$H_j^{(0)} = \text{diag} \left( \frac{1}{j} 10^{-m}, j \times 10^{-m} \right), \quad G_j^{(0)} = \text{diag} \left( \frac{1}{j} 10^{-m}, \frac{1}{j} 10^{-m}, j \times 10^{-m} \right), \quad j = 1, 2, 3,$$

$$A_j^{(0)} = \text{diag}(0, 10^{-m}, 1), \quad A_j^{(0)} = \text{diag}(10^{-9}, 10^{-m}, 1 + 10^{-3}),$$

$$A_j^{(0)} = \text{diag}(10^{-3}, 10^{-m+1}, 0.5),$$

and

$$V = I - 2vv^T, \quad v = \frac{1}{\sqrt{3}} [1, 1, 1]^T.$$ 

The coefficient matrices of (1.1) are constructed by

$$H_j = V^T H_j^{(0)} V, \quad A_j = V^T A_j^{(0)} V, \quad G_j = V^T G_j^{(0)} V = B_j B_j^T, \quad j = 1, 2, 3.$$ 

The unique symmetric p.s.d. solution set $\{X_j\}_{j=1}^3$ to (1.1) is given by $X_j = V^T X_j^{(0)} V$ for $j = 1, 2, 3$ with

$$X_j^{(0)} = 2 \left[ P_j^{(0)} + \left( P_j^{(0)} + 4 \tilde{H}_j^{(0)} \tilde{G}_j^{(0)} \right)^{1/2} \right] \tilde{G}_j^{(0)}^{-1} \text{ (diagonal)}$$

and

$$P_j^{(0)} = \tilde{A}_j^{(0)} + \tilde{H}_j^{(0)} \tilde{G}_j^{(0)} - I_3 \text{ (diagonal)},$$

where

$$\tilde{A}_j^{(0)} = A_j^{(0)} (I_3 + \tilde{G}_j^{(0)} H_j^{(0)})^{-1} \tilde{A}_j^{(0)},$$

$$\tilde{G}_j^{(0)} = G_j^{(0)} + A_j^{(0)} \tilde{G}_j^{(0)} (I_3 + H_j^{(0)} \tilde{G}_j^{(0)})^{-1} A_j^{(0)}^\top,$$

$$\tilde{H}_j^{(0)} = \tilde{H}_j^{(0)} + \tilde{A}_j^{(0)} (I_3 + H_j^{(0)} \tilde{G}_j^{(0)})^{-1} A_j^{(0)} \tilde{A}_j^{(0)},$$

and

$$\tilde{A}_j^{(0)} = A_j^{(0)} (I_3 + G_j^{(0)} H_j^{(0)})^{-1} A_j^{(0)},$$

$$\tilde{G}_j^{(0)} = G_j^{(0)} + A_j^{(0)} G_j^{(0)} (I_3 + H_j^{(0)} G_j^{(0)})^{-1} A_j^{(0)}^\top,$$

$$\tilde{H}_j^{(0)} = H_j^{(0)} + A_j^{(0)} (I_3 + H_j^{(0)} G_j^{(0)})^{-1} H_j^{(0)} A_j^{(0)}.$$
Let
\[
\Delta H_j^{(0)} = \begin{bmatrix}
\frac{1}{j} 10^m & -5j & 7 \\
-5j & j & 3 \\
7 & 3 & j \times 10^{-m}
\end{bmatrix} \times 10^{-k},
\Delta A_j^{(0)} = \begin{bmatrix}
3j & -\frac{4}{5} & \frac{8}{7} \\
-6j & \frac{2}{5} & -\frac{9}{7} \\
2j & 7j & \frac{5}{7}
\end{bmatrix} \times 10^{-k},
\]
\[
\Delta G_j^{(0)} = \begin{bmatrix}
\frac{1}{j} 10^{-m} & -\frac{1}{j} 10^{-m} & \frac{2}{j} 10^{-m} \\
-\frac{1}{j} 10^{-m} & \frac{5}{j} 10^{-m} & -j \times 10^{-m} \\
\frac{2}{j} 10^{-m} & -j \times 10^{-m} & 3j \times 10^{-m}
\end{bmatrix} \times 10^{-k}, \quad j = 1, 2, 3.
\]

The coefficient matrices of the perturbed P-DARE (1.4) are given by
\[
\tilde{H}_j = H_j + V^T(\Delta H_j^{(0)})V, \quad \tilde{A}_j = A_j + V^T(\Delta A_j^{(0)})V,
\]
\[
\tilde{G}_j = G_j + V^T(\Delta G_j^{(0)})V \equiv \tilde{B}_j^{\top} \tilde{B}_j.
\]

By applying the file “DARE” of Control System Toolbox in MATLAB, one can compute the unique symmetric p.s.d. solution set
\[
\{\tilde{X}_j\}_{j=1}^3 \text{ to (1.4).}
\]

Denote
\[
\delta_x = \|(\tilde{X}_1 - X_1, \tilde{X}_2 - X_2, \tilde{X}_3 - X_3)\|_F, \quad n_x = \|(X_1, X_2, X_3)\|_F.
\]

Let \(\epsilon_1\) be the quantity defined by (A.21), where \(l, p_d, q_d\) are given by (3.24), (3.26), and (3.27), respectively, and let
\[
\epsilon_1 = \frac{1}{\ell} \left( \delta_h + 2 \max_{1 \leq j \leq 3} \{\|K_j\|_2\} \delta_a + \max_{1 \leq j \leq 3} \{\|K_j\|_2^2\} \delta_g \right).
\]

Some numerical results on relative and absolute perturbation bounds are listed in Tables 5.1–5.3, where the bounds \(\epsilon_1, \tilde{\epsilon}_1, \) and \(\xi_*\) are defined by (A.21), (5.1), and (A.32). The relative condition number \(c_{rel} \equiv c_{rel}(X_1, X_2, X_3)\) and the absolute condition number \(c_{abs} \equiv c_{abs}(X_1, X_2, X_3)\) are computed by (4.24). The cases when the conditions of Theorem 3.1 are violated are denoted by asterisks.

The results listed in Tables 5.1–5.3 show that the relative perturbation bounds \(\epsilon_1/n_x\) and \(\xi_*/n_x\), as well as the absolute perturbation bounds \(\epsilon_1\) and \(\xi_*\), are fairly sharp. The immediate upper bound \(\tilde{\epsilon}_1\) for \(\epsilon_1\) has almost the same order as \(\epsilon_1\) in this example, which might be used to estimate \(\epsilon_1\) economically when the size of the system is too large.

**Appendix A. Proof of Theorem 3.1.** For simplicity, we now consider the case of \(p = 3\).

Step 1. Perturbation equation. Let
\[
X = \text{diag}\{X_j\}_{j=1}^3, \quad A = \text{cyc}\{A_j\}_{j=1}^3, \quad G = \text{diag}\{G_j\}_{j=1}^3, \quad H = \text{diag}(H_2, H_3, H_1).
\]

Let \( (A.2) \)

\[ X = A^H X (I + GX)^{-1} A + H. \]

Let

\( (A.3) \)

\[ F = \text{diag}\{(I + G_j X_j)^{-1}\}^3_{j=1} \equiv \text{diag}\{F_j\}^3_{j=1}, \]

\( (A.4) \)

\[ \Phi = \text{cyc}\{(I + G_j X_j)^{-1} A_j\}^3_{j=1} \equiv \text{cyc}\{\Phi_j\}^3_{j=1}. \]

Moreover, let

\( (A.5) \)

\[ \Psi = \text{diag}\{X_j (I + G_j X_j)^{-1}\}^3_{j=1} \equiv \text{diag}\{\Psi_j\}^3_{j=1}, \]

\( (A.6) \)

\[ K = \text{cyc}\{X_j (I + G_j X_j)^{-1} A_j\}^3_{j=1} \equiv \text{cyc}\{K_j\}^3_{j=1}, \]

\( (A.7) \)

\[ \Theta = \text{diag}\{(I + G_j X_j)^{-1} (I + \Delta G_j X_j (I + G_j X_j)^{-1})^{-1}\}^3_{j=1} \equiv \text{diag}\{\Theta_j\}^3_{j=1}. \]

By \([20, (4.7)–(4.12)]\), we have the perturbation equation

\( (A.8) \)

\[ \Delta X - \Phi^H \Delta X \Phi = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X), \]
where
\[
\Delta X - \Phi^H \Delta X \Phi = \text{diag}(\Delta X_1 - \Phi_1^H \Delta X_2 \Phi_2, 
\Delta X_2 - \Phi_2^H \Delta X_3 \Phi_3, \Delta X_3 - \Phi_3^H \Delta X_1 \Phi_1),
\]
(A.9)
\[
E_1 = \Delta H + K^H \Delta A + \Delta A^H K - K^H \Delta G K = \text{diag}(E_{12}, E_{13}, E_{11})
\]
(A.10a) with
\[
E_{1j} = \Delta H_j + K_j^H \Delta A_j + \Delta A_j^H K_j - K_j \Delta G_j K_j \in \mathcal{H}_n
\]
(A.10b) for \( j = 1, 2, 3; \)
\[
E_2 = \Delta A^H \Psi \Delta A + K^H \Delta G \Psi (I + \Delta G \Psi)^{-1} \Delta G K
\]
\[-K^H \Delta G \Psi (I + \Delta G \Psi)^{-1} \Delta A - \Delta A^H \Psi (I + \Delta G \Psi)^{-1} \Delta G (K + \Psi \Delta A)
\]
(A.11a) = \text{diag}(E_{22}, E_{23}, E_{21})
with
\[
E_{2j} = \Delta A_j^H \Psi_j \Delta A_j + K_j^H \Delta G_j \Psi_j (I + \Delta G_j \Psi_j)^{-1} \Delta G_j K_j
\]
\[-K_j^H \Delta G_j \Psi_j (I + \Delta G_j \Psi_j)^{-1} \Delta A_j
\]
\[-\Delta A_j^H \Psi_j (I + \Delta G_j \Psi_j)^{-1} \Delta G_j (K_j + \Psi_j \Delta A_j)
\]
\[\in \mathcal{H}_n, \quad \text{for } j = 1, 2, 3; \]
\[
h_1(\Delta X) = \Delta \Phi^H \Delta X \Phi + \Phi^H \Delta X \Delta \Phi + \Phi^H \Delta X \Delta \Phi
\]
with
(A.12a) \[
\Delta \Phi = F(I + \Delta G \Psi)^{-1}(\Delta A - \Delta G K) \equiv \text{cyc} \{\Delta \Phi_j\}_{j=1}^3,
\]
in which
(A.12b) \[
\Delta \Phi_j = F_j(I + \Delta G_j \Psi_j)^{-1}(\Delta A_j - \Delta G_j K_j) \quad \text{for } j = 1, 2, 3;
\]
so we have
(A.13a) \[
h_1(\Delta X) = \text{diag}(h_{12}(\Delta X), h_{13}(\Delta X), h_{11}(\Delta X))
\]
with
(A.13b) \[
h_{1j}(\Delta X) = \Delta \Phi_j^H \Delta X_j \Phi_j + \Phi_j^H \Delta X_j \Delta \Phi_j + \Phi_j^H \Delta X_j \Delta \Phi_j \in \mathcal{H}_n, \quad \text{for } j = 1, 2, 3
\]
and
(A.14a) \[
h_2(\Delta X) = -(A + \Delta A)^H \Theta^H \Delta X \Theta (G + \Delta G) \Delta X \Theta (I + (G + \Delta G) \Delta X \Theta)^{-1} (A + \Delta A)
\]
\[= \text{diag}(h_{22}(\Delta X), h_{23}(\Delta X), h_{21}(\Delta X))\]
Consequently, from (A.8), (A.9), (A.10a), (A.11a), (A.13a), and (A.14a), we obtain the perturbation equation

(A.15) \[ \Delta X_{j-1} - \Phi_j^H \Delta X_j \Phi_j = E_{1j} + E_{2j} + h_{1j}(\Delta X) + h_{2j}(\Delta X) \]

for \( j = 1, 2, 3 \), where \( E_{1j}, E_{2j}, h_{1j}(\Delta X) \), and \( h_{2j}(\Delta X) \) are defined by (A.10b), (A.11b), (A.13b), and (A.14b), respectively.

By using the operators \( \mathbf{L}, \mathbf{P}, \) and \( \mathbf{Q} \) (see (3.4), (3.6), and (3.7)), the perturbation equation (A.15) can be expressed by

\[
\begin{align*}
(\Delta X_1, \Delta X_2, \Delta X_3) &= \mathbf{L}^{-1}(E_{12}, E_{13}, E_{11}) + \mathbf{L}^{-1}(E_{22}, E_{23}, E_{21}) \\
&\quad + \mathbf{L}^{-1}(h_{12}(\Delta X), h_{13}(\Delta X), h_{11}(\Delta X)) \\
&\quad + \mathbf{L}^{-1}(h_{22}(\Delta X), h_{23}(\Delta X), h_{21}(\Delta X)),
\end{align*}
\]

(A.16)

where

\[
\mathbf{L}^{-1}(E_{12}, E_{13}, E_{11}) = \mathbf{L}^{-1}(\Delta H_2, \Delta H_3, \Delta H_1) + \mathbf{P}(\Delta A_2, \Delta A_3, \Delta A_1) \\
&\quad - \mathbf{Q}(\Delta G_2, \Delta G_3, \Delta G_1).\]

(A.17)

Define the function \( \mu(\Delta X_1, \Delta X_2, \Delta X_3) \) on \( \mathcal{H}_n^3 \) by

\[
\mu(\Delta X_1, \Delta X_2, \Delta X_3) = \mathbf{L}^{-1}(E_{12}, E_{13}, E_{11}) + \mathbf{L}^{-1}(E_{22}, E_{23}, E_{21}) \\
&\quad + \mathbf{L}^{-1}(h_{12}(\Delta X), h_{13}(\Delta X), h_{11}(\Delta X)) \\
&\quad + \mathbf{L}^{-1}(h_{22}(\Delta X), h_{23}(\Delta X), h_{21}(\Delta X)),
\]

(A.18)

which can be regarded as a continuous mapping \( \mathcal{M} : \mathcal{H}_n^3 \rightarrow \mathcal{H}_n^3 \), and the set of solutions to (A.16) is just the set of fixed points of the mapping \( \mathcal{M} \).

**Step 2. Estimates of some fixed points of \( \mathcal{M} \).** Define \( \ell, p_d, q_d \) and \( \varphi \) by (3.8) and (3.9), respectively. Note that the quantity \( \ell \) can be equivalently defined by

\[
\ell = \min_{(w_1, w_2, w_3) \in \mathcal{H}_n^3} \frac{\| \mathbf{L}(W_1, W_2, W_3) \|_F}{\| (W_1, W_2, W_3) \|_F}.
\]

Moreover, we define

\[
\delta_j = \frac{\| \Delta A_j \|_2 + \| K_j \|_2 \| \Delta G_j \|_2}{1 - \| \Psi_j \|_2 \| \Delta G_j \|_2}, \quad j = 1, 2, 3,
\]

(A.19)

where \( \Delta A_j, \Delta G_j, K_j, \Psi_j \) are defined by (3.3) and (3.2). Here we assume that \( \| \Delta G_j \|_2 \) satisfy

\[
1 - \| \Psi_j \|_2 \| \Delta G_j \|_2 > 0, \quad j = 1, 2, 3.
\]

(A.20)

Observe the following facts.
where
\[ (A.21) \]
\[ (A.22) \]
\[ (A.23) \]
(I) By (A.17), we have
\[ \|L^{-1}(E_{12}, E_{13}, E_{11})\|_{F} \leq \frac{1}{T} \| (\Delta H_{1}, \Delta H_{2}, \Delta H_{3}) \|_{F} + p_{d} \| (\Delta A_{1}, \Delta A_{2}, \Delta A_{3}) \|_{F} + q_{d} \| (\Delta G_{1}, \Delta G_{2}, \Delta G_{3}) \|_{F} = \epsilon_{1} \]
(II) By (A.11b), we have
\[ (A.22) = \left\{ \sum_{j=1}^{3} \| \Psi_{j} \|_{2} \left[ \left( \| \Delta A_{j} \|_{2} + \| K_{j} \|_{2} \| \Delta G_{j} \|_{2} \right) + \| K_{j} \|_{2} \| \Delta G_{j} \|_{2} \right]^{2} \right\} \frac{1}{2} \]
(III) By (A.13b), we have
\[ (h_{12}(\Delta X), h_{13}(\Delta X), h_{11}(\Delta X)) \|_{F} \leq \left\{ \sum_{j=1}^{3} 2 \| \Phi_{j} \|_{2} \| \Delta \Phi_{j} \|_{2} \| \Delta X_{j} \|_{F} \right\} \frac{1}{2} \]
and, by (A.12b), we have
\[ \| \Delta \Phi_{j} \|_{2} \leq \frac{\| F_{j} \|_{2}(\| \Delta A_{j} \|_{2} + \| K_{j} \|_{2} \| \Delta G_{j} \|_{2})}{1 - \| \Psi_{j} \|_{2} \| \Delta G_{j} \|_{2}} = \| F_{j} \|_{2} \delta_{j} \]
where \( \delta_{j} \) (\( j = 1, 2, 3 \)) are defined by (A.19). Thus we have
\[ (A.23) \]
(IV) By (A.14b), we have
\[ (h_{22}(\Delta X), h_{23}(\Delta X), h_{21}(\Delta X)) \|_{F} \]
\[ \leq \left\{ \sum_{j=1}^{3} \left[ \| \Theta_{j} \|_{2}^{2}(\| A_{j} \|_{2} + \| \Delta A_{j} \|_{2})^{2}(\| G_{j} \|_{2} + \| \Delta G_{j} \|_{2}) \| \Delta X_{j} \|_{F}^{2} \right] \right\} \frac{1}{2} \]
Observe that, by (A.7),
\[
\|\Theta_j\|_2 \leq \frac{\|F_j\|_2}{1 - \|\Psi_j\|_2\|\Delta G_j\|_2}, \quad j = 1, 2, 3.
\]
Hence we have
\[
\| (h_{22}(\Delta X), h_{23}(\Delta X), h_{21}(\Delta X)) \|_F \\
\leq \left\{ \sum_{j=1}^{3} \left[ \frac{\|F_j\|_2(\|A_j\|_2 + \|\Delta A_j\|_2)^2(\|G_j\|_2 + \|\Delta G_j\|_2)^2}{(1 - \|\Psi_j\|_2\|\Delta G_j\|_2)^3} \left( \frac{1}{1 - \frac{1 - \|F_j\|_2(\|G_j\|_2 + \|\Delta G_j\|_2)\|\Delta X_j\|_F}{1 - \|\Psi_j\|_2\|\Delta G_j\|_2}} \right)^2 \right] \right\}^{\frac{1}{2}}
\]
(A.24) \leq \left\{ \sum_{j=1}^{3} \left( \frac{\alpha^2\gamma\|\Delta X_j\|_F^2}{1 - \gamma\|\Delta X_j\|_F} \right) \right\}^{\frac{1}{2}},
\]
where \(\alpha\) and \(\gamma\) are defined by
\[
\alpha = \max_{1 \leq j \leq 3} \left\{ \frac{\|F_j\|_2(\|A_j\|_2 + \|\Delta A_j\|_2)}{1 - \|\Psi_j\|_2\|\Delta G_j\|_2} \right\}, \quad \gamma = \max_{1 \leq j \leq 3} \left\{ \frac{\|F_j\|_2(\|G_j\|_2 + \|\Delta G_j\|_2)}{1 - \|\Psi_j\|_2\|\Delta G_j\|_2} \right\},
\]
and it is assumed that
\[
1 - \gamma\|\Delta X_j\|_F > 0, \quad j = 1, 2, 3.
\]
Consequently, from (A.18), (A.21)–(A.24), the function \(\mu(\Delta X_1, \Delta X_2, \Delta X_3)\) satisfies
\[
\|\mu(\Delta X_1, \Delta X_2, \Delta X_3)\|_F \leq \epsilon_1 + \frac{\epsilon_2}{\ell} + \frac{1}{\ell} \left\{ \sum_{j=1}^{3} \left( \frac{\alpha^2\gamma\|\Delta X_j\|_F^2}{1 - \gamma\|\Delta X_j\|_F} \right) \right\}^{\frac{1}{2}}
\]
(A.27) + \frac{1}{\ell} \left\{ \sum_{j=1}^{3} \left( \frac{\alpha^2\gamma\|\Delta X_j\|_F^2}{1 - \gamma\|\Delta X_j\|_F} \right) \right\}^{\frac{1}{2}}.
\]
Let
\[
\epsilon = \epsilon_1 + \frac{\epsilon_2}{\ell}, \quad \zeta = \max_{1 \leq j \leq 3} \left\{ \|F_j\|_2\|\Delta \varphi + (\|F_j\|_2\delta_j)\|_2 \right\}.
\]
Then, from (A.27) and (A.28), we have
\[
\|\mu(\Delta X_1, \Delta X_2, \Delta X_3)\|_F \leq \epsilon + \frac{1}{\ell} \left( \zeta\|\Delta X_1, \Delta X_2, \Delta X_3\|_F + \frac{\alpha^2\gamma\|\Delta X_1, \Delta X_2, \Delta X_3\|_F^2}{1 - \gamma\|\Delta X_1, \Delta X_2, \Delta X_3\|_F} \right).
\]
Consider the equation
\[
\xi = \epsilon + \frac{1}{\ell} \left( \zeta \xi + \frac{\alpha^2\gamma\xi^2}{1 - \gamma\xi} \right),
\]
that is,
\[
\gamma(\ell - \zeta + \alpha^2)\xi^2 - (\ell - \zeta + \ell\gamma\epsilon)\xi + \ell\epsilon = 0.
\]

It can be verified that, if $\epsilon$ satisfies
\[
\epsilon \leq \frac{(\ell - \zeta)^2}{\ell \gamma \left( \ell - \zeta + 2\alpha + \sqrt{(\ell - \zeta + 2\alpha)^2 - (\ell - \zeta)^2} \right)},
\]
then the positive scalar $\xi_*$ expressed by
\[
\xi_* = \frac{2\ell\epsilon}{\ell - \zeta + \ell\gamma\epsilon + \sqrt{(\ell - \zeta + \ell\gamma\epsilon)^2 - 4\ell\gamma(\ell - \zeta + \alpha^2)\epsilon}}
\]
is a solution of (A.30).

It is known that the product space $\mathcal{H}_0^3$ with the Frobenius norm $\| \cdot \|_F$ is a Banach space. We now consider the set $S_{\xi_*} \subset \mathcal{H}_0^3$ defined by
\[
S_{\xi_*} = \{ (\Delta X_1, \Delta X_2, \Delta X_3) \in \mathcal{H}_0^3 : \| (\Delta X_1, \Delta X_2, \Delta X_3) \|_F \leq \xi_* \}.
\]
$S_{\xi_*}$ is obviously a bounded closed convex set of $\mathcal{H}_0^3$. Moreover, from (A.29) and the fact that $\xi_*$ is a solution of (A.30), we see that, if $(\Delta X_1, \Delta X_2, \Delta X_3) \in S_{\xi_*}$, then
\[
\| \mu(\Delta X_1, \Delta X_2, \Delta X_3) \|_F \leq \xi_*,
\]
which shows that the continuous mapping $M$ expressed by (A.18) maps $S_{\xi_*}$ into $S_{\xi_*}$. Thus, by the Schauder fixed-point theorem, the mapping $M$ has a fixed point $(\Delta X_1^*, \Delta X_2^*, \Delta X_3^*) \in S_{\xi_*}$. Note that, if the scalar $\zeta$ defined by (A.28) satisfies
\[
\ell - \zeta > 0,
\]
then any $(\Delta X_1, \Delta X_2, \Delta X_3) \in S_{\xi_*}$ satisfies the condition (A.26). In fact, for any $(\Delta X_1, \Delta X_2, \Delta X_3) \in S_{\xi_*}$, we have
\[
1 - \gamma \| \Delta X \|_F \geq 1 - \gamma \| (\Delta X_1, \Delta X_2, \Delta X_3) \|_F
\geq 1 - \gamma \xi_* \geq 1 - \gamma \frac{2\ell\epsilon}{\ell - \zeta + \ell\gamma\epsilon} \quad \text{(by (A.32))}
\geq \frac{\ell - \zeta - \ell\gamma\epsilon}{\ell - \zeta + \ell\gamma\epsilon} \geq \frac{\ell - \zeta - (\ell - \zeta)^2}{\ell - \zeta + 2\alpha}\frac{2\ell - \zeta + 2\alpha}{(\ell - \zeta + \ell\gamma\epsilon)(\ell - \zeta + 2\alpha)} > 0.
\]

**Step 3.** The periodic matrix set \(\{X_j + \Delta X_j^*\}_{j=1}^3\). Let \(\{X_j\}_{j=1}^3\) be the unique Hermitian p.s.d. solution set to the P-DARE (1.1), and let \((\Delta X_1^*, \Delta X_2^*, \Delta X_3^*) \in S_{\xi_*}\) be the fixed point of the mapping $M$ by (A.18). Let
\[
X = \text{diag}\{X_j\}_{j=1}^3, \quad \Delta X = \text{diag}\{\Delta X_j\}_{j=1}^3, \quad Y = X + \Delta X^* \equiv \text{diag}\{Y_j\}_{j=1}^3.
\]

Then the Hermitian matrix $Y$ satisfies
\[
Y = \tilde{A}^H Y (I + \tilde{G} Y)^{-1} \tilde{A} - \tilde{H} = 0.
\]
We now rewrite (A.35) as
\[
Y - \left[ (I + \tilde{G} Y)^{-1} \tilde{A} \right]^H Y (I + \tilde{G} Y)^{-1} \tilde{A}
= \tilde{H} + \left[ Y (I + \tilde{G} Y)^{-1} \tilde{A} \right]^H \tilde{G} Y (I + \tilde{G} Y)^{-1} \tilde{A}.
\]
Observe that
\[(I + \tilde{G}Y)^{-1}\tilde{A} = [I + (G + \Delta G)(X + \Delta X^*)]^{-1}(A + \Delta A)\]
(A.37)
where \(\Phi = (I + GX)^{-1}A\) is d-stable (by Lemma 2.2 and Theorem 1.1), and \(\Phi_1\) can be expressed by
\[
\Phi_1 = F[\Delta A - \Omega(I + \Omega)^{-1}(A + \Delta A)]
\]
with
\[
\Omega = \Delta G\Psi + G\Delta X^*F + \Delta G\Delta X^*F.
\]
(A.38)

A simple calculation gives \(\Phi_1 = \text{cyc}\{\Phi_{1j}\}_{j=1}^3\), where
\[
\Phi_{1j} = F_j(I + \Delta G_j)\|\Psi_j\|_2 + G_j\Delta X_j^*F_j + \Delta G_j\Delta X_j^*F_j)^{-1}
\]
\[
(\Delta A_j - \Delta G_j\Phi_j - \Delta G_j\Delta X_j^*\Phi_j)
\]
for \(j = 1, 2, 3\), and
\[
\|\Phi_{1j}\|_2 \leq \frac{\|F_j\|_2[\|\Delta A_j\|_2 + \|K_j\|_2\|\Delta G_j\|_2 + \|\Phi_j\|_2(\|G_j\|_2\|\Delta G_j\|_2)\xi_*]}{1 - [\|\Psi_j\|_2\|\Delta G_j\|_2 + \|F_j\|_2(\|G_j\|_2 + \|\Delta G_j\|_2)\xi_*]}
\]
\[
\leq \frac{\|F_j\|_2\delta_j + \|\Phi_j\|_2\gamma\xi_*}{1 - \gamma\xi_*} \quad \text{(by (A.19) and (A.25))},
\]
(A.41)
where it is assumed that
\[
\|\Phi_{1j}\|_2 \leq \frac{\|F_j\|_2[\|\Delta A_j\|_2 + \|K_j\|_2\|\Delta G_j\|_2 + \|\Phi_j\|_2(\|G_j\|_2\|\Delta G_j\|_2)\xi_*]}{1 - [\|\Psi_j\|_2\|\Delta G_j\|_2 + \|F_j\|_2(\|G_j\|_2 + \|\Delta G_j\|_2)\xi_*]}
\]
(A.42)

Hence, by (A.41) and Lemma 2.4, if
\[
\max_{1 \leq j \leq p} \left\{ \frac{\|F_j\|_2\delta_j + \|\Phi_j\|_2\gamma\xi_*}{1 - \gamma\xi_*} \right\} < \frac{\ell}{\varphi + \sqrt{\varphi^2 + \ell}}
\]
(A.43)
then \(\{\Phi_j + \Phi_{1j}\}_{j=1}^3\) is pd-stable. Further, by Lemma 2.2, \(\Phi + \Phi_1\) (i.e., \((I + \tilde{G}Y)^{-1}\tilde{A}\) by (A.37)) is d-stable.

We now consider the DARE (A.35) in the classical case. The matrix \(Y\) is a d-stabilizing solution to the DARE. By [8], the solution \(Y\) is unique. Moreover, the Hermitian matrix \(Y\), as a solution to (A.36), is p.s.d. [8]. Thus we have proved that under the conditions (A.20), (A.31), (A.33), (A.42), and (A.43), there is a unique Hermitian p.s.d. solution set \(\{X_j + \Delta X_j^*\}_{j=1}^3\) to the perturbed P-DARE (1.4). Note that the condition (A.33) can be deduced from the condition (A.43). In fact, from the inequality (A.43),
\[
\|F_j\|_2\delta_j < \frac{\ell}{\varphi + \sqrt{\varphi^2 + \ell}}, \quad j = 1, 2, 3,
\]
which implies
\[
2\varphi\|F_j\|_2\delta_j + (\|F_j\|_2\delta_j)^2 < \ell
\]
(A.44)
for \(j = 1, 2, 3\). Observe that, by (A.28) and (A.44), we obtain \(\zeta < \ell\), that is, the condition (A.33).
Appendix B. (I) Proof of (3.24).

Let \( \hat{L} \) be a natural extension of the linear operator \( L \) on \( \mathbb{C}^p \), that is,

\[
\hat{L}(Z_1, \ldots, Z_p) = (Z_1 - \Phi_2^H Z_2 \Phi_2, \ldots, Z_{p-1} - \Phi_p^H Z_p \Phi_p, Z_p - \Phi_1^H Z_1 \Phi_1),
\]

where \( Z_j \in \mathbb{C}^n \) for \( j = 1, \ldots, p \). Then the matrix \( L \) defined by (2.6) is a matrix representation of \( \hat{L} \) on \( \mathbb{C}^{pn^2} \). By the definition of the operator norm, we have

\[
\| L^{-1} \|^{-1} = \min_{W_j \in \mathbb{R}^n} \left( \sum_{j=1}^{p} \| W_{j-1} - \Phi_j^H W_j \Phi_j \|_F^2 \right)^{1/2} / \left( \sum_{j=1}^{p} \| W_j \|_F^2 \right)^{1/2}
\]

\[
\geq \min_{Z_j \in \mathbb{C}^n} \left( \sum_{j=1}^{p} \| Z_{j-1} - \Phi_j^H Z_j \Phi_j \|_F^2 \right)^{1/2} / \left( \sum_{j=1}^{p} \| Z_j \|_F^2 \right)^{1/2}
\]

\[
= \| \hat{L}^{-1} \|^{-1} = \| L^{-1} \|_2^{-1}.
\]

We shall prove that the equality in (B.2) holds. Let \( (Z_1^*, \ldots, Z_p^*) \in \mathbb{C}^p \) be a singular “vector” such that

\[
\| L^{-1} \|_2^{-1} = \left( \sum_{j=1}^{p} \| Z_j^* - \Phi_j^H Z_j^* \Phi_j \|_F^2 \right)^{1/2} / \left( \sum_{j=1}^{p} \| Z_j^* \|_F^2 \right)^{1/2}.
\]

Then \( (Z_1^H, \ldots, Z_p^H) \) is also a singular “vector” satisfying (B.3). Let \( W_j^* = Z_j^* + Z_j^H \) for \( j = 1, \ldots, p \). Obviously, \( W_1^*, \ldots, W_p^* \) are Hermitian. Consequently, if \( W_j^* \neq 0 \) for some \( j \in \{1, \ldots, p\} \), then we have

\[
\| L^{-1} \|_2^{-1} = \left( \sum_{j=1}^{p} \| W_j^* - \Phi_j^H W_j^* \Phi_j \|_F^2 \right)^{1/2} / \left( \sum_{j=1}^{p} \| W_j^* \|_F^2 \right)^{1/2} = \| L^{-1} \|^{-1}.
\]

If \( W_j^* = 0 \) for all \( j = 1, \ldots, p \), then \( Z_j^* = -(Z_j^*)^H \). In such a case, \( iZ_1^*, \ldots, iZ_p^* \) are Hermitian, and we also have

\[
\| L^{-1} \|_2^{-1} = \left( \sum_{j=1}^{p} \| iZ_j^* - \Phi_j^H (iZ_j^*) \Phi_j \|_F^2 \right)^{1/2} / \left( \sum_{j=1}^{p} \| iZ_j^* \|_F^2 \right)^{1/2} = \| L^{-1} \|^{-1}.
\]
For the real case, i.e., the case when all coefficient matrices are real, the relations (B.1)–(B.4) still hold, where we need only to replace $\mathcal{H}_n$, $\mathbb{C}_n$, and the superscript "$^H$" by $\mathcal{S}_n$, $\mathbb{R}_n$, and the superscript "$^T$" respectively. However, the equality (B.5) no longer holds because the matrix $iZ_j^*$ is Hermitian but not real symmetric. In order to remedy this defect, we first prove the following lemma.

**Lemma B.1.** Suppose that $\Phi_j \in \mathbb{R}_n(j = 1, \ldots, p)$, $\{\Phi_j\}_{j=1}^p$ is pd-stable, and $B_j, C_j \in \mathcal{S}_n(\text{or } \mathcal{H}_n)$ with $B_j \geq C_j$ (i.e., $B_j - C_j \geq 0$) for $j = 1, \ldots, p$. If $X_j$ and $Y_j$ satisfy

(B.6) \[ X_{j-1} - \Phi_j^T X_j \Phi_j = B_j, \quad j = 1, \ldots, p, \]

(B.7) \[ Y_{j-1} - \Phi_j^T Y_j \Phi_j = C_j, \quad j = 1, \ldots, p, \]

respectively, then $X_j, Y_j$ are real symmetric (or Hermitian) and $X_j \geq Y_j$ for $j = 1, \ldots, p$.

**Proof.** Let $\text{vec}(X_j) = x_j, \text{vec}(B_j) = b_j, \text{vec}(Y_j) = y_j, \text{vec}(C_j) = c_j$ for $j = 1, \ldots, p$. Then (B.6) and (B.7) can be written as

(B.8) \[ Lx = b \quad \text{and} \quad Ly = c, \]

where $L$ is defined by (2.6), $x = (x_1^T, \ldots, x_p^T)^T$, $b = (b_1^T, \ldots, b_p^T)^T$, $y = (y_1^T, \ldots, y_p^T)^T$, and $c = (c_1^T, \ldots, c_p^T)^T$. By the assumption, $\lambda(L) \subset \mathcal{D}$, and $L$ is invertible. For any $b_j \in \mathbb{R}^{n^2}$ (or $\mathbb{C}^{n^2}$) so that $B_j = \text{unvec}(b_j) \in \mathcal{S}_n$ (or $\mathcal{H}_n$) (here unvec denotes the inverse operator of vec) for $j = 1, \ldots, p$, the solution $x = (x_1^T, \ldots, x_p^T)^T$ in (B.8) is uniquely solvable. Let $X_j = \text{unvec}(x_j)$, for $j = 1, \ldots, p$. Then $\{X_j\}_{j=1}^p$ satisfies (B.6). Taking the transpose (or conjugate transpose) of (B.6), it follows that $\{X_j^H\}_{j=1}^p$ (or $\{X_j^\top\}_{j=1}^p$) is also a solution set of (B.6). By the uniqueness, the solution set $\{X_j\}_{j=1}^p$ of (B.6) satisfies $X_j = X_j^\top \in \mathcal{S}_n$ (or $X_j = X_j^H \in \mathcal{H}_n$) for $j = 1, \ldots, p$. Similarly, the solution set $\{Y_j\}_{j=1}^p$ of (B.7) satisfies $Y_j = Y_j^\top \in \mathcal{S}_n$ (or $Y_j = Y_j^H \in \mathcal{H}_n$) for $j = 1, \ldots, p$. Denote

(B.9) \[ \Phi = \text{cyc}\{\Phi_j\}_{j=1}^p, \quad X = \text{diag}\{X_j\}_{j=1}^p, \quad Y = \text{diag}\{Y_j\}_{j=1}^p, \quad B = \text{diag}\{B_j\}_{j=1}^p, \quad \text{and} \quad C = \text{diag}\{C_j\}_{j=1}^p. \]

Subtracting (B.6) from (B.7), we have

(B.10) \[ (X - Y) - \Phi^\top (X - Y) \Phi = B - C \geq 0. \]

Applying Proposition 2.1 of [8] to (B.10), we obtain $X_j \geq Y_j$ for $j = 1, \ldots, p$. \[ \square \]

Now suppose that $\{Z_j^*, Z_p^*\}$ is the singular “vector” satisfying (B.3), where $Z_j^* = -Z_j^\top$ are $n \times n$ real skew-symmetric matrices for $j = 1, \ldots, p$. Let

(B.11) \[ N_j = Z_j^* - \Phi_j^T Z_j^* \Phi_j \quad (\text{real skew-symmetric}) \]

for $j = 1, \ldots, p$, and let $N_j = U_j D_j U_j^\top$ be the orthogonal spectral decomposition such that $D_j$ is block diagonal with $1 \times 1$-zero blocks and $2 \times 2$-blocks

\[ D_{j,ii} = \begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix} \]

for $j = 1, \ldots, p$. According to a technique developed by Byers and Nash [4], we construct the symmetric matrices

(B.12) \[ M_j = U_j E_j U_j^\top, \quad j = 1, \ldots, p, \]
where $E_j$ is the block diagonal matrix with the same $1 \times 1$-zero block as $D_j$ and

$$E_{j,ii} = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}$$

provided that $D_{j,ii}$ is of the form

$$\begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix}.$$  

It is easy to see that

$$M_j \geq iN_j \geq -M_j, \quad j = 1, \ldots, p.$$  

Let $\{W_j\}_{j=1}^p$ ($W_j \in \mathcal{S}_n$) be the symmetric solution set satisfying

$$W_{j-1}^* - \Phi_j W_j^* \Phi_j = M_j, \quad j = 1, \ldots, p.$$  

Applying Lemma B.1 to (B.14) and

$$(B.14)$$

Let

$$(B.13)$$

and by using (B.13), we obtain $W_j^* \geq iZ_j^* \geq -W_j^*$. Hence, by Lemma 7 of [4], we get

$$\|W_j^*\|_F \geq \|Z_j^*\|_F, \quad j = 1, \ldots, p.$$  

By (B.3), (B.11), and (B.15),

$$\|L^{-1}\|_2^{-1} \geq \left(\sum_{j=1}^p \|N_j\|_F^2\right)^{1/2} \left(\sum_{j=1}^p \|W_j\|_F^2\right)^{1/2} \left(\sum_{j=1}^p \|W_j^* - \Phi_j W_j^* \Phi_j\|_F^2\right)^{1/2}$$

$$= \|L^{-1}\|_2^{-1} \quad \text{(by (B.12) and (B.14))},$$

which shows that, in the real case and when $W_j^* = 0$ for all $j = 1, \ldots, p$, the equality

$$\|L^{-1}\|_2^{-1} = \|L^{-1}\|_2^{-1}$$

also holds.

**II** Proof of (3.25) and (3.26).

The complex case. Since $K_j^H N_j + N_j^H K_j \in \mathcal{H}_n$ for any $N_j \in \mathcal{C}_n$, we have

$$L^{-1}(H_2, \ldots, H_p, H_1) = \hat{L}^{-1}(H_2, \ldots, H_p, H_1),$$

where $H_j = K_j^H N_j + N_j^H K_j \in \mathcal{H}_n$.

By the definition of the operator norm, we have

$$\|P\| = \max_{N_j \in \mathcal{C}_n} \frac{\|P(N_1, \ldots, N_p)\|_F}{\|(N_1, \ldots, N_p)\|_F}$$

$$= \max_{N_j \in \mathcal{C}_n} \frac{\|\hat{L}^{-1}(K_2^H N_2 + N_2^H K_2, \ldots, K_p^H N_p + N_p^H K_p, K_1^H N_1 + N_1^H K_1)\|_F}{\|(N_1, \ldots, N_p)\|_F}$$

$$= \max_{\text{vec}(N_j) = z_j \in \mathbb{C}^n} \sqrt{\sum_{j=1}^p \|z_j\|_2^2}.$$
Denote $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}^{n^2}$, for $j = 1, \ldots, p$, and $x = (x_1^T, \ldots, x_p^T)^T, y = (y_1^T, \ldots, y_p^T)^T$. Let

\begin{align*}
 L^{-1} \left[ \text{cyc} \left\{ (I \otimes K_j^H)^p \right\}_{j=1}^p \right]^T &= \Omega_1 + i\Omega_2, \\
 L^{-1} \left[ \text{cyc} \left\{ (K_j^T \otimes I)^p \right\}_{j=1}^p \right]^T &= \Theta_1 + i\Theta_2,
\end{align*}

where $\Omega_1, \Omega_2, \Theta_1, \text{ and } \Theta_2$ are real matrices. By (B.16), (B.17), and the technique proposed by [14], we have

\[
\|P\| = \max_{[x^T, y^T]^T \neq 0} \frac{\| (\Omega_1 + i\Omega_2)(x + iy) + (\Theta_1 + i\Theta_2)(x - iy) \|_2}{\sqrt{\|x\|_2^2 + \|y\|_2^2}} \leq \frac{\sqrt{\|x\|_2^2 + \|y\|_2^2}}{\sqrt{\|x\|_2^2 + \|y\|_2^2}} = 1.
\]

The real case. By replacing $\mathcal{H}_n, \mathbb{C}_n$ and the superscript “$H$” by $\mathbb{S}_n, \mathbb{R}_n$, and the superscript “$T$,” respectively, (B.16) becomes

\[
\|P\| = \left\| L^{-1} \left[ \text{cyc} \left\{ [I \otimes K_j^T + (K_j^T \otimes I)]^p \right\}_{j=1}^p \right]^T \right\|_2.
\]

(III) Proof of (3.27).

Obviously,

\[
\|Q\| = \max_{M_1, \ldots, M_p} \frac{\|L^{-1}(K_2^H M_2 K_2, \ldots, K_p^H M_p K_p, K_1^H M_1 K_1)\|_F}{\|(M_1, \ldots, M_p)\|_F} \\
\leq \max_{N_1, \ldots, N_p} \frac{\|\hat{L}^{-1}(K_2^H N_2 K_2, \ldots, K_p^H N_p K_p, K_1^H N_1 K_1)\|_F}{\|(N_1, \ldots, N_p)\|_F} \\
= \frac{\|\hat{L}^{-1}(K_2^H N_2^* K_2, \ldots, K_p^H N_p^* K_p, K_1^H N_1^* K_1)\|_F}{\|(N_1^*, \ldots, N_p^*)\|_F}.
\]

Let $(Z_1^*, \ldots, Z_p^*) = \hat{L}^{-1}(K_2^H N_2^* K_2, \ldots, K_p^H N_p^* K_p, K_1^H N_1^* K_1)$. By the definition (B.1) of $\hat{L}$, we have

\[
(Z_1^*, \ldots, Z_p^*) - (\Phi_1^H Z_1^* \Phi_2, \ldots, \Phi_p^H Z_p^* \Phi_2, \Phi_1^H Z_1^* \Phi_1) \\
= (K_2^H N_2^* K_2, \ldots, K_p^H N_p^* K_p, K_1^H N_1^* K_1),
\]

which implies

\[
(Z_1^H, \ldots, Z_p^H) - (\Phi_1^H Z_1^H \Phi_2, \ldots, \Phi_p^H Z_p^H \Phi_2, \Phi_1^H Z_1^H \Phi_1) \\
= (K_2^H N_2^H K_2, \ldots, K_p^H N_p^H K_p, K_1^H N_1^H K_1).
\]

Thus we have

\[
(Z_1^H, \ldots, Z_p^H) = \hat{L}^{-1}(K_2^H N_2^H K_2, \ldots, K_p^H N_p^H K_p, K_1^H N_1^H K_1).
\]
By the fact that \( \|(Z_1^*, \ldots, Z_p^*)\|_F = \|(Z_1^{*H}, \ldots, Z_p^{*H})\|_F \), we obtain
\[
\|\hat{L}^{-1}(K_2^H N_2^H K_2, \ldots, K_p^H N_p^H K_p, K_1^H N_1^H K_1)\|_F \\
= \|\hat{L}^{-1}(K_2^H N_2^* K_2, \ldots, K_p^H N_p^* K_p, K_1^H N_1^* K_1)\|_F.
\]

We now define the operator \( \hat{Q} \) on \( \mathbb{C}^n \) by
\[
\hat{Q}(N_1, \ldots, N_p) = \hat{L}^{-1}(K_2^H N_2^* K_2, \ldots, K_p^H N_p^* K_p, K_1^H N_1^* K_1).
\]

It is easy to see that the matrix
\[
Q = L^{-1} [\text{cyc} \{K_j \otimes K_j\}_{j=1}^p]^T
\]
is a matrix representation of \( \hat{Q} \). Combining (B.19) with (B.20) shows that \((N_1^*, \ldots, N_p^*)\) and \((N_1^{*H}, \ldots, N_p^{*H})\) are the singular “vectors” of \( \hat{Q} \) corresponding to its largest singular value. Let
\[
W_j^* = Z_j^* + Z_j^{*H} \quad \text{for} \quad j = 1, \ldots, p.
\]

If \( W_j^* \neq 0 \) for some \( j \in \{1, \ldots, p\} \), then \((W_1^*, \ldots, W_p^*)\) is also a singular “vector” of \( \hat{Q} \) corresponding to its largest singular value. Hence we have
\[
\|Q\|_2 = \frac{\|\hat{L}^{-1}(K_2^H W_2^* K_2, \ldots, K_p^H W_p^* K_p, K_1^H W_1^* K_1)\|_F}{\|(W_1^*, \ldots, W_p^*)\|_F} \\
\|Q\|_2 = \frac{\|L^{-1}(K_2^H W_2^* K_2, \ldots, K_p^H W_p^* K_p, K_1^H W_1^* K_1)\|_F}{\|(W_1^*, \ldots, W_p^*)\|_F} = \|Q\|.
\]

If \( W_j^* = 0 \) for all \( j = 1, \ldots, p \), then set \( H_j^* = iZ_j^* \in \mathcal{H}_n \) for \( j = 1, \ldots, p \), and thus \((H_1^*, \ldots, H_p^*)\) is also a singular “vector” of \( \hat{Q} \) corresponding to \( \|Q\|_2 \). Hence, we have
\[
\|Q\|_2 = \frac{\|\hat{L}^{-1}(K_2^H H_2^* K_2, \ldots, K_p^H H_p^* K_p, K_1^H H_1^* K_1)\|_F}{\|(H_1^*, \ldots, H_p^*)\|_F} \\
\|Q\|_2 = \frac{\|L^{-1}(K_2^H H_2^* K_2, \ldots, K_p^H H_p^* K_p, K_1^H H_1^* K_1)\|_F}{\|(H_1^*, \ldots, H_p^*)\|_F} = \|Q\|.
\]

From (B.21) and (B.22) we obtain \( \|Q\| = \|Q\|_2 \).

Similarly, we can prove the expression (3.27) in the real case.

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