Robust and Minimum Norm Partial Pole Assignment in Vibrating Structures with Aerodynamics Effects

Biswa N. Datta*, IEEE Fellow, Wen-Wei Lin† and Jenn-Nan Wang‡

Abstract—The paper considers a practical solution of the partial eigenvalue assignment problem in a cubic matrix polynomial arising from modeling of vibrating structures with aerodynamics effects, with a special attention to numerically robust feedback design and obtaining feedback matrices with minimum norms. To this end, a direct parametric method that works exclusively with the coefficient matrices of the cubic polynomial without requiring transformation to a standard state-space form is proposed first. The computational requirements of the method are minimal - all that are needed are solutions of a small Sylvester equation and a small linear algebraic system, both of the same order as the number of eigenvalues to be reassigned. The parametric nature of the method is then exploited to deal with the issues of robustness and minimum feedback norms, by formulating the problem as an unconstrained optimization problem. The numerical effectiveness of the method is demonstrated by results of a numerical experiment performed on simulated data obtained from the Boeing Company.

I. INTRODUCTION

The vibrating structures with aerodynamic effects can be modeled by means of systems of nonhomogeneous matrix second-order differential equations of the type (see [7]):

\[ M \ddot{q} + (C_1 + \zeta(s)C_2) \dot{q} + (K_1 + \zeta(s)K_2) q = G(s, t), \]

where \( M \) is the inertia matrix, \( C_1 \) and \( K_1 \) are the structural damping and stiffness matrices; and \( C_2 \) and \( K_2 \) are aerodynamic damping and stiffness matrices, respectively. The nonhomogeneous term \( G(s, t) \) represents the forcing function which is the combination of the generalized forces and gust inputs. In practice, very often the matrices \( M, K_1, K_2 \) are real positive definite and \( C_1, C_2 \) are real symmetric. In this paper, we will, however, assume throughout that \( M, K_1, K_2 \) are real symmetric and \( M \) is only nonsymmetric. The factor \( \zeta(s) \) in (1) is called the Wagner lift-growth buildup function which is due to an instantaneous change in angle of attack [7]. For our study, we take \( \zeta(s) \) in the form

\[ \zeta(s) = \rho + \frac{s}{s - \omega} \]

* Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, USA. Email: datta@math.niu.edu

† Partially supported by NSF Grant ECS-007441.

‡ Department of Mathematics, National Tsing-Hua University, Hsinchu 300, Taiwan. Email: wwlin@am.nthu.edu.tw

Partially supported by the National Science Council of Taiwan.

Partially supported by the National Science Council of Taiwan.

The system (1) leads to the open-loop cubic pencil

\[ P(\lambda) = MA^3 + (C_1 + \rho C_2 - \omega M)\lambda^2 + ((K_1 + \rho K_2) - \omega(C_1 + \rho C_2)) \lambda + \rho(K_2 - \omega(K_1 + \rho K_2)) \]

where

\[ C = C_1 + \rho C_2 - \omega M, \]
\[ K = (K_1 + \rho K_2) - \omega(C_1 + \rho C_2) + \rho C_2, \]
\[ L = \rho K_2 - \omega(K_1 + \rho K_2). \]

We would like to point out that if we take \( \rho = 0 \) in \( \zeta(s) \) and substitute back to (1), we get the usual second-order system without aerodynamic effect. On the other hand, if we choose \( \rho = 0 \) in (3) and substitute them into (2), we get a third order system. However, it does not lead to an inconsistency, since (2) is derived by multiplying \( s - \omega \) to (1), and \( s \) is interpreted as the derivative with respect to \( t \).

Now, by applying the control force \( H(s, t) = B(\lambda^2 + \zeta(s)G^2)q \) to (1), we obtain a controlled system, which gives rise to the closed-loop cubic pencil

\[ P_c(\lambda) = MA^3 + (C_1 - B\lambda^2 F + \rho C_2 - \omega M)\lambda^2 + ((K_1 - B\lambda G_1 + \rho K_2) - \omega(C_1 - B\lambda F + \rho C_2)) \lambda + \rho(K_2 - \omega(K_1 - B\lambda G_1)) \]

Here \( B \in \mathbb{R}^{n \times p} \) is the control matrix and \( F, G_1, G_2 \in \mathbb{R}^{n \times p} \) are gain matrices, where \( 1 \leq p \leq n \). Without loss of generality, we assume throughout that \( B \) has full column rank.

Let \( \{\lambda_k\}_{k=1}^{3n} \) be the spectrum of \( P(\lambda) \). Clearly, this is a self-conjugate set. Now let \( \{\mu_k\}_{k=1}^{3n} \) be another self-conjugate set with \( 1 \leq k \leq 3n \). It is well-known that the dynamical behavior of a vibrating structure represented by (1) is governed by the eigenvalues and eigenvectors of the pencil (2). Since in practice for a large system, only a few eigenvalues are "troublesome", it makes sense to reassign those small number of eigenvalues by using suitable feedback control while keeping
the remaining large number invariant. This gives rise to the Partial Pole Assignment (PPA) Problem for the system (1):

Given the matrices $M$, $C_1$, $C_2$, $K_1$, and $K_2$ of the system (1) and a control matrix $B$ of full-rank, and a part of the open-loop spectrum, say $\{\lambda_1, \ldots, \lambda_k\}$, find real gain matrices $F$, $G_1$, and $G_2$ such that the spectrum $\sigma(P_2(\lambda)) = \{\{\mu_{i_1}\}_{i_1=1}^{m_1}, \{\lambda_i\}_{i=1}^{m_2}\}$.

For the standard first-order state-space system, the methods for PPA include (i) a projection algorithm [2] and (ii) a Sylvester equation technique [2]. For the standard second-order systems; that is, for systems (1) with $\zeta(s) = 0$, explicit expressions for the feedback vectors in the single-input problem were first developed in [3] using a Cauchy-matrix, and solutions of the multi-input problem were then obtained in [7] and [2]. A Sylvester equation method was developed in the last paper. The idea in [2] has been recently generalized to the matrix second-order systems with aerodynamic effects in [7]; that is for the cubic polynomial (3).

A distinguished and very desirable feature, common to all the methods for a second-order system mentioned above, is that the problem is solved directly in matrix second-order setting; no transformation to a first-order state-space form is required. Thus, a possible ill-conditioned matrix transformation is avoided and certain exploitable structures such as bandness, sparsity, symmetry, and positive definiteness, etc., very often offered by practical problems, are exploited in computations.

In this paper, we first generalize the Sylvester equation approach in [2] to the solution of PPA problem for the cubic polynomial (2)-(3), and then, based on this approach, develop a technique for simultaneously making the closed-loop system numerically robust and obtaining feedback gains with minimum norms.

Note that, it is not enough only to develop an algorithm for feedback design, numerical robustness issues also have to be dealt with for practical considerations. Since in the multi-input feedback control problem, the feedback matrices are not unique, this non-uniqueness property should be exploited in the design of a controller. On one hand, it is advisable to determine gain matrices with small norms. Intuitively, this corresponds to small control signals and, hence, less energy consumption. On the other hand, one would like to reduce the sensitivity of the closed-loop eigenvalues, which can be achieved by minimizing the associated eigenvalue condition number. The latter problem is termed as robust pole assignment problem. From practical point of view, it is desirable to simultaneously minimize the gain matrices and the condition number of the closed-loop eigenvalues.

While there are now several excellent methods for pole-placement in standard state-space forms, (see, e.g., [2], [7], [2]), very few papers considered the robustness and minimum norm feedback issues of the problem. A well-known paper dealing with robustness alone is [2]. The methods of this paper were later modified and re-interpreted in [2]. A scheme for minimum-norm feedback design, based on solution of a Sylvester equation was developed in [7], [2]. Only the paper [2] considered both robustness and minimum norm feedback issues for periodic systems. For quadratic matrix polynomials, algorithms for robustness alone were developed in [7] and [2]. All these papers above, however, deal with complete assignment only. These issues have not so far been considered for partial pole-placement, neither for cubic nor for quadratic matrix polynomials.

To solve the robust and minimum gain partial pole assignment for the cubic pencil $P(\lambda)$, we formulate the problem as an unconstrained optimization problem. The objective function is the convex combination of the measure of robustness and the magnitude of gain matrices which are related to the solution of a Sylvester equation (see Section II). This optimization problem can be solved quite effectively by the gradient-based minimization techniques. For better accuracy, one can use Newton's method with the line search. The later requires computation of the gradient of the objective functions. A computational procedure for doing so has been derived, but omitted here for the sake of space limitation.

The major computational requirements of the proposed method are solutions of (i) a small Sylvester equation, (ii) a small linear algebraic system, and (iii) an unconstrained minimization problem. There exist excellent numerical methods for performing these tasks. These minimal computational requirements along with the fact that the method is "direct" in nature and the knowledge of only a small number of eigenvalues and eigenvectors is sufficient to implement the method, make it viable for practical use to stabilize and control large vibrating structures with aerodynamics effects.

II. SOLUTION OF THE PARTIAL POLE ASSIGNMENT

To begin with, let $\{(\lambda_i, x_i)\}_{i=1}^{m_1}$ be the eigenpairs of $P(\lambda)$, i.e. $P(\lambda_i)x_i = 0$ for $1 \leq i \leq 3n$. Denote

$\begin{align*}
\Lambda &= \text{diag}(\lambda_1, \cdots, \lambda_k),
\tilde{\Lambda} &= \text{diag}(\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_{3n}),
J &= \text{diag}(\Lambda, \tilde{\Lambda}),
X &= [x_1, \cdots, x_k, \tilde{x}_1, \cdots, \tilde{x}_k, x_{k+1}, \cdots, x_{3n}],
U &= [X, \tilde{X}],
\end{align*}$

where $1 \leq k < 3n$ and $\{\lambda_i\}_{i=1}^{k}$ and $\{\lambda_{i}\}_{i=k+1}^{3n}$ are two self-conjugate sets. Assume that $(U, J)$ is a Jordan pair of $P(\lambda)$, i.e.

$M U J^3 + C U J^2 + K U J + L U = 0$

and the $3n \times 3n$ matrix

$$
\begin{bmatrix}
U \\
U J \\
U J^2
\end{bmatrix}
$$

is nonsingular. The following result, proved in [2], shows that, for any arbitrary matrix $\xi$, the feedback matrices defined by (5) below are such that the pairs $\{(\lambda_i, x_i)\}_{i=k+1}^{3n}$ remains eigenpairs of the closed-loop pencil $P_2(\lambda)$.

Theorem 2.1: Let $\sigma(\Lambda) \cap \sigma(\tilde{\Lambda}) = \emptyset$. Define the gain matrices as

$F = MXA\xi,$

$G_1 = \frac{1}{\rho}(\omega(K_1 + \rho K_2)x\xi + (1 - \omega)[MXA]^2 + (C_1 + \rho C_2)X\xi),$ 

$G_2 = \frac{1}{\rho}(\omega(K_1 + \rho K_2)X\xi + \omega[MXA]^2 + (C_1 + \rho C_2)X\xi),$ 

where

$$
(5)
$$
where \( \xi \in \mathbb{C}^{k \times r} \). Then \((\tilde{X}, \tilde{\lambda})\) is an eigenmatrix pair of \( P_{c}(\lambda) \), i.e.

\[
P_{c}(\tilde{\lambda}) \tilde{X} = M \tilde{X} \tilde{\lambda} + (C_{2} - B F^{T} + \rho C_{2} - \omega M) \tilde{X} \tilde{\lambda} + \omega (C_{1} - B F^{T}) \tilde{X} \tilde{\lambda} - \omega C_{2} \tilde{X} \tilde{\lambda} + \omega (K_{2} - B G_{2}^{T}) \tilde{X} \tilde{\lambda} - \omega (K_{1} - B G_{1}^{T}) + \omega (K_{2} - B G_{2}^{T}) \tilde{X} \tilde{\lambda} = 0.
\]

To complete the solution of our PPA problem, we still need to assign \( \{\lambda_{i}\}_{j=1}^{h} \) into a self-conjugate set of complex numbers \( \{\mu_{j}\}_{j=1}^{k} \) by choosing \( \xi \) properly. We shall show that this can be done by solving a \( k \times k \) Sylvester matrix equation. Assume that

\[
\{\mu_{j}\}_{j=1}^{k} \cap \sigma(P) = \emptyset,
\]

and write

\[
\{\mu_{j}\}_{j=1}^{k} = \{\{\mu_{2r-1}, \mu_{2r}\}_{r=1}^{m}, \{\mu_{j}\}_{j=2m+1}^{k}\},
\]

where \( 0 \leq m \leq k/2 \), \( \{\mu_{2r-1}, \mu_{2r}\}_{r=1}^{m} \) are pairs of conjugate complex numbers with nonzero imaginary parts, and \( \{\mu_{j}\}_{j=2m+1}^{k} \) are all real numbers. Let

\[
H = (h_{1}, \ldots, h_{k})
\]

be a \( p \times k \) complex matrix with column vectors \( h_{j} \) satisfying

\[
\begin{cases}
h_{2r-1} = \overline{h}_{2r} & \text{for } 1 \leq r \leq m, \\
h_{j} \in \mathbb{R}^{p 	imes 1} & \text{for } 2m + 1 \leq j \leq k.
\end{cases}
\]

It is clear that \( H \) is a linear function of \( \beta = [h_{1}, h_{2}^{T}, \ldots, h_{2m}^{T}, h_{2m+1}^{T}, \ldots, h_{k}^{T}]^{T} \in \mathbb{R}^{kp} \), where \( h_{1}, h_{2}, \ldots, h_{k} \) are the real and imaginary parts of \( h_{j} \) for \( 1 \leq j \leq m \), respectively. Denote this function by \( H(\beta) \). In view of (6), \( P(\mu_{j}) \) is nonsingular. Define

\[
y_{j} = P(\mu_{j})^{-1} B h_{j}, \quad 1 \leq j \leq k.
\]

That is, \( y_{j} \) satisfies

\[
P(\mu_{j}) y_{j} = M y_{j} \mu_{j}^{2} + C y_{j} \mu_{j} + K y_{j} + L y_{j} = B h_{j}, \quad 1 \leq j \leq k.
\]

Notice that \( y_{j} \neq 0 \) for all \( 1 \leq j \leq k \).

We will now show that the vector \( y_{j} \) is the eigenvector of the closed-pencil \( P_{c}(\lambda) \) corresponding to \( \mu_{j} \). Under some suitable conditions, this leads to a Sylvester equation. To begin, let us denote

\[
\Gamma = \text{diag}(\mu_{1}, \ldots, \mu_{k}) \quad \text{and} \quad Y = [y_{1}, \ldots, y_{k}].
\]

Then

\[
P_{c}(\Gamma) Y = M Y \Gamma^{2} + [(C_{1} - B F^{T} + \rho C_{2} - \omega M) Y] \Gamma^{2} + \omega (C_{1} - B F^{T}) Y - \omega C_{2} Y + \omega (K_{2} - B G_{2}^{T}) Y - \omega (K_{1} - B G_{1}^{T}) \tilde{X} \tilde{\lambda} - \omega (K_{2} - B G_{2}^{T}) \tilde{X} \tilde{\lambda} = 0.
\]

To guarantee this will happen provided that (i) the open-loop system is, it suffices to choose \( H \) such that \( H = \xi \Gamma \). In summary, to replace unwanted eigenvalues \( \{\lambda_{j}\}_{j=1}^{h} \) by \( \{\mu_{j}\}_{j=1}^{k} \), we need to solve the following two equations in sequence:

\[
\begin{cases}
\xi \Gamma - \Lambda \Psi = \xi \Gamma B H \\
\xi \Psi = H.
\end{cases}
\]

Once \( \xi \) is obtained from (10), this matrix can be substituted in (5) to obtain the required gain matrices, solving completely the partial pole-assignment problem. Notice also that \( F, G_{1} \) and \( G_{2} \) can be shown to be real matrices using the same arguments as in [7].

Remark: For the solvability of the last equation in (10), it suffices to choose \( H \) so that \( \Psi \) is nonsingular. It can be shown, using the idea of the generalized Cauchy matrix [7], that this will happen provided that (i) the open-loop system is partially controllable with respect to \( A_{i} \); (ii) \( x_{i}^{T} B \neq 0 \) for all \( i \); and (iii) none of the eigenvectors \( \lambda_{i}, i = 1, \ldots, k \) is zero.

Algorithm 2.1: An Algorithm for Partial Pole Assignment in Vibrating Structures with Aerodynamic Effects

Inputs: (i) The matrices \( M, C_{1}, C_{2}, K_{1}, \) and \( K_{2} \). (ii) The non-zero scalars \( \rho \) and \( \omega \). (iii) The partial spectrum
\{\lambda_1, \ldots, \lambda_k\} of \(P(\lambda)\) in (2) and the associated eigenvectors 
\(x_1, \ldots, x_k\), (iv) A set of self-conjugate scalars \(\mu_1, \ldots, \mu_k\),
(v) The control matrix \(A\).

\textbf{Outputs:} The feedback matrices \(F, G_1\) and \(G_2\) such that \(
\sigma(P_2(\lambda)) = \{\mu_1, \ldots, \mu_k, \lambda_{k+1}, \ldots, \lambda_{3n}\}\), where \(P_2(\lambda)\) is given by (4).

\textbf{Assumptions:} The system is partially controllable with respect to the set \(\{\lambda_1\}_{k=1}^{n}\), the sets \(\{\lambda_1\}_{k=1}^{n}\) and \(\{\mu_i\}_{k=1}^{n}\) are disjoint, and none of the numbers \(\lambda_i, i = 1, \ldots, k\) is zero.

\textbf{Step 1:} Form \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k)\); \(X = (x_1, \ldots, x_k)\) and \(\Gamma = \text{diag}(\mu_1, \ldots, \mu_k)\).

\textbf{Step 2:} Solve the Sylvester equation: \(\Psi \Gamma - \Lambda \Psi = AXTBH\) for \(\Psi\), choosing the matrix \(H\) arbitrarily.

\textbf{Step 3:} Solve the linear system: \(\xi^T \Psi = H\) for \(\xi\).

\textbf{Step 4:} Form \(F, G,\) and \(G_2\) using (5).

\section{III. Computing the Minimum-Norm and Robust Feedback}

By virtue of the formulas of the gain matrices (5) and the form \(\xi^T = \Psi^{-1}\), we obtain that the magnitudes of the gain matrices satisfy
\[
\|F\|_2 + \|G_1\|_2 + \|G_2\|_2 \leq \tilde{C} \|H(\beta)\|_2 \|\Psi^{-1}(\beta)\|_2, \quad (11)
\]
where \(\tilde{C}\) is a constant independent of \(\beta\). Next we consider the robustness of the partial pole assignment for the cubic pencil \(P(\lambda)\). The robustness is measured in terms of the condition number \(\kappa(P_2(\lambda))\) of the eigenvalues of the closed-loop pencil \(P_2(\lambda)\). To define \(\kappa(P_2(\lambda))\), we denote
\[
Y = (y_1, \ldots, y_k, x_{k+1}, \ldots, x_{3n}),
\]
where \(y_j, 1 \leq j \leq k\) are defined by (8), and
\[
D = \text{diag}(\mu_1, \ldots, \mu_k, \lambda_{k+1}, \ldots, \lambda_{3n}).
\]

For simplicity, we assume that \((Y, D, Z)\) is a Jordan pair of \(P_\lambda(\lambda)\). That is, \((Y, D, Z)\) with
\[
Z = \begin{bmatrix}
Y \\
YD \\
MYD^2
\end{bmatrix}^{-1}
\]
form a Jordan triple of \(P_\lambda(\lambda)\), where \(M = I \otimes M\) and
\[
S = \begin{bmatrix}
Y \\
YD \\
YD^2
\end{bmatrix}.
\]

It can then be shown that \(\kappa(P_\lambda(\lambda)) = \|Y(\beta)\|_2 \|Z(\beta)\|_2\). Therefore, we can formulate both the robust and minimum gain partial pole assignment problem as the single minimization problem:
\[
\min_{\beta} \{\alpha \|H(\beta)\|_2 \|\Psi^{-1}(\beta)\|_2 + (1 - \alpha)\kappa(P_\lambda(\beta))\}, \quad (12)
\]
where \(0 < \alpha < 1\) is the weighting factor. If \(\alpha = 0\), then we have the pure robust pole assignment problem, while, if \(\alpha = 1\), then we obtain the minimum gain pole assignment problem.

Next we observe that
\[
\|H(c\beta)\|_2 \|\Psi^{-1}(c\beta)\|_2 = \|H(\beta)\|_2 \|\Psi^{-1}(\beta)\|_2 + \kappa(P_\lambda(\beta)) \quad \text{for any} \ c \in \mathbb{R} \ \text{with} \ c \neq 0.
\]
This ambiguity will cause difficulty in solving the minimization problem (12). One possible way to cure this problem is to put constraints on (12) by normalizing the eigenmatrix \(Y\). However, this constrained minimization problem is very hard to handle numerically. Here we will adopt an idea in [2] where Byers and Nash consider the pure robust pole assignment for first order systems by an optimization approach without constraint. In their method, instead of taking the condition number to be the objective function, they consider an alternative objective function which is closely related to the original one. It turns out these two objective functions produce equivalent optima. For our case, we define
\[
\phi(\beta) = \frac{1}{2} \left(\|Y(\beta)\|_2^2 + \|Z(\beta)\|_2^2\right).
\]
It is clear that \(\phi(\beta) \geq \kappa(P_\lambda(\beta))\). Moreover, it is shown in [7] that \(\phi(\beta)\) and \(\kappa(P_\lambda(\beta))\) have the same minimizer and the minimum values are equal. Therefore, it is reasonable to reformulate our problem as the following minimization problem:
\[
\min_{\beta} \{\alpha \|H(\beta)\|_2 \|\Psi^{-1}(\beta)\|_2 + (1 - \alpha)\phi(\beta)\} \quad (13)
\]
which is an unconstrained minimization problem where the scaling invariant property has been eliminated.

This unconstrained minimization problem now can be solved using a standard numerical method such as Newton's method with line search. This leads to the following simple scheme.

\section{An Optimization Technique For Robust and Minimum Feedback Norm}

Let \(\beta^*\) be the solutions of (13), and \(\beta^{(0)}\) be an initial approximation.

Set \(l = 0\).

\textbf{Step 1.} If \(\beta^{(l)}\) and \(\beta^*\) are within tolerance error, stop.

\textbf{Step 2.} Find a direction \(p^{(l)}\) such that the objective function \(f(\beta) = \alpha \|H(\beta)\|_2 \|\Psi^{-1}(\beta)\|_2 + (1 - \alpha)\phi(\beta)\) decreases locally along \(p^{(l)}\).

\textbf{Step 3.} Perform the line search to yield \(f(\beta^{(l)} + t^{(l)} p^{(l)}) < f(\beta^{(l)})\) with \(t^{(l)} > 0\). Set \(\beta^{(l+1)} = \beta^{(l)} + t^{(l)} p^{(l)}\) and go to Step 1.

\textbf{Step 4.} Set \(\beta^{(l+1)} = \beta^{(l)} + t^{(l)} p^{(l)}\) and return to Step 1.
IV. RESULTS OF A NUMERICAL EXPERIMENT

In this section, we present results of a numerical experiment performed on a simulated data provided by the Boeing Company. All computations were performed using MATLAB version 5.3 on a Linux server with machine precision $2.2 \times 10^{-16}$. For implementation of our optimization algorithm, we simply used MATLAB function fminbnd. The matrices $M$, $C_1$, $C_2$, $K_1$ and $K_2$ are the order 42 with norms of orders between $10^5$ and $10^9$. There are 126 eigenvalues of the system (counting multiplicities).

Let
\[
\{\lambda_j\}_{j=1}^{126} = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{11}, \lambda_{12}\}
\]
be a set of 6 pairs of complex conjugate unwanted eigenvalues and let
\[
\{\mu_j\}_{j=1}^{126} = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_{11}, \mu_{12}\}
\]
be the set of prescribed values to be assigned. Their specific values are given in Table I.

We choose the input matrix
\[
B = (b_1, b_2, b_3, b_4)
\]
(i.e., $p = 4$) with
\[
b_T^T = \frac{1}{\sqrt{21}}[1, 0, -1, 0, 1, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0]
\]
except $b_2(2, 1) = -\frac{1}{2}$; $b_2^T = \frac{1}{\sqrt{7}}[1, 1, 1, 1, 1]$, and $b_4^T = b_4^T$ except $b_4(4, 2) = \frac{1}{2}$. The initial guess $\beta^{(0)}$ is taken as
\[
\beta^{(0)} = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T
\]
which corresponds to
\[
H = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
The superscripts $(0)$ and $(f)$ are used to indicate the initial and final approximations, respectively, and $(\wedge)$ is used for the computed value of a number or matrix.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\lambda_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-1.9416357e+00 + 5.7143524e+01i$</td>
</tr>
<tr>
<td>2</td>
<td>$-1.9416357e+00 - 5.7143524e+01i$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.93183335e-01 + 4.1683155e+01i$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.93183335e-01 - 4.1683155e+01i$</td>
</tr>
<tr>
<td>5</td>
<td>$-2.6340898e+00 + 3.5636998e+01i$</td>
</tr>
<tr>
<td>6</td>
<td>$-2.6340898e+00 - 3.5636998e+01i$</td>
</tr>
<tr>
<td>7</td>
<td>$-2.5838996e+00 + 2.8411096e+01i$</td>
</tr>
<tr>
<td>8</td>
<td>$-2.5838996e+00 - 2.8411096e+01i$</td>
</tr>
<tr>
<td>9</td>
<td>$-4.039691e-01 + 2.415725e+01i$</td>
</tr>
<tr>
<td>10</td>
<td>$-4.039691e-01 - 2.415725e+01i$</td>
</tr>
<tr>
<td>11</td>
<td>$-9.9717738e-01 - 1.5327875e+01i$</td>
</tr>
<tr>
<td>12</td>
<td>$-9.9717738e-01 + 1.5327875e+01i$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\max \left\{ \left| \lambda_k - \lambda_k^{(f)} \right| \middle| \lambda_k \right\end{align*}
\]

Interpretation: Tables II and III display implementational results of Algorithm 2.1, which correspond to those without using an optimization technique. The relative errors of the assigned eigenvalues are acceptable; however, the norms of the gain matrices are somewhat large.

The results of Tables IV and V correspond to the use of the optimization technique. Both the relative errors of the assigned eigenvalues and the norms of the feedback matrices are improved this time. We expect that using the line search method.
V. CONCLUSION

The dynamical behavior of vibrating structures are determined by the eigenvalues and eigenvectors of the associated mathematical models modeling the structures. For large systems, a realistic scenario is that only a few eigenvalues and eigenvectors are troublesome. Thus a practical feedback design scheme for such systems should be one that is capable of reassigning only those small number of eigenvalues to suitable locations by using feedback control while keeping the remaining large number of eigenvalues of the open-loop pencil unchanged. Of course, the design has to be carried out using only available resources such as a limited number of eigenvalues and eigenvectors that can be measured in a vibration laboratory or computed using numerically effective computational schemes. Furthermore, for computational and other practical considerations, it is desirable that the closed-loop system is numerically robust; that is, the closed-loop eigenvalues are as insensitive as possible to small changes in the coefficient matrices and that the feedback gains should have small norms; because, the design with large feedback gains are not practically implementable. In this paper, a new parametric method for such a feedback design is developed for vibrating structures with aerodynamics effects. A distinguished feature of the method is that it works exclusively with the coefficient matrices of the associated mathematical model, thus enabling ones to take full advantages in a computational setting of the exploitable structures, such as the sparsity, definiteness, bandness, and others, very often offered by practical problems. Furthermore, under this set up, matrix inversion of a possible ill-conditioned matrix arising from transformation of the system to a standard first-order state space system can be avoided.

This attractive feature along with the minimal computational requirements of the method; namely, solutions of a small (i) Sylvester equation, (ii) algebraic linear system, and (iii) an unconstrained optimization problem, for which there exist excellent numerical methods, make the proposed scheme practical for stabilizing and controlling large amplitude vibrations in large structures with aerodynamics effects. It is hoped that paper will provide an incentive for further research on vibration control along this line.

TABLE V

<table>
<thead>
<tr>
<th></th>
<th>$|F^{(1)}|_2$</th>
<th>$|C_{1}^{(1)}|_2$</th>
<th>$|C_{2}^{(1)}|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1502287866e+06</td>
<td>1.4502290913e+08</td>
<td>1.3469566815e+07</td>
</tr>
</tbody>
</table>

procedure, further improvements would be possible.