REGULARIZATION OF LINEAR DISCRETE-TIME PERIODIC DESCRIPTOR SYSTEMS BY DERIVATIVE AND PROPORTIONAL STATE FEEDBACK

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Abstract. In this paper, we consider the regularization problem for the linear time-varying discrete-time periodic descriptor systems by derivative and proportional state feedback controls. Sufficient conditions are given under which derivative and proportional state feedback controls can be constructed so that the periodic closed-loop systems are regular and of index at most one. The construction procedures used to establish the theory are based on orthogonal and elementary matrix transformations and can, therefore, be developed to a numerically efficient algorithm. The problem of finite pole assignment of periodic descriptor systems is also studied.

Key words. linear periodic descriptor systems, regularization, derivative and proportional feedback

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1. Introduction. We consider linear time-varying discrete-time periodic descriptor systems of the form

\[ \begin{align*}
E_j x_{j+1} &= A_j x_j + B_j u_j, \\
y_j &= C_j x_j,
\end{align*} \tag{1.1} \]

where \( x_0 \) is given and the matrices \( E_j, A_j \in \mathbb{R}^{n \times n}, B_j \in \mathbb{R}^{n \times m} \) (\( m \leq n \)), \( C_j \in \mathbb{R}^{k \times n} \) are periodic with period \( p \geq 1 \), that is, \( E_j = E_{j+p}, A_j = A_{j+p}, B_j = B_{j+p} \), and \( C_j = C_{j+p} \) for all \( j \). Throughout this paper we assume that the control matrices \( B_j \) are all of full column rank and the matrices \( E_j \) are allowed to be singular.

The number of contributions on linear time-varying discrete-time periodic systems has been increasing in recent times; see, for example, \([5, 15, 20, 21, 22, 24, 25, 28, 29, 30, 31, 32]\) and references therein. This increasing interest in such systems is motivated by the large variety of processes that can be modelled through linear discrete-time periodic systems (e.g., multirate sampled-data systems, chemical processes, periodically time-varying filters and networks, seasonal phenomena, and so on \([1, 2, 4, 16, 26, 27, 33]\)). The dynamics of linear discrete-time periodic descriptor systems (1.1) depend critically on the regularity and the eigenstructure of the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) which form the homogeneous systems of (1.1), i.e.,

\[ E_j x_{j+1} = A_j x_j. \tag{1.2} \]

The matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are called to be regular if \( \det[C((\alpha_j, \beta_j)_{j=1}^p)] \neq 0 \),
where

\[
C((\alpha_j, \beta_j)_{j=1}^p) \equiv \begin{bmatrix}
\alpha_1 E_1 & 0 & \cdots & 0 & -\beta_1 A_1 \\
-\beta_2 A_2 & \alpha_2 E_2 & 0 & 0 & \vdots \\
& \ddots & \ddots & \ddots & \vdots \\
& & \ddots & \ddots & 0 \\
0 & 0 & -\beta_p A_p & \alpha_p E_p & \end{bmatrix},
\]

in which \(\alpha_j, \beta_j\) are complex variables for \(j = 1, \ldots, p\).

**Definition 1.1.** Let \((E_j, A_j)\)\(^{p}_{j=1}\) be \(n \times n\) regular matrix pairs. If there are complex numbers \(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p\) with

\[
\left( \prod_{j=1}^p \alpha_j, \prod_{j=1}^p \beta_j \right) \equiv (\pi\alpha, \pi\beta) \neq (0, 0)
\]

satisfying \(\det[C((\alpha_j, \beta_j)_{j=1}^p)] = 0\), then we say that \((\pi\alpha, \pi\beta)\) is an eigenvalue pair of \((E_j, A_j)\)\(^{p}_{j=1}\).

Note that if \((\pi\alpha, \pi\beta)\) is an eigenvalue of \((E_j, A_j)\)\(^{p}_{j=1}\), then \((\pi\alpha, \pi\beta)\) and \((\tau \pi\alpha, \tau \pi\beta)\) represent the same eigenvalue pair for any nonzero \(\tau\). If \(\pi\beta \neq 0\), then \(\lambda = \pi\alpha/\pi\beta\) is a finite eigenvalue; otherwise \((\pi\alpha, 0)\) is an infinite eigenvalue. The set of all eigenvalue pairs of \((E_j, A_j)\)\(^{p}_{j=1}\) is denoted by \(\sigma((E_j, A_j)\)\(^{p}_{j=1}\))

\[
\sum_{k=0}^{n} c_k \pi^{n-k}_\alpha \pi^{-k}_\beta,
\]

where \(c_0, \ldots, c_n\) are complex numbers uniquely determined by \((E_j, A_j)\)\(^{p}_{j=1}\). For the regular matrix pairs \((E_j, A_j)\)\(^{p}_{j=1}\) this implies that at least one of the \(c_k\)'s is nonzero, and hence we see from Definition 1.1 that there are exact \(n\) eigenvalue pairs (counting multiplicity) for \((E_j, A_j)\)\(^{p}_{j=1}\).

It was shown in \([29]\) that the solvability of (1.2) is equivalent to the condition that the pencil

\[
\alpha E - \beta A := \begin{bmatrix}
\alpha E_1 & 0 & \cdots & 0 & -\beta A_1 \\
-\beta A_2 & \alpha E_2 & 0 & 0 & \vdots \\
& \ddots & \ddots & \ddots & \vdots \\
& & \ddots & \ddots & 0 \\
0 & 0 & -\beta A_p & \alpha E_p & \end{bmatrix}
\]

is regular, i.e., \(\det(\alpha E - \beta A) \neq 0\). From (1.5) it is easy to check that

\[
\sigma((E_j, A_j)\)\(^{p}_{j=1}\) = \{(\alpha^p, \beta^p) \mid \det(\alpha E - \beta A) = 0\}.
\]

Hence, from (1.7) the solvability condition of (1.2) becomes the regularity of the matrix pairs \((E_j, A_j)\)\(^{p}_{j=1}\).

In order to alter the dynamics of the periodic descriptor systems (1.1), it is usually to use proportional state feedback to modify the matrices \(A_j\), that is, the control
vectors are taken to be \( u_j = F_j x_j + v_j \) for \( j = 1, \ldots, p \). The closed-loop matrix pairs then become

\[
\{(E_j, A_j + B_j F_j)\}^p_{j=1}.
\]

Similarly, if we interchange the role of \( E_j \) and \( A_j \), then we can also use derivative state feedback to modify the matrices \( E_j \). The closed-loop matrix pairs become

\[
\{(E_j + B_j G_j, A_j + B_j F_j)\}^p_{j=1},
\]

where the control vectors are taken to be \( u_j = -G_j x_{j+1} + v_j, \) \( j = 1, \ldots, p \). If a full state feedback of the form \( u_j = -G_j x_j + F_j x_j + v_j \) is considered (see [6] for the case of \( p = 1 \)), then it leads to periodic descriptor systems with the periodic matrix pairs of the form

\[
\{(E_j + B_j G_j, A_j + B_j F_j)\}^p_{j=1}.
\]

For the case of period \( p = 1 \), one has the time-invariant case \( E_j = E, A_j = A, B_j = B, C_j = C \). It is well known that for a regular matrix pair \((E,A)\) (i.e., \( \det(\alpha E - \beta A) \neq 0 \) for some \( (\alpha, \beta) \in \mathbb{C}^2 \)) there exist nonsingular matrices \( P \) and \( Q \) which transform \( E \) and \( A \) into the Kronecker canonical form [17]:

\[
PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}.
\]

Here \( J \) is a Jordan matrix corresponding to the finite eigenvalues of \((E,A)\) and \( N \) is a nilpotent Jordan matrix corresponding to the infinite eigenvalues. The index of the matrix pair is the index of nilpotency of the nilpotent matrix \( N \), i.e., \((E,A)\) is of index \( \nu \), denoted by \( \nu = \text{ind}_\infty(E,A) \), if \( N^{\nu-1} \neq 0 \) and \( N^\nu = 0 \). By convention, if \( E \) is nonsingular, the pencil is said to be of index zero. If a matrix pair is regular and of index at most 1, the corresponding time-invariant continuous system

\[
E\frac{dx}{dt} = Ax(t) + Bu(t)
\]

has a unique solution for all admissible controls \( u(t) \) with consistent initial conditions. In theory, such a system can be separated into purely dynamical and purely algebraic parts, and moreover, the algebraic part can be eliminated to give a reduced-order standard system. If the index is larger than 1, however, impulses can arise in the response of the system and the system can lose causality if the control is not sufficiently smooth [18]. Therefore, it is desirable to use a feedback control that ensures that the closed-loop system is regular and of index at most one, and furthermore, has the required finite poles. In the last few years, there has been an increasing interest in developing numerical algorithms for the regularization and the finite pole assignment of descriptor time-invariant systems by proportional and derivative feedback. See, for example, [6, 7, 8, 9, 10, 11, 12, 13, 23] and references therein.

In this paper, we focus on the following regularization and pole assignment problems: For given periodic matrix triples \( \{(E_j, A_j, B_j)\}^p_{j=1} \), we first construct periodic derivative and proportional matrices \( G_j \) and \( F_j \) such that the periodic matrix pairs \( \{(E_j + B_j G_j, A_j + B_j F_j)\}^p_{j=1} \) of the periodic closed-loop systems are regular and of
index at most one (see the definitions in the next section). Then we construct periodic feedback matrices $G_f$ and $F_f$ such that the periodic closed-loop systems not only are regular and have the required finite poles, but also have index at most one. To the best of our knowledge, for the case of period $p \geq 2$, these problems have not been investigated much in the literature.

Our contribution in this paper is threefold. First, in Theorem 2.5 we give an equivalent condition for the periodic matrix pairs $\{(E_j, A_j)\}_{j=1}^{p}$ to be regular and of index at most one. Second, in Theorems 3.1 and 4.3 we specify sufficient conditions under which derivative and proportional state feedback can be constructed so that the periodic closed-loop systems are regular and of index at most one. Third, in Theorem 5.1, we give the solvability condition for the finite pole assignment problem of the periodic matrix triples. The main proofs given in this paper can provide a numerically method for constructing the required feedback matrices, which is based on orthogonal and elementary matrix transformations.

This paper is organized as follows. In section 2 we introduce some notations and definitions, and give some preliminary results. In section 3 we present a canonical form under matrix transformations. In section 4, we use this canonical form to construct derivative and proportional feedback so that the periodic closed-loop systems are regular and of index at most one. The problem of finite pole assignment with derivative and proportional feedback is presented in section 5.

2. Preliminaries. In this section we introduce some notations and definitions, and give some preliminary results. Throughout this paper we use the following notations. For any given periodic matrix triples $\{(E_j, A_j, B_j)\}_{j=1}^{p}$ we use, alternatively, the script notations

$$
\tilde{E}_j \equiv \tilde{E}(E_j, \ldots, E_{j+p-1}; A_{j+1}, \ldots, A_{j+p-1}) := \begin{bmatrix}
E_j & 0 & 0 & \cdots & \cdots & 0 \\
-A_{j+1} & E_{j+1} & 0 & \cdots & 0 \\
0 & -A_{j+2} & E_{j+2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & -A_{j+p-1} & E_{j+p-1}
\end{bmatrix},
$$

(2.1a)

$$
\tilde{A}_j \equiv \tilde{A}(A_j) := \begin{bmatrix}
0_{n \times (p-1)n} & A_j \\
0_{(p-1)n \times (p-1)n} & 0_{(p-1)n \times n}
\end{bmatrix},
$$

(2.1b)

$$
B_j \equiv B(B_j, \ldots, B_{j+p-1}) := \text{diag}(B_j, \ldots, B_{j+p-1}).
$$

(2.1c)

We also denote the null space of a matrix $M$ by $\mathcal{N}(M)$, and use $S_\infty(M)$ to denote a full rank matrix whose columns span the null space $\mathcal{N}(M)$. The indices “$j$” for all periodic coefficient matrices are chosen in $\{1, \ldots, p\}$ modulo $p$ without ambiguity.

In terms of the above notations, we now characterize the regular periodic matrix pairs as follows.

**Lemma 2.1.** The following statements are equivalent.

1. The periodic matrix pairs $\{(E_j, A_j)\}_{j=1}^{p}$ are regular.
2. The matrix pair $(\mathcal{E}, \mathcal{A})$ in (1.6) is regular, i.e., $\det(\alpha \mathcal{E} - \beta \mathcal{A}) \neq 0$.
3. The matrix pair $(\tilde{E}_j, \tilde{A}_j)$ is regular, for some $j \in \{1, \ldots, p\}$.
4. The matrix pairs $(\tilde{E}_j, \tilde{A}_j)$ are regular, for all $j = 1, \ldots, p$. 


$Q_j$ and $Z_j$ such that

\[
Q_j E_j Z_j = \begin{bmatrix}
b_{11}^j & \cdots & b_{1n}^j \\
\vdots & \ddots & \vdots \\
b_{n1}^j & \cdots & b_{nn}^j
\end{bmatrix}, \quad Q_j A_j Z_j^{-1} = \begin{bmatrix}
a_{11}^j & \cdots & a_{1n}^j \\
\vdots & \ddots & \vdots \\
a_{n1}^j & \cdots & a_{nn}^j
\end{bmatrix},
\]

where $Z_0 = Z_p$, from which we can derive that

\[
(2.2a) \quad \det[C((\alpha_j, \beta_j)_{j=1}^p)] = \prod_{i=1}^n \left( \prod_{j=1}^p \alpha_j b_{ii}^j - \prod_{j=1}^p \beta_j a_{ii}^j \right),
\]

\[
(2.2b) \quad \det(\alpha E - \beta A) = \prod_{i=1}^n \left( \alpha^p \prod_{j=1}^p b_{ii}^j - \beta^p \prod_{j=1}^p a_{ii}^j \right),
\]

\[
(2.2c) \quad \det(\alpha \tilde{E}_j - \beta \tilde{A}_j) = \prod_{i=1}^n \left( \alpha^{j+p-1} \prod_{k=j}^p b_{ii}^k - \beta \alpha^{p-1} \prod_{k=j}^p a_{ii}^k \right),
\]

for $j = 1, \ldots, p$.

It is easily seen that any equation in (2.2) which is not identical to zero, i.e., 
$(\prod_{i=1}^p b_{ii}^j, \prod_{i=1}^p a_{ii}^j) \neq (0,0)$, for $i = 1, \ldots, n$, implies that the other equations in (2.2) 
are also not identical to zero. This completes the proof. \qed

In a similar fashion to the Kronecker canonical form for a regular matrix pair, we 
can transform regular periodic matrix pairs into periodic Kronecker canonical forms.

**Lemma 2.2.** Suppose that the periodic matrix pairs $\{(E_j, A_j)\}_{j=1}^p$ in systems (1.1) are regular. Then there exist nonsingular matrices $X_j$ and $Y_j, j = 1, \ldots, p$, such that

\[
(3.3) \quad X_j E_j Y_j = \begin{bmatrix} I & 0 \\ 0 & E_0^j \end{bmatrix}, \quad X_j A_j Y_j^{-1} = \begin{bmatrix} A_j^f & 0 \\ 0 & I \end{bmatrix},
\]

where $Y_0 = Y_p, A_j^{f+p-1} A_j^{f+p-2} \cdots A_j^f \equiv J_j, (j = 1, \ldots, p)$ is a Jordan matrix corresponding to the finite eigenvalues of $\{(E_j, A_j)\}_{j=1}^p$ and $E_0^f E_0^{f+1} \cdots E_0^{f+p-1} \equiv N_j, (j = 1, \ldots, p)$ is a nilpotent Jordan matrix corresponding to the infinite eigenvalues of $\{(E_j, A_j)\}_{j=1}^p$.

Proof. By periodic Schur decomposition theorem and the reordering of eigenvalues [5, 20] there are unitary matrices $Q_j, P_j, j = 1, \ldots, p$ so that

\[
(3.4) \quad Q_j E_j P_j = \begin{bmatrix} E_{j,1} & E_{j,3} \\ 0 & E_{j,2} \end{bmatrix}, \quad Q_j A_j P_j^{-1} = \begin{bmatrix} A_{j,1} & A_{j,3} \\ 0 & A_{j,2} \end{bmatrix},
\]

are upper triangular, and moreover $E_{j,1}$ and $A_{j,2}$ are nonsingular and all diagonal elements of $E_{j,2} E_{j+1,2} \cdots E_{j+p-1,2}$ are zero for $j = 1, 2, \ldots, p$. We then let

\[
(3.5a) \quad \begin{bmatrix} E_{j,1}^{-1} & 0 \\ 0 & A_{j,2}^{-1} \end{bmatrix} \begin{bmatrix} E_{j,1} & E_{j,3} \\ 0 & E_{j,2} \end{bmatrix} = \begin{bmatrix} I & \hat{E}_{j,3} \end{bmatrix},
\]

\[
(3.5b) \quad \begin{bmatrix} E_{j,1}^{-1} & 0 \\ 0 & A_{j,2}^{-1} \end{bmatrix} \begin{bmatrix} A_{j,1} & A_{j,3} \\ 0 & A_{j,2} \end{bmatrix} = \begin{bmatrix} \hat{A}_{j}^f & A_{j,3} \end{bmatrix}.
\]
Next, we prove that there exists periodic matrices \( U_j \) and \( V_j \), \( j = 1, 2, \ldots, p \) such that

\[
\begin{align*}
(2.6a) & \quad \begin{bmatrix} I & U_j \end{bmatrix} \begin{bmatrix} I & \hat{E}_{j,3} \\ 0 & \hat{E}_{j}^0 \end{bmatrix} \begin{bmatrix} I & V_j \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \hat{E}_{j}^0 \end{bmatrix}, \\
(2.6b) & \quad \begin{bmatrix} I & U_j \end{bmatrix} \begin{bmatrix} \hat{A}_j^f & \hat{A}_{j,3} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & V_{j-1} \end{bmatrix} = \begin{bmatrix} \hat{A}_j^f & 0 \\ 0 & I \end{bmatrix}.
\end{align*}
\]

Comparing the both sides of (2.6a) and (2.6b) we have

\[
\begin{align*}
(2.7a) & \quad V_j + U_j \hat{E}_j^0 + \hat{E}_{j,3} = 0, \\
(2.7b) & \quad \hat{A}_j^f V_{j-1} + U_j + \hat{A}_{j,3} = 0
\end{align*}
\]

for \( j = 1, 2, \ldots, p \), where \( V_0 = V_p \). Eliminating \( U_j \) in (2.7) we get

\[
(2.8) \quad V_j = \hat{A}_j^f V_{j-1} \hat{E}_j^0 + \hat{A}_{j,3} \hat{E}_j^0 - \hat{E}_{j,3}
\]

for \( j = 1, 2, \ldots, p \), from which we obtain

\[
(2.9) \quad V_p = \left( \hat{A}_p^f \hat{A}_{p-1}^f \cdots \hat{A}_1^f \right) V_p \left( \hat{E}_1^0 \hat{E}_2^0 \cdots \hat{E}_p^0 \right) + D_p,
\]

where

\[
D_p = (\hat{A}_{p,3} \hat{E}_p^0 - \hat{E}_{p,3}) + \hat{A}_p^f (\hat{A}_{p-1,3} \hat{E}_{p-1}^0 - \hat{E}_{p-1,3}) \hat{E}_p^0 + \cdots + \\
\left( \hat{A}_p^f \hat{A}_{p-1}^f \cdots \hat{A}_2^f \right) (\hat{A}_{1,3} \hat{E}_1^0 - \hat{E}_{1,3}) (\hat{E}_2^0 \hat{E}_3^0 \cdots \hat{E}_p^0).
\]

Notice that \( (\hat{E}_1^0 \hat{E}_2^0 \cdots \hat{E}_p^0) \) is an upper triangular matrix with all diagonal elements zero, we can uniquely determine the matrix \( V_p \) from (2.9). Then, from (2.8) and (2.6b) we can uniquely determine \( V_j \) for \( j = 1, 2, \ldots, p-1 \), and \( U_j \) for \( j = 1, 2, \ldots, p \), respectively.

Finally, by the well-known Jordan decomposition theorem we know that there exist nonsingular matrices \( G_j, Z_j, j = 1, \ldots, p \) such that

\[
\begin{align*}
(2.10a) & \quad G_j^{-1} \left( \hat{A}_{j+p-1}^f \hat{A}_{j+p-2}^f \cdots \hat{A}_j^f \right) G_j = J_j \text{ (Jordan form)}, \\
(2.10b) & \quad Z_j^{-1} \left( \hat{E}_j^0 \hat{E}_{j+1}^0 \cdots \hat{E}_{j+p-1}^0 \right) Z_j = N_j \text{ (nilpotent Jordan form)}.
\end{align*}
\]

Now let

\[
X_j := \begin{bmatrix} G_{j+1}^{-1} & 0 \\ Z_j^{-1} & I \end{bmatrix} \begin{bmatrix} I & U_j \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{j,1}^{-1} & 0 \\ 0 & A_{j,2}^{-1} \end{bmatrix} Q_j,
\]

\[
Y_j := P_j \begin{bmatrix} I & V_j \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{j+1} & 0 \\ 0 & Z_{j+1} \end{bmatrix},
\]

and

\[
\begin{align*}
(2.11) & \quad E_j^0 := Z_j^{-1} \hat{E}_j^0 Z_{j+1}, \\
(2.12) & \quad A_j^f := G_{j+1}^{-1} \hat{A}_j^f G_j.
\end{align*}
\]
Then from (2.4)–(2.12) we have
\[
X_j E_j Y_j = \begin{bmatrix} I & 0 \\ 0 & E_j^0 \end{bmatrix}, \quad X_j A_j Y_{j-1} = \begin{bmatrix} A_j^f & 0 \\ 0 & I \end{bmatrix}
\]
with \( \prod_{k=1}^{j+p-1} A_k^f = J_j \), a Jordan matrix, and \( \prod_{k=j}^{j+p-1} E_k^0 = N_j \), a nilpotent Jordan matrix, for \( j = 1, \ldots, p \).

As an application of this lemma, let us consider the periodic system (1.1) with \( u_j = 0 \), i.e., the free periodic system
\[
E_j x_{j+1} = A_j x_j.
\]
Using Lemma 2.2 we can reduce the system (2.13) into a forward and a backward part:
\[
\begin{align*}
x_j^{f+1} &= A_j^f x_j^f, \\
x_j^{b+1} &= A_j^b x_j^b,
\end{align*}
\]
\[
\begin{align*}
E_j^0 x_j^{b+1} &= x_j^b, \\
E_{j+p}^0 &= E_j^0,
\end{align*}
\]
where
\[
x_j = Y_{j-1} \begin{bmatrix} x_j^f \\ x_j^b \end{bmatrix},
\]
provided the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are regular, then we obtain from (2.14) and (2.15) the set of \( p \) subsampled systems:
\[
x_{j+(i+1)p}^f = J_j x_{j+ip}^f, \quad i = 0, 1, 2, \ldots
\]
\[
N_j x_{j+(i+1)p}^b = x_{j+ip}^b, \quad i = 0, 1, 2, \ldots
\]
for \( j = 1, 2, \ldots, p \), which are time invariant. This shows that the dynamical properties of the system (2.13) depend critically on the eigenstructure of the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \). Especially, if \( N_j = 0 \) for \( j = 1, 2, \ldots, p \), then \( x_j^b = 0 \) for all \( j = 0, 1, \ldots \), and so in such case the system (2.13) is reduced into a reduced-order standard periodic system (2.14).

By Lemma 2.2, we can characterize the nilpotency of the regular periodic matrix pairs by the index of \((\tilde{E}_j, \tilde{A}_j)\).

**Lemma 2.3.** Assume that the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are regular and have the periodic Kronecker canonical forms as shown in (2.3), then the nilpotency of the nilpotent matrix \( E_j^0 \cdots E_{j+p-1}^0 \equiv N_j \) (i.e., \( N_j^{\nu_j-1} \neq 0 \) and \( N_j^{\nu_j} = 0 \)) is just equal to \( \ind_{\infty}(\tilde{E}_j, \tilde{A}_j) \), which denotes the index of \((\tilde{E}_j, \tilde{A}_j)\), for \( j = 1, \ldots, p \).

**Proof.** Let
\[
X_j = \diag(X_j, \ldots, X_{j+p-1}), \quad Y_j = \diag(Y_j, \ldots, Y_{j+p-1}),
\]
where \( X_j \) and \( Y_j \) are defined in Lemma 2.2, and we define \( X_{k+p} = X_k \) and \( Y_{k+p} = Y_k \) for all \( k \). Then it follows from (2.3) that
\[
X_j \tilde{E}_j Y_j = \begin{bmatrix}
\hat{E}_j & 0 & 0 & \cdots & 0 \\
-\hat{A}_{j+1} & \hat{E}_{j+1} & 0 & \cdots & 0 \\
0 & -\hat{A}_{j+2} & \hat{E}_{j+2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \hat{E}_{j+p-2} & 0 \\
0 & \cdots & \cdots & 0 & -\hat{A}_{j+p-1} \hat{E}_{j+p-1}
\end{bmatrix},
\]
\( \begin{equation} \mathcal{X}_j \tilde{A}_j \mathcal{Y}_j = \tilde{A}(A_j) := \begin{bmatrix} 0_{n \times (p-1)n} & \hat{A}_j \\ 0_{(p-1)n \times (p-1)n} & 0_{(p-1)n \times n} \end{bmatrix}, \end{equation} \)

where

\( \hat{E}_j = X_j E_j Y_j = \begin{bmatrix} I \\\n0 \end{bmatrix}, \quad \hat{A}_j = X_j A_j Y_j^{-1} = \begin{bmatrix} A^f_j \\
0 \end{bmatrix}. \)

Notice the special structure of (2.16) and (2.17). Using the elementary row transformations, we can find nonsingular matrices \( \mathcal{R}_j \) such that

\( \begin{equation} \begin{bmatrix} I_{(p-1)n \times (p-1)n} & 0 \\ 0 & \hat{E}_j \end{bmatrix} = \mathcal{R}_j \begin{bmatrix} I_{(p-1)n \times (p-1)n} & \hat{A}_j \\ 0 & 0 \end{bmatrix}, \end{equation} \)

which implies that

\( \text{ind}_\infty (\hat{E}_j, \hat{A}_j) = \text{ind}_\infty (\mathcal{K}(\hat{E}_j), \mathcal{K}(\hat{A}_j)) = \text{ind}_\infty \left( \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} A^f_{j+p-1} \cdots A^f_j \\ 0 \end{bmatrix} \right), \)

and hence, the nilpotency \( \nu_j \) of the nilpotent matrix \( N_j = E^0_j \cdots E^0_{j+p-1} \) is equal to \( \text{ind}_\infty (\hat{E}_j, \hat{A}_j) \), for \( j = 1, \ldots, p \). \( \square \)

According to the result of Lemma 2.3 the indexes of the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) can be defined as follows.

**Definition 2.1.** The indexes of regular periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are defined by

\( \begin{equation} \nu_j = \text{ind}_\infty (\hat{E}_j, \hat{A}_j), \quad j = 1, 2, \ldots, p. \end{equation} \)

If \( \nu_j \leq 1 \) for all \( j = 1, \ldots, p \), i.e., \( E_j \) are all nonsingular or \( N_j = 0 \), for all \( j \), then the periodic matrix pairs are said to be of index at most one.

**Remark.** (i) It is worthwhile to point out that the indexes \( \nu_j \) for regular periodic matrix pairs are not necessarily equal. For example, the periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^2 \) with \( A_j = I_2, j = 1, 2 \) and

\( E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \)

have indexes \( \nu_1 = 1 \) and \( \nu_2 = 2 \).

(ii) As shown in the preceding part of this paper, the monodromy matrices

\( J_j = \prod_{k=j+jp-1}^j A^f_k \) and \( N_j = \prod_{k=1}^{j+p-1} E^0_k, \quad j = 1, \ldots, p, \)
play an important role in the representation of solutions of (1.1). From Lemma 2.3 it is reasonable to define the indexes of \( \{(E_j, A_j)\}_{j=1}^p \) by (2.20). Note that the indexes of the enlarged cyclic forms as in (1.6) are not appropriate to define the indexes of \( \{(E_j, A_j)\}_{j=1}^p \). To see this, let us consider the above-given data again. A short calculation gives rise to

\[
\text{ind}_\infty \left( \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix} \right) = \text{ind}_\infty \left( \begin{bmatrix} E_2 & 0 \\ 0 & E_1 \end{bmatrix}, \begin{bmatrix} 0 & A_2 \\ A_1 & 0 \end{bmatrix} \right) = 3,
\]

which is neither equal to the nilpotency of \( E_1E_2 \) nor to the nilpotency of \( E_2E_1 \).

From (2.18) and (2.19), we immediately get the following result.

**Corollary 2.4.** If periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are regular and of index at most 1, then \( \text{rank} \tilde{E}_j \) is independent of \( j \). Moreover, the number of finite eigenvalues of \( \{(E_j, A_j)\}_{j=1}^p \) is equal to \( \gamma - (p-1)n \), where \( \gamma = \text{rank} \tilde{E}_j \).

We note that if regular periodic matrix pairs have some higher indexes, then, generally speaking, \( \text{rank} \tilde{E}_j \) is dependent on \( j \). This can be illustrated by the simple example. Let \( p = 3 \), \( A_j = I_3 \), and

\[
E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

It is easy to verify that \( \text{rank} \tilde{E}_1 = 6 \) and \( \text{rank} \tilde{E}_2 = \text{rank} \tilde{E}_3 = 7 \).

According to the result of Lemma 2.3 and Definition 2.1 the following equivalent condition follows by Lemma 1 of [6] (see also [21]) immediately.

**Theorem 2.5.** The periodic matrix pairs \( \{(E_j, A_j)\}_{j=1}^p \) are regular and of index at most 1 if and only if

\[
\text{rank} \left[ \tilde{E}_j, \tilde{A}_j S_\infty(\tilde{E}_j) \right] = pn \quad \text{for} \quad j = 1, 2, \ldots, p. \tag{2.21}
\]

For the linear time-invariant descriptor systems \( Ex_{k+1} = Ax_k + Bu_k \), the condition

\[
\text{rank} [\lambda E - A, B] = n \quad \forall \ \lambda \in \mathbb{C} \quad \text{and} \quad \text{rank} [E, AS_\infty(E), B] = n \tag{2.22}
\]

give sufficient conditions for the solvability of regularization and pole assignment problems [6, 21]. By Lemma 2.3 and (2.22) it is motivated to give conditions for investigating the regularization problem and pole assignment problem of the linear time-varying periodic descriptor systems (1.1).

**Definition 2.2.** The periodic matrix triples \( \{(E_j, A_j, B_j)\}_{j=1}^p \) satisfy conditions (C1) and (C2) if

\[
\text{rank} [\lambda E - A, B] = pn \quad \forall \ \lambda \in \mathbb{C}; \tag{C1}
\]

\[
\text{rank} [\tilde{E}_j, \tilde{A}_j S_\infty(\tilde{E}_j), B_j] = pn \quad \text{for} \quad j = 1, \ldots, p. \tag{C2}
\]

Here \( E, A \) and \( B \equiv B_1 \) are given in (1.6) and (2.1), respectively.

**Remark.** A natural question can be asked here: can we extend the condition \( \text{rank} [E, AS_\infty(E), B] = n \) directly to the enlarged cyclic triples \( (E, A, B) \) by

\[
\text{rank} [E, AS_\infty(E), B] = pn, \tag{C2'}
\]
or equivalently,

$$\text{rank}[E_j, A_j S_{\infty}(E_{j-1}), B_j] = n, \quad j = 1, \ldots, p.$$ 

In fact, (C2) is sufficient for (C2). From the reduction in Lemma 5 of [6] and (C2) we can w.l.o.g. suppose that the periodic matrix triples \{(E_j, A_j, B_j)\}_{j=1}^p have the following forms:

$$E_j = \begin{bmatrix} E_{j1}^1 & 0 & 0 \\ E_{j1}^2 & E_{j2}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_j = \begin{bmatrix} A_{11}^j & A_{12}^j & A_{13}^j \\ A_{21}^j & A_{22}^j & A_{23}^j \\ A_{31}^j & A_{32}^j & A_{33}^j \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ 0 \\ B_2^j \end{bmatrix},$$

with a compatible partitioning, where \(E_{j1}^1, B_2^j\) are nonsingular, \(E_{j2}^2\) and \((A_{33}^j)\) are of full column rank for \(j = 1, \ldots, p\). It is easy to check that rank\([E_j, A_j S_{\infty}(E_j), B_j]\) = \(pn\), for \(j = 1, \ldots, p\). However, (C2) is not a necessary condition for (C2). For example, if we let \(E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\), \(E_2 = A_1 = A_2 = I_2\), \(B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\). One can check that \((E_j, A_j, B_j)\)\(p\)\(_{j=1}^\) satisfy (C2), but not (C2). Therefore, (C2) is weaker than (C2). In our main theorem (Theorem 4.3) we will show that (C2) implies the periodic feedback closed-loop systems are regular and of index at most one.

The following two lemmas are simple but useful for the proof of the main result in sections 4 and 5.

**Lemma 2.6.** [6] Let \((E, A, B)\) satisfy (C1) or (C2) with \(p = 1\), where \(E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\). Then

(i) \((QEP, QAP, QBV)\) satisfies (C1) or (C2) for any nonsingular \(P, Q,\) and \(V\);
(ii) \((E + BG, A + BF, B)\) satisfies (C1) or (C2) for any \(G\) and \(F \in \mathbb{R}^{m \times n}\) with \(\mathcal{N}(E) \subset \mathcal{N}(E + BG)\);
(iii) rank\([E, A\Sigma, B]\) = \(n\) for any matrix in the form \(\Sigma = [S_{\infty}(E), R]\), where \(R \in \mathbb{R}^{n \times l}\).

**Lemma 2.7.** Assume that the periodic matrix triples \{(E_j, A_j, B_j)\}_{j=1}^p satisfy (C1) or (C2). Then

(i) \((Q_j E_j P_j, Q_j A_j P_{j-1}, Q_j B_j V_j)\)\(p\)\(_{j=1}^\) satisfy (C1) or (C2) for any nonsingular matrices \(P_j, Q_j \in \mathbb{R}^{n \times n},\) and \(V_j \in \mathbb{R}^{m \times m}\);
(ii) \((E_j + B_j G_j, A_j + B_j F_j, B_j)\)\(p\)\(_{j=1}^\) satisfy (C1) or (C2) for any matrices \(G_j, F_j \in \mathbb{R}^{m \times n}\) with \(\mathcal{N}(E_j) \subset \mathcal{N}(E_j + \cdots + E_{j+p-1}; A_{j+1}, \ldots, A_{j+p-1})\), where \(E_j = E_j + B_j G_j\ and \ A_j = A_j + B_j F_j\).
Proof. By Lemma 2.6(i) we get (i). From Lemma 2.6(ii), (iii), and (2.24), (ii) follows immediately.

3. Canonical forms of \( \{(E_j, A_j, B_j)\}_{j=1}^p \). In this section we present an algorithm to reduce the periodic matrix triples \( \{(E_j, A_j, B_j)\}_{j=1}^p \) into canonical forms by using orthogonal and elementary transformations. In the next section we show how to exploit these canonical forms to construct the required regularizing feedback.

Before describing the algorithm we introduce some convenient notations. We denote by \( M(m, n) \), \( O(n) \), \( L(n) \), and \( R(n) \) the sets of \( m \times n \) matrices, \( n \times n \) orthogonal, lower triangular, and upper triangular matrices, respectively. If \( m = n \), we simplify \( M(n) := M(m, n) \). Let \( T \) be a row or column transformation which is applied to a submatrix of a given matrix. Then we use \( T \) to denote the natural extension of \( T \) to be applied to the whole matrix. For example, let

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\]

with \( C_{ii} \in M(n_i), i = 1, 2, 3 \). Let \( Q_2 \in O(n_2) \) such that \( Q_2 C_{22} \) is upper triangular and

\[
R_1 = \begin{bmatrix}
I_{n_1} & -C_{11}^{-1} C_{13} \\
0 & I_{n_3}
\end{bmatrix},
\]

which is the transformation to eliminate \( C_{13} \) by \( C_{11} \), i.e., \( [C_{11}, C_{13}] R_1 = [C_{11}, 0] \). Then we have

\[
\bar{Q}_2 = \begin{bmatrix}
I_{n_1} \\
Q_2 \\
I_{n_3}
\end{bmatrix}, \quad \bar{R}_1 = \begin{bmatrix}
I_{n_1} & 0 & -C_{11}^{-1} C_{13} \\
0 & I_{n_2} & 0 \\
0 & 0 & I_{n_3}
\end{bmatrix}.
\]

Algorithm 3.1.

Input: periodic matrix triples \( \{(E_j, A_j, B_j)\}_{j=1}^p \) with \( E_j, A_j \in M(n) \) and \( B_j \in M(n, m) \) satisfying that for \( j = 1, \ldots, p \), \( \text{rank}(B_j) = m \) and

\[
\begin{pmatrix}
-A_j & E_j & B_j \\
-A_{j+1} & E_{j+1} & B_{j+1} \\
\ldots & \ldots & \ldots \\
-A_{j+p-2} & E_{j+p-2} & B_{j+p-2}
\end{pmatrix} = (p-1)n.
\]

Output: nonsingular matrices \( Q_j, P_j \in M(n) \), feedback matrices \( G_j, F_j \in M(m, n) \), and canonical forms

\[
Q_j(E_j + B_j G_j)P_j = \begin{bmatrix}
0_{m \times (n-i_j)} & 0_{m \times l_j} \\
0_{n_{j+1} \times i_j} & E^j_{11} \\
\vdots & \ddots \\
E^j_{i_l} & \ddots & E^j_{pp}
\end{bmatrix}, \quad Q_j B_j = \begin{bmatrix}
B^j_{11} \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]
\[
\begin{align*}
Q_j(A_j + B_j F_j) P_{j-1} &= \begin{bmatrix}
A_{L(11)}^j & \cdots & A_{L(pp)}^j \\
\vdots & \ddots & \vdots \\
A_{L(p1)}^j & \cdots & A_{L(pp)}^j
\end{bmatrix} \begin{bmatrix}
0_{m \times (n-j-1)} \\
A_{R(11)}^j \\
\vdots \\
A_{R(p1)}^j \\
0_{m \times j-1}
\end{bmatrix},
\end{align*}
\]
for \( j = 1, \ldots, p \), where
(i) \( B_{11}^k \in \mathcal{R}(m) \) nonsingular,
(ii) \( E_{kk}^j \in \mathcal{M}(n_{k-1}^{j+1}) \) nonsingular for \( k = 1, 2, \ldots, p - 1 \),
(iii) \( E_{pp}^j \in \mathcal{R}(I_j) \) nonsingular,
(iv) \( A_{L(kk)}^j \in \mathcal{M}(n_{k-1}^{j+1}, m_k^{j}) \) nonsingular for \( k = 1, 2, \ldots, p - 1 \),
(v) \( A_{R(kk)}^j \in \mathcal{M}(n_{k-1}^{j+1}, n_k^{j}) \) nonsingular for \( k = 1, 2, \ldots, p - 1 \),
(vi) \( A_{L(pp)}^j \) and \( A_{R(pp)}^j \) are \((l_j + n_{p-1}^{j+1}) \times (n - l_j - 1 - \sum_{k=1}^{p-1} m_k^{j}) \) and \((l_j + n_{p-1}^{j+1}) \times l_j - 1 \) matrices, respectively.

Here \( l_j, n_k^{j}, m_k^{j}, \) and \( \hat{l}_j \) are nonnegative integers, which are determined by
\[
\begin{align*}
l_j &= \text{rank}[E_j, B_j] - m, \quad n_{0}^{j+1} = n - m - l_j, \\
\hat{l}_j &= l_j - \sum_{k=1}^{p-1} n_{k}^{j+1}, \\
n_{k}^{j} &= kn + m + l_{j-1} - \sum_{i=1}^{k-1} n_{i}^{j}, \\
\text{rank} \begin{bmatrix}
E_{j-1} & -A_{j} \\
-\cdots & \ddots \cdots & B_{j-1} \\
-\cdots & \ddots & \ddots & B_{j+k-1} \\
A_{j+k-1} & E_{j+k-1} & \cdots & B_{j+k-1}
\end{bmatrix},
\end{align*}
\]
(3.6) \( k = 1, 2, \ldots, p - 1 \),
(3.7) \( m_{k}^{j} = n_{k-1}^{j+1} - n_{k}^{j}, \quad k = 1, 2, \ldots, p - 1 \).

Initialization Step 0:

For \( j = 1, \ldots, p \)
set \( Q_j = P_j = I_n \),
endfor \( j \):

For \( j = 1, \ldots, p \),
(1.1) find \( Q_j^0 \in \mathcal{O}(n) \) such that \( Q_j^0 B_j = [B_{11}^j 0] \) with \( B_{11}^j \in \mathcal{R}(m) \) nonsingular,
\( Q_j := Q_j^0 Q_j \),
(1.2) partition \( Q_j E_j \) as
\[
\begin{bmatrix}
E_{a}^j \\
E_{b}^j
\end{bmatrix} \}
m = Q_j E_j,
(I.3) find \( Q^j_b \in \mathcal{O}(n-m) \) and \( P^j_b \in \mathcal{O}(n) \) such that
\[
Q^j_b E^j_b P^j_b = \begin{bmatrix} 0 & 0 \\ 0 & \hat{E}^j_{11} \end{bmatrix} := \hat{E}^j_{00}
\]
with \( \hat{E}^j_{11} \in \mathcal{R}(l_j) \) nonsingular and \( l_j = \text{rank}[E_j, B_j] - m \), and update
\[
E^j_a := E^j_a P^j_b, \quad Q^j := \overline{Q^j} Q^j, \quad P^j := P^j P^j_b,
\]
end for \( j \);

For \( j = 1, \ldots, p \),

(I.5) partition \( A^j_0 \) as
\[
\begin{bmatrix} A^j_a \\ A^j_b \end{bmatrix} := A^j_0,
\]
and partition \( A^j_b \) as
\[
\begin{bmatrix} \hat{A}^j_{L(11)} \\ \hat{A}^j_{R(11)} \end{bmatrix} := A^j_b,
\]
end for \( j \).

Induction \hspace{1cm} \text{Step } k:

For \( k = 1, \ldots, p-1 \),

For \( j = 1, 2, \ldots, p \),

(K.1) if \( k \geq 2 \), then for \( i = 1, \ldots, k-1 \), partition \( \hat{A}^j_{L(k,i)} \) and \( \hat{A}^j_{R(k,i)} \), respectively, as
\[
\begin{bmatrix} A^j_{L(k,i)} \\ A^j_{L(k+1,i)} \end{bmatrix} n^j_{k-1}^{i+1} := \hat{A}^j_{L(k,i)}, \quad \begin{bmatrix} A^j_{R(k,i)} \\ A^j_{R(k+1,i)} \end{bmatrix} n^j_{k-1}^{i+1} := \hat{A}^j_{R(k,i)},
\]
end for \( i \); endif;

(K.2) partition \( \begin{bmatrix} \hat{A}^j_{L(k,k)} \\ \hat{A}^j_{R(k,k)} \end{bmatrix} \) as
\[
\begin{bmatrix} \phi^j_{k,1} \\ \phi^j_{k,2} \\ \phi^j_{k,3} \\ \phi^j_{k,4} \end{bmatrix} n^j_{k-1}^{i+1} := \begin{bmatrix} \hat{A}^j_{L(k,k)} \\ \hat{A}^j_{R(k,k)} \end{bmatrix},
\]
end for \( j \);

For \( j = 1, 2, \ldots, p \),
(K.3) find \( U_k^j \in \mathcal{O}(n_{k-1}^{j+1}) \) and \( V_k^{j-1} \in \mathcal{O}\left(n - l_{j-1} - \sum_{i=1}^{k-1} m_i\right) \) such that
\[
U_k^j \Phi_{k,1}^j V_k^{j-1} = \begin{bmatrix} A_{L(k,k)}^j & 0 \end{bmatrix},
\]
where \( A_{L(k,k)}^j = \begin{bmatrix} 0 & L_{kk}^j \end{bmatrix} \) with \( L_{kk}^j \in \mathcal{L}(m_k^j) \) nonsingular, update
\[
\Phi_{k,2}^j \leftarrow U_k^j \Phi_{k,2}^j, \quad \Phi_{k,3}^j \leftarrow \Phi_{k,3}^j V_k^{j-1}, \quad E_{k-1,k-1}^j := U_k^j E_{k-1,k-1}^j, \\
Q_j := \Phi_{k,4}^j V_j^{-1}, \quad P_{j-1} := P_{j-1} \Phi_{k,4}^j V_j^{-1},
\]
and if \( k \geq 2 \), then for \( i = 1, \ldots, k-1 \), update
\[
A_{L(k,i)}^j := U_k^j A_{L(k,i)}^j, \quad A_{R(k,i)}^j := U_k^j A_{R(k,i)}^j.
\end{for}
endfor \( i \); endfor

(K.4) partition \( \Phi_{k,2}^j \) as
\[
\begin{bmatrix} \Phi_{k,2a}^j \\ \Phi_{k,2b}^j \end{bmatrix}, \quad \text{with } m_k^j := \Phi_{k,2},
\]
(K.5) find an elementary transformation \( T_k^{j-1} \) to eliminate \( \Phi_{k,2b}^j \) by \( L_{kk}^j \) and update
\[
P_{j-1} := P_{j-1} T_k^{j-1} \quad \text{and} \quad \begin{bmatrix} \Phi_{k,3}^j & \Phi_{k,4}^j \end{bmatrix} := \begin{bmatrix} \Phi_{k,3}^j & \Phi_{k,4}^j \end{bmatrix} T_k^{j-1},
\]
(K.6) find \( V_k^{j-1} \in \mathcal{O}(l_{j-1} - \sum_{i=1}^{k-1} n_i^j) \) such that \( \Phi_{k,2a}^j V_k^{j-1} = \begin{bmatrix} R_{kk}^j \\ 0 \end{bmatrix} \) with
\[
R_{kk}^j \in \mathcal{R}(n_k^j) \text{ nonsingular, set } A_{R(k,k)}^j := \begin{bmatrix} R_{kk}^j \\ 0 \end{bmatrix}, \quad \text{and update } \Phi_{k,4}^j := \Phi_{k,4}^j V_k^{j-1}, P_{j-1} := P_{j-1} V_k^{j-1},
\]
(K.7) find \( Q_k^{j-1} \in \mathcal{O}(l_{j-1} - \sum_{i=1}^{k-1} n_i^j) \) such that \( \hat{E}_{kk}^{j-1} := Q_k^{j-1} (\hat{E}_{kk}^{j-1} V_k^{j-1}) \) is upper triangular, update
\[
\begin{bmatrix} \Phi_{k,3}^{j-1} & \Phi_{k,4}^{j-1} \end{bmatrix} := \begin{bmatrix} Q_k^{j-1} \Phi_{k,3}^{j-1} & \Phi_{k,4}^{j-1} \end{bmatrix},
\]
and if \( k \geq 2 \), then for \( i = 1, \ldots, k-1 \), update
\[
\hat{A}_{L(k+1,i)}^{j-1} := Q_k^{j-1} \hat{A}_{L(k+1,i)}^{j-1}, \quad \hat{A}_{R(k+1,i)}^{j-1} := Q_k^{j-1} \hat{A}_{R(k+1,i)}^{j-1},
\end{for}
endfor \( i \); endfor

(K.8) partition \( \Phi_{k,3}^j \) and \( \Phi_{k,4}^j \), respectively, as
\[
\begin{bmatrix} \hat{A}_{L(k,k)}^j \\ \hat{A}_{L(k+1,k+1)}^j \end{bmatrix} := \Phi_{k,3}^j, \quad \begin{bmatrix} \hat{A}_{R(k,k)}^j \\ \hat{A}_{R(k+1,k+1)}^j \end{bmatrix} := \Phi_{k,4}^j,
\]
endfor \( j \);
For \( j = 1, 2, \ldots, p \),
(K.9) partition \( \hat{E}_{k,k}^j \) as
\[
\begin{bmatrix} E_{k,k}^j & \hat{E}_{k,k+1}^j \\ 0 & \hat{E}_{k+1,k+1}^j \end{bmatrix} n_k^{j+1} := \hat{E}_{k,k}^j,
(K.10) find an elementary transformation $S_{k+1}$ to eliminate $\hat{E}_{k+1,k+1}$ by $\hat{E}_{k+1,k+1}$, update $Q_j := S_{k+1}^jQ_j$, and for $i = 1, \ldots, k+1$, update

$$\hat{A}_{L(k+1,i)} := S_{k+1}^j \hat{A}_{L(k+1,i)}^j, \quad \hat{A}_{R(k+1,i)} := S_{k+1}^j \hat{A}_{R(k+1,i)}^j,$$

endfor $i$
endfor $j$.

For $j = 1, \ldots, p$,

set $E_{j,p} = \hat{E}_{p,p}$
for $i = 1, \ldots, p$,

set $A_{L(p,i)} := \hat{A}_{L(p,i)}^j$, \quad $A_{R(p,i)} := \hat{A}_{R(p,i)}^j$
endfor $i$
endfor $j$.

In Figure 3.1 we illustrate the canonical forms of $E_{j-1}$, $A_j$, and $E_j$ computed by Algorithm 3.1 for the case of $p = 4$:

$$E_{j-1} \downarrow$$

$$\begin{bmatrix}
E_{11}^{j-1} & E_{22}^{j-1} & E_{33}^{j-1} & E_{44}^{j-1} \\
E_{11}^{j} & E_{22}^{j} & E_{33}^{j} & E_{44}^{j} \\
n_1^j & n_2^j & n_3^j & l_{j-1}
\end{bmatrix}$$

The row number of $A_{R(44)}^j$ is $l_j + n_{j+1} + l_{j-1} - \sum_{k=2}^{j-1}(m_k^j + n_k^j)$

$$\begin{bmatrix}
0 & R_{11}^j & n_1^j & 0 \\
L_{11}^j & m_1^j & n_1^j & 0 \\
* & 0 & R_{22}^j & n_2^j \\
* & L_{22}^j & m_2^j & n_2^j \\
* & * & 0 & R_{33}^j \\
* & * & L_{33}^j & m_3^j \\
* & * & * & A_{L(44)}^j \\
* & * & * & A_{R(44)}^j
\end{bmatrix}$$

$$\begin{bmatrix}
\downarrow & \uparrow
\end{bmatrix}$$

$$A_j \uparrow$$

$$\begin{bmatrix}
E_{11}^j & E_{12}^j & E_{13}^j & E_{14}^j \\
E_{21}^j & E_{22}^j & E_{23}^j & E_{24}^j \\
E_{31}^j & E_{32}^j & E_{33}^j & E_{34}^j \\
E_{41}^j & E_{42}^j & E_{43}^j & E_{44}^j
\end{bmatrix}$$

$$\begin{bmatrix}
l_{j-1} & l_j & l_{j-1} & l_j \\
E_j & E_j & E_j & E_j
\end{bmatrix}$$

FIG. 3.1. The canonical forms for $p = 4.$

**Theorem 3.1.** If the periodic matrix triples $\{(E_j, A_j, B_j)\}_{j=1}^p$ satisfy $\text{rank}(B_j) = m$ and conditions (3.1), then the properties (i)--(vi) of outputs computed by Algorithm 3.1 hold and are completely determined by the relations given by (3.4)--(3.7).
Remark. Note that (3.4)–(3.7) show that the sizes of submatrices $E_{kk}^j$, $A_{L(kk)}^j$, and $A_{R(kk)}^j$ ($k = 1, \ldots, p$) computed by Algorithm 3.1 are uniquely determined by the original periodic matrix triples $\{(E_j, A_j, B_j)\}_{j=1}^p$.

Proof of Theorem 3.1. I. The numbers $l_j = \text{rank}[E_j, B_j] - m$ and $n_j+1 = n - m - l_j$ in (3.4) are obtained by (I.3) and (I.6) of Algorithm 3.1 immediately. The number $\hat{l}_j$ in (3.5) is just the size of $E_{pp}^j$.

The proof of (3.6). Algorithm 3.1 computes $Q_j$, $P_j$, $F_j$ and $G_j$, $j = 1, \ldots, p$ such that

\[(3.6)\]

\[E_j^1 := Q_j^1(E_j + B_j G_j) P_j = \begin{bmatrix}
0_{m \times (n - l_j)} & 0_{m \times l_j} \\
0_{n_j+1 \times l_j} & E_b^j
\end{bmatrix}, \quad Q_j B_j = \begin{bmatrix}
B_{11}^j \\
0 \\
\vdots \\
0
\end{bmatrix} := B_j^1,
\]

\[(3.9)\]

\[A_j^1 := Q_j(A_j + B_j F_j) P_{j-1} = \begin{bmatrix}
0_{m \times (n - l_j)} & 0_{m \times l_j} \\
A_L^j & A_R^j
\end{bmatrix},
\]

where $E_b^j := \text{diag}(E_{11}^j, \ldots, E_{p-1,p-1}^j, E_{pp}^j)$,

\[A_L^j := \begin{bmatrix}
A_{L(11)}^j & 0 \\
\vdots & \ddots \\
A_{L(p1)}^j & \cdots & A_{L(pp)}^j
\end{bmatrix}, \quad A_R^j := \begin{bmatrix}
A_{R(11)}^j & 0 \\
\vdots & \ddots \\
A_{R(p1)}^j & \cdots & A_{R(pp)}^j
\end{bmatrix}
\]
as in (3.2) and (3.3), respectively. From (I.3), (K.7), (K.9), and (K.10) it follows that $E_{kk}^j \in \mathcal{M}(n_k^{j+1})$ nonsingular, $k = 1, \ldots, p$, and $E_{pp}^j$ is upper triangular. By (K.3) and (K.6), respectively, we have that $A_{L(k,k)}^j$ has the form $A_{L(k,k)}^j = \begin{bmatrix} 0 \\ L_{kk}^j \end{bmatrix}$ with $L_{kk}^j \in \mathcal{L}(m_k^j)$ nonsingular, and $A_{R(k,k)}^j = \begin{bmatrix} R_{kk}^j \end{bmatrix}$ with $R_{kk}^j \in \mathcal{R}(n_k^j)$ nonsingular for $k = 1, \ldots, p - 1$. (We will prove that $R_{kk}^j \in \mathcal{R}(n_k^j)$ is nonsingular later!)

Consider the matrix

\[(3.10)\]

\[C_{j,\ell}^1 = \begin{bmatrix}
E_{j-1}^1 & -A_j^1 & E_j^1 & \cdots & -A_j^{j-2} & E_{j-2}^j & B_{j-1}^1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
E_{j+\ell-2}^1 & -A_j^{j+\ell-2} & E_{j+\ell-2}^j & \cdots & -A_j^{j+\ell-2} & E_{j+\ell-2}^j & B_{j+\ell-2}^1
\end{bmatrix}
\]

for $j$ fixed and $\ell = 2, \ldots, p$.

Noting the special structures of $E_{j+i}^1$, $A_{j+i}^1$, and $B_{j+i}^1$, for each given $\ell$ we can use $L_{kk}^{j+i}$ and $R_{kk}^j$ ($i = \ell - 2, \ldots, 0$, $k = 1, 2, \ldots, \ell - i - 1$) as pivots to eliminate the nonzero blocks in the same column of $A_{L(k,k)}^{j+i}$ and $A_{R(k,k)}^{j+i}$, respectively, of the $C_{j,\ell}^1$ in (3.10) by row transformations, and finally we get the following forms, for $\ell = 2, \ldots, p$,

\[(3.11)\]

\[C_{j,\ell} = \begin{bmatrix}
E_{j-1}^\ell & -A_j^\ell & E_j^\ell & \cdots & -A_j^{j-2} & E_{j-2}^\ell & B_{j-1}^1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
E_{j+\ell-2}^\ell & -A_j^{j+\ell-2} & E_{j+\ell-2}^\ell & \cdots & -A_j^{j+\ell-2} & E_{j+\ell-2}^\ell & B_{j+\ell-2}^1
\end{bmatrix},
\]
where
\[
E_{j+i}^{\ell} = \begin{bmatrix}
0_{m \times (n-l_{j+1})} & 0_{m \times l_{j+1}} \\
0_{n_{j+1} \times l_{j+1}} & E_{b}^{j+i}
\end{bmatrix}, \quad A_{j+i}^{\ell} = \begin{bmatrix}
0_{m \times (n-l_{j+i-1})} & 0_{m \times l_{j+i-1}} \\
A_{L}^{j+i} & A_{R}^{j+i}
\end{bmatrix},
\]
in which
\[
E_{b}^{j+i} := \text{diag}\{0, \ldots, 0, E_{j+i}^{j+i}, \ldots, E_{j+i}^{j+i}\}
\]
for \(i = -1, 0, 1, \ldots, \ell - 2\) and
\[
A_{L}^{j+i} := \begin{bmatrix}
A_{L}^{j+i} \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & A_{L}^{j+i}
\end{bmatrix},
\]
\[
A_{R}^{j+i} := \begin{bmatrix}
A_{R}^{j+i} \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & A_{R}^{j+i}
\end{bmatrix}
\]
for \(i = 0, 1, \ldots, \ell - 2\).

From the rank conditions (3.1) and (3.12)–(3.15) it is easy to derive the rank of the matrix \(C_{j+\ell}^{\ell} \) in (3.11), and therefore the rank of \(C_{j+\ell}^{\ell} \) in (3.10) is equal to
\[
(\ell - 1)n + m + l_{j-1} - \sum_{i=1}^{\ell-1} n_{i}.
\]
By Lemma 2.6 it follows from (3.8), (3.9), and (3.16) that
\[
n_{j-1}^{\ell} = (\ell - 1)n + m + l_{j-1} - \sum_{i=1}^{\ell-2} n_{i}.
\]

By Lemma 2.6 it follows from (3.8), (3.9), and (3.16) that
If we let \( \ell - 1 = k \) in (3.17) for \( k = 1, \ldots, p - 1 \), then (3.17) shows that (3.6) holds.

The proof of (3.7). From the rank conditions (3.1) we have that the submatrices \([-A_j, E_j, B_j]\) of \( C_j \) in (3.11) with \( l = p \) are of full row rank, i.e., \( \text{rank}[-A_j, E_j, B_j] = n \). From the special structures of \( E_j, A_j, B_j \) it follows that \( R_{kk}^j \in \mathcal{R}(n_{kk}^j) \) nonsingular, for \( k = 1, \ldots, p - 1 \) and \( j = 1, \ldots, p \).

II. The properties (i)–(v) of the outputs of Algorithm 3.1 hold by (3.4), (3.5), and (3.7) immediately. To prove (vi), we denote by \( r(A) = m \) and \( c(A) = n \) the row and column numbers of \( A \in \mathcal{M}(m, n) \), respectively. From the properties (ii), (iv), and (v) it follows that

\[
\begin{align*}
    r(A_{L(pp)}) &= r(A_{R(pp)}) = l_j - \sum_{k=2}^{p-1} (m_{k} + n_k^j) \\
    &= l_j - \sum_{k=2}^{p-1} n_{k-1}^{j+1} \quad \text{(by (3.7))} \\
    &= n_{p-1}^{j+1} + \left( l_j - \sum_{k=1}^{p-1} n_k^j \right) = n_{p-1}^{j+1} + \hat{l}_j \quad \text{(by (3.5))} \\
    &= r \left( \begin{bmatrix} E_{p-1-1}^j & 0 \\ E_{pp}^j \end{bmatrix} \right).
\end{align*}
\]

Similarly, from the properties (ii), (iv), and (v) we also have

\[
\begin{align*}
    c(A_{L(pp)}) &= n - l_{j-1} - \sum_{k=1}^{p-1} m_k^j, \quad c(A_{R(pp)}) = \hat{l}_{j-1} = l_{j-1} - \sum_{k=1}^{p-1} n_k^j. \quad \Box
\end{align*}
\]

4. Regularization of \( \{ (E_j, A_j, B_j) \}_{j=1}^p \). In this section we will use the canonical forms of \( \{ (E_j, A_j, B_j) \}_{j=1}^p \) computed by Algorithm 3.1 to construct derivative and proportional feedback controls so that the closed-loop systems are regular and of index at most one.

By Theorem 3.1 there are \( Q_j, P_j, G_j, \) and \( F_j \) such that

\[
E_j^1 := Q_j(E_j + B_jG_j)P_j, \quad A_j^1 := Q_j(E_j + B_jF_j)P_j, \quad B_j^1 := Q_jB_j
\]

have the canonical forms as in (3.2) and (3.3). For convenience, by Lemma 2.7, hereafter, we suppose w.l.o.g. that \( \{ (E_j, A_j, B_j) \}_{j=1}^p \) are of the canonical forms as in (3.2) and (3.3), and moreover, we assume that \( \{ (E_j, A_j, B_j) \}_{j=1}^p \) satisfy (C2) in (2.24).

Since the sizes of submatrices of (3.2) and (3.3) play an important role in the regularization of the periodic matrix triples, in the following lemma we will prove under condition (C2) that \( c(A_{L(pp)}) \geq r(E_{p-1-1}^j), \ j = 1, \ldots, p \).

LEMMA 4.1. Suppose the periodic matrix triples \( \{ (E_j, A_j, B_j) \}_{j=1}^p \) satisfy (C2). Then it holds that

\[
n_{p-1}^{j+1} \leq n - l_{j-1} - \sum_{i=1}^{j-1} m_i^j, \quad j = 1, \ldots, p,
\]

i.e., \( r(E_{p-1-1}^j) \leq c(A_{L(pp)}) \) (see also Figure 3.1). Moreover, let

\[
\delta_j := n - l_j - \sum_{i=1}^{p-1} m_i^{j+1} - n_{p-1}^{j+2} \geq 0,
\]

where \( \delta_j \) is called the degree of invariance of the\( j \)th period.
where $\delta_0 = \delta_p$. Then it holds that

\begin{equation}
\sum_{j=1}^{p} \delta_j = pm.
\end{equation}

**Proof.** Partition $S_\infty(\tilde{E}_j)$ into

\begin{equation}
S_\infty(\tilde{E}_j) = \begin{bmatrix} S_j^T & S_{j+1}^T & \ldots & S_{j+p-1}^T \end{bmatrix}^T
\end{equation}

and rewrite the equation $\tilde{E}_j S_\infty(\tilde{E}_j) = 0$ in the forms

\begin{equation}
\begin{aligned}
E_j S_j &= 0, \\
E_{j+\ell} S_{j+\ell} &= A_{j+\ell} S_{j+\ell-1}, \quad \ell = 1, \ldots, p-1.
\end{aligned}
\end{equation}

Partitioning $S_{j+\ell-1}$ compatibly with $A_{j+\ell}$ by

\begin{equation}
S_{j+\ell-1} = \begin{bmatrix} (S_{L(1)}^{j+\ell-1})^T, \ldots, (S_{L(p)}^{j+\ell-1})^T, (S_{R(1)}^{j+\ell-1})^T, \ldots, (S_{R(p)}^{j+\ell-1})^T \end{bmatrix}^T
\end{equation}

and comparing both sides of $E_{j+p-1} S_{j+p-1} = A_{j+p-1} S_{j+p-2}$ in (4.5) with $\ell = p-1$ we have

\begin{equation}
S_{L(1)}^{j+p-2} = 0, \quad S_{R(1)}^{j+p-2} = 0,
\end{equation}

\begin{equation}
E_{11}^{j+p-1} S_{R(1)}^{j+p-1} = A_{L(22)}^{j+p-1} S_{L(2)}^{j+p-2} + A_{R(22)}^{j+p-1} S_{R(2)}^{j+p-2}.
\end{equation}

Using (4.7) and comparing both sides of $E_{j+p-2} S_{j+p-2} = A_{j+p-2} S_{j+p-3}$ of (4.5) with $\ell = p-2$ we get

\begin{equation}
S_{L(1)}^{j+p-3} = 0, \quad S_{R(1)}^{j+p-3} = 0,
\end{equation}

\begin{equation}
S_{L(2)}^{j+p-3} = 0, \quad S_{R(2)}^{j+p-3} = 0,
\end{equation}

\begin{equation}
E_{22}^{j+p-2} S_{R(2)}^{j+p-2} = A_{L(33)}^{j+p-2} S_{L(3)}^{j+p-3} + A_{R(33)}^{j+p-3} S_{R(3)}^{j+p-3}.
\end{equation}

In such a way, in general, we have for each $\ell = 2, \ldots, p-1$, that

\begin{equation}
S_{L(i)}^{j+p-\ell} = 0, \quad S_{R(i)}^{j+p-\ell} = 0, \quad i = 1, \ldots, \ell - 1,
\end{equation}

\begin{equation}
E_{\ell-1,\ell-1}^{j+p-\ell+1} S_{R(\ell-1)}^{j+p-\ell+1} = A_{L(\ell\ell)}^{j+p-\ell+1} S_{L(\ell)}^{j+p-\ell} + A_{R(\ell\ell)}^{j+p-\ell+1} S_{R(\ell)}^{j+p-\ell}.
\end{equation}

Finally, using (4.9) and comparing both sides of $E_j S_j = 0$ and $E_{j+1} S_{j+1} = A_{j+1} S_j$ in (4.5) with $\ell = 1$ we get

\begin{equation}
S_{L(i)}^{j} = S_{R(i)}^{j} = 0, \quad i = 1, \ldots, p-1, \quad S_{R(p)}^{j} = 0,
\end{equation}

\begin{equation}
E_{p-1,p-1}^{j+1} S_{R(p-1)}^{j+1} = A_{L(pp),a}^{j+1} S_{L(p)},
\end{equation}

where $A_{L(pp)}^{j+1}$ is partitioned into

\begin{equation}
A_{L(pp)}^{j+1} = \begin{bmatrix} A_{L(pp),a}^{j+1} \\ A_{L(pp),b}^{j+1} \end{bmatrix} \tilde{t}_{j+1}^{j+2}.
\end{equation}
On the other hand, it follows from (C2) and (4.4) that
\[
\text{rank}[E_j, A_j S_{j+p-1}, B_j] = n.
\]

Note that the matrices \(E_j, A_j,\) and \(B_j\) are assumed to have the special structures as shown in (3.2) and (3.3). Consequently, from (4.14) we have
\[
\text{rank}(R_{11}^j S_{R(1)}^{j+p-1}) = n_1^j.
\]

This, together with \(R_{11}^j\) nonsingular, shows that \(S_{R(1)}^{j+p-1}\) has full row rank.

Now we rewrite (4.8) as
\[
E_{11}^{j+p-1} S_{R(1)}^{j+p-1} = \begin{bmatrix}
R_{22}^{j+p-1} S_{R(2)}^{j+p-2} \\
L_{22}^{j+p-1} S_{L(2)}^{j+p-2}
\end{bmatrix},
\]
which implies that \(S_{R(2)}^{j+p-2}\) must have full row rank because \(E_{11}^{j+p-1}\) and \(R_{22}^{j+p-1}\) are nonsingular and \(S_{R(1)}^{j+p-1}\) is of full row rank. Continuing this process, by (4.10) we can derive step by step that \(S_{j+p-\ell}^{j+p-1}\) must have full row rank for \(\ell = 2, \ldots, p-1\). Finally, it follows from (4.12) and \(S_{R(p-1)}^{j+p-1}\) of full row rank that \(A_{L(p,\ell)}^{j+p} S_{R(\ell)}^{j+p-1}\) in (4.13) must have full row rank, and hence it must have
\[
n_1^{j+p-2} \leq n - l_j - \sum_{i=1}^{p-1} m_i^{j+1}.
\]

Therefore, (4.1) holds, for \(j = 1, \ldots, p\). Using the equality \(n_{k-1}^{j+1} = n_k^j + m_k^j\) in (3.7) it is easy to verify that
\[
\sum_{j=1}^{p} \left( \sum_{i=1}^{p-1} m_i^{j+1} + n_{p-1}^{j+2} \right) = \sum_{j=1}^{p} n_0^j.
\]

This, together with \(n_0^j = n - m - l_{j-1}\) in (3.4), implies
\[
\sum_{i=1}^{p} \delta_j = pm.
\]

Lemma 4.1 shows that the integers \(\{\delta_j\}_{j=1}^{p}\) in (4.2) are nonnegative and satisfy (4.3). We now use \(\{\delta_j\}_{j=1}^{p}\) starting with a nonnegative integer \(r_1\), recursively, to construct a sequence \(\{r_j, s_j\}_{j=1}^{p}\) by
\[
\begin{align*}
s_{j+1} & = \delta_j - r_j, \quad j = 1, \ldots, p, \\
r_{j+1} & = m - s_{j+1}, \quad j = 1, \ldots, p-1,
\end{align*}
\]
where \(s_1 = s_{p+1}\). Under certain condition of \(r_1\), we will show that the integers \(\{r_j, s_j\}_{j=1}^{p}\) defined by (4.16) are all nonnegative, which can determine the number of finite eigenvalues of periodic regularizing closed-loop systems. Let
\[
L := \min_{1 \leq j \leq p} \left( \sum_{\ell=1}^{j-1} \delta_{\ell} - (j-1)m \right), \quad U := \max_{1 \leq j \leq p} \left( \sum_{\ell=1}^{j-1} \delta_{\ell} - (j-1)m \right),
\]
where \(\delta_{\ell}\) are given by (4.2). Then the following lemma holds.
Lemma 4.2. If $L$ and $U$ defined by (4.17) satisfy
\begin{equation}
U \leq L + m,
\end{equation}
and there is a nonnegative integer $r_1$ such that $U \leq r_1 \leq L + m$, then the integers of sequence $\{r_j, s_j\}_{j=1}^p$ defined by (4.16) are all nonnegative and satisfy $0 \leq r_j, s_j \leq m$ for $j = 1, \ldots, p$.

Proof. Since the nonnegative integer $r_1$ satisfies $U \leq r_1 \leq L + m$, from (4.17) and (4.18) we have
\begin{equation}
\sum_{\ell=1}^{j-1} \delta_\ell - (j - 1)m \leq r_1 \leq \sum_{\ell=1}^{j-1} \delta_\ell - (j - 2)m
\end{equation}
for $j = 1, \ldots, p$. From (4.19) we recursively get
\begin{equation}
m \geq r_1 \geq 0, \quad s_{j+1} := \delta_j - r_j \geq 0, \quad r_{j+1} := m - s_{j+1} \geq 0,
\end{equation}
for $j = 1, \ldots, p - 1$. Furthermore, from (4.16) and (4.3) of Lemma 4.1 we have
\begin{equation}
s_1 := s_{p+1} = \delta_p - r_p = pm - \sum_{\ell=1}^{p-1} \delta_\ell - r_p = m - r_1 \geq 0.
\end{equation}
This, together with (4.20), gives rise to
\begin{equation}
0 \leq r_j, s_j \leq m, \quad j = 1, \ldots, p.
\end{equation}

Since the submatrix $A_{L(p),a}^{j+1}$ of $A_{L(p)}^{j+1}$ in (4.13) is of full row rank, there is an orthogonal matrix $P_p^j$ such that
\begin{equation}
A_{L(p)}^{j+1} P_p^j = \begin{bmatrix} \Delta_{p,1}^{j+1} & 0 \\ \Delta_{p,2}^{j+1} & \Delta_{p,3}^{j+1} \end{bmatrix},
\end{equation}
where $\Delta_{p,1}^{j+1} \in \mathcal{L}(n_{p-1}^{j+1})$ nonsingular. The Lemma 4.2 ensures that the integers $s_j$ and $r_j$, $j = 1, \ldots, p$, defined by (4.16) are all nonnegative and satisfy $0 \leq r_j, s_j \leq m$ provided $r_1 \in [U, L + m]$. Now, for any nonnegative integer $r_1$ with $U \leq r_1 \leq L + m$ we define
\begin{equation}
G_j = (B_{11}^j)^{-1} \begin{bmatrix} 0 & K_j & 0 \\ K_j & l_j & \end{bmatrix}, \quad F_j = (B_{11}^j)^{-1} \begin{bmatrix} 0 & H_j & 0 \\ H_j & \delta_{j-1} & \end{bmatrix} P_{j-1},
\end{equation}
where
\begin{align*}
K_j & = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_j} \end{bmatrix} \in \mathcal{M}(m, \delta_j), \\
H_j & = \begin{bmatrix} I_{s_j} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}(m, \delta_{j-1}), \\
P_j & = \text{diag}\left\{ I_{m^{j+1}+\cdots+m_{j+1}}, P_p^{j}, I_{l_j} \right\}, \quad d_j = n - \delta_j - l_j.
\end{align*}
Then we have
\begin{equation}
E_j^1 := (E_j + B_j G_j) P_j = \begin{bmatrix} 0_{m \times (n - l_j - \delta_j)} & K_j & 0_{m \times l_j} \\ 0_{n_{j+1} \times l_j} & E_{11}^j \\ \vdots & \ddots & E_{pp}^j \end{bmatrix},
\end{equation}
\( A_j^1 := (A_j + B_j F_j) P_{j-1} \)

\[(4.25)\]

\[
\begin{bmatrix}
A_j^1_{L,(11)} & \cdots & A_j^1_{L(p-1,1)} \\
\vdots & \ddots & \vdots \\
A_j^1_{L,(p,1)} & \cdots & A_j^1_{L(p,p-1)}
\end{bmatrix}
\begin{bmatrix}
\Delta_j^2_{p,1} \\
\Delta_j^2_{p,2} \\
\Delta_j^2_{p,3}
\end{bmatrix}
\begin{bmatrix}
H_j \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
A_j^1_{R,(11)} \\
\vdots \\
A_j^1_{R,(p,1)} & \cdots & A_j^1_{R(pp)}
\end{bmatrix}
\end{array}
\]

We now prove our main theorem.

Theorem 4.3. Suppose the periodic matrix triples \( \{ (E_j, A_j, B_j) \}_{j=1}^p \) satisfy (C2) and \( U \leq L + m \), where \( U \) and \( L \) are given by (4.17). Then for any nonnegative integer \( r_1 \) with \( 0 \leq r_1 \leq L + m \), the feedback matrices \( G_j, F_j \) constructed by (4.23) make the periodic closed-loop systems \( \{ (E_j + B_j G_j, A_j + B_j F_j) \}_{j=1}^p \) regular and have the index at most one. Moreover, the number of finite eigenvalues of \( \{ (E_j + B_j G_j, A_j + B_j F_j) \}_{j=1}^p \) is equal to

\[ f := r_j + \hat{l}_j = r_j + l_j - \sum_{k=1}^{p-1} n_k^{j+1} \]

for any \( j \in \{1, \ldots, p\} \).

Proof. Let

\[(4.26) \quad E_j^0 := E_j + B_j G_j, \quad A_j^0 := A_j + B_j F_j, \quad j = 1, \ldots, p, \]

and let

\[(4.27) \quad \tilde{E}_j^0 := \tilde{E}(E_j^0, \ldots, E_{j+p-1}^0, A_j^0, \ldots, A_{j+p-1}^0), \tilde{A}_j^0 := \tilde{A}(A_j^0), \]

\[
\tilde{E}_j^1 := \tilde{E}(E_j^1, \ldots, E_{j+p-1}^1, A_j^1, \ldots, A_{j+p-1}^1), \tilde{A}_j^1 := \tilde{A}(A_j^1)
\]

for \( j = 1, \ldots, p \). Since

\[
\text{rank}[\tilde{E}_j^0 P_j, (\tilde{A}_j^0 P_j) P_j^T S_\infty(\tilde{E}_j^0)] = \text{rank}[\tilde{E}_j^1, \tilde{A}_j^1 S_\infty(\tilde{E}_j^1)],
\]

where \( P_j = \text{diag}\{P_1, \ldots, P_{j+p-1}\} \), by Theorem 2.5, in order to prove Theorem 4.3, it is sufficient to prove that

\[(4.28) \quad \text{rank}[\tilde{E}_j^1, \tilde{A}_j^1 S_\infty(\tilde{E}_j^1)] = pn, \quad j = 1, \ldots, p. \]

For simplicity, here we only prove (4.28) for \( j = 1 \), but the others can be shown in a similar way. For \( j = 1 \), we first construct a basis for the null space \( N(\tilde{E}_1^1) \) of the forms

\[(4.29) \quad S_\infty(\tilde{E}_1^1) \equiv S_1 = [S_1^T, S_2^T, \ldots, S_p^T]^T \]

with

\[(4.30) \quad S_k = [0_{n \times (n-l_k-r_p)}, S_{k1}, \ldots, S_{kk}, 0, \ldots, 0], \quad k = 1, \ldots, p-1, \]

and

\[(4.31) \quad S_p = \begin{bmatrix} I_{n-I_p-r_p} & 0 & \cdots & 0 \\ 0 & S_{p1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{p,p-1} \end{bmatrix},\]

where

\[(4.32) \quad S_{kk} = \begin{bmatrix} 0 & 0 & m_{k+1}^1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & m_{p-1}^k + n_k - \sum_{i=1}^{p-1} m_i^{k+1} + n_k^{k+1} \\
S_{L(p)}^k & 0 & n_k - l_k - \sum_{i=1}^{p-1} m_i^{k+1} \\ 0 & \vdots & \vdots \\ 0 & 0 & m_{p-1}^k + n_k - l_k - \sum_{i=1}^{p-1} n_i^{k+1} \\
n_{p+2}^k & S_{L(p)}^k & 0 \end{bmatrix}, \quad S_{L(p)}^k = \begin{bmatrix} I_{n_{p-1}^k} \\ 0 \end{bmatrix}, \quad g_1^k = n_k^1 - n_{k+2}^1,
\]

\[(4.33) \quad S_{k+i, k+i} = \begin{bmatrix} 0 & 0 & 0 & m_{k+i+1}^1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & m_{p-1}^k + n_k - \sum_{i=1}^{p-1} m_i^{k+i+1} + n_k^{k+i+1} \\
S_{L(p-i)}^{k+i} & 0 & 0 & n_k - l_{k+i} - \sum_{i=1}^{p-1} n_i^{k+i+1} \\
0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & n_{k+i+1}^{k+i+1} \\
S_{R(p-i)}^{k+i} & 0 & 0 & 0 \\
\ast & 0 & 0 & 0 \\
n_{p-i}^{k+i+1} & n_{p-i}^{k+i+1} & n_{p-i}^{k+i+1} & g_{i+1}^k \end{bmatrix}, \quad S_{L(p-i)}^{k+i} = I_{n_{p-1}^k}, \quad g_{i+1}^k = n_k^1 - n_{k+i+2}^1, \quad k = 1, 2, \ldots, p-1, \\
i = 1, 2, \ldots, p-k-1,
\]

and

\[(4.34) \quad S_{p-p-i} = [0, \ldots, 0, 0, \ldots, 0, (S_{R(p-i)}^p)^T, * , \ldots, *]^T, \quad i = 1, \ldots, p-1,
\]

in which the submatrices $S_{R(p-i)}^{k+i}$, $k = 1, \ldots, p-1$, $i = 1, \ldots, p-k-1$, in (4.33) and $S_{R(p-i)}^i$, $i = 1, \ldots, p-1$, in (4.34) are to be determined.
From (4.29), (4.24), and (4.25) we have, for \( k = 1, \ldots, p - 1, \)

\[
E_{k+1}^1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ S^{k+1}_{R(p-1)} \\ * \end{bmatrix} = A_{k+1}^1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ S^k_{L(p)} \\ * \\ 0 \end{bmatrix},
\]

(4.35)

especially

\[
E_{p-1, p-1}^{k+1} S^{k+1}_{R(p-1)} = \Delta_{p, 1}^{k+1};
\]

thus, \( S^{k+1}_{R(p-1)} \in \mathcal{M}(n_{k+1}^p) \) and “\(*\)” below it are uniquely determined with \( S^{k+1}_{R(p-1)} \) nonsingular, since \( E_{p-1, p-1}^{k+1}, E_{p, p}^k \) and \( \Delta_{p, 1}^{k+1} \) are nonsingular. Especially, taking \( k = p - 1 \) in (4.35) we have that the matrix \( S_{p, p-1} \) in (4.34) is uniquely determined with \( S_{R(p-1)}^p \) nonsingular. Again, from (4.29), (4.24), and (4.25) we have, for \( k = 1, \ldots, p - 2, \)

\[
E_{k+2}^1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ S^{k+2}_{R(p-2)} \\ * \\ * \end{bmatrix} = A_{k+2}^1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ S^k_{L(p)} \\ 0 \\ 0 \end{bmatrix},
\]

(4.36)

especially

\[
E_{p-2, p-2}^{k+2} S^{k+2}_{R(p-2)} = \begin{bmatrix} R_{p-1, p-2}^{k+1} S^{k+1}_{R(p-1)} & 0 \\ 0 & L_{p-1, p-1}^{k+2} S^k_{L(p-1)} \end{bmatrix}.
\]

This, together with \( R_{p-1, p-1}^{k+1}, L_{p-1, p-1}^{k+1}, \) and \( S^k_{L(p-1)} = I_{m_{p-1}^{k+2}} \) all nonsingular, implies that the matrix \( S^{k+2}_{R(p-2)} \in \mathcal{M}(n_{p-2}^p) \) and all “\(*\)” below it are uniquely determined with \( S^{k+2}_{R(p-2)} \) nonsingular. Especially, taking \( k = p - 2 \) in (4.36) we have the matrix \( S_{p, p-2} \) in (4.34) is also uniquely determined with \( S_{R(p-2)}^p \) nonsingular.
In general, for \( i = 2, \ldots, p - 1, \) \( k = 1, \ldots, p - i, \) comparing the both sides of

\[
E^1_{k+i} \begin{bmatrix}
0 \\
\vdots \\
0 \\
S^k_{R(p-i)} \\
\ast \\
\vdots \\
\ast 
\end{bmatrix} = A^1_{k+i} \begin{bmatrix}
0 \\
\vdots \\
0 \\
S^k_{R(p-i-1)} \\
\ast \\
\vdots \\
\ast 
\end{bmatrix}
\]

(4.37)

step by step, the matrices \( S^k_{R(p-i)} \in \mathcal{M}(n^{k+i+1}) \) and all \("\ast\"\) below it are uniquely determined with \( S^k_{R(p-i)} \) nonsingular. Furthermore, the matrix \( S_{p,p-i} \) in (4.34) is then uniquely determined with \( S^p_{R(p-i)} \) nonsingular.

From (4.32) and (4.35)–(4.37) we have, for each \( k = 1, \ldots, p - 1, \) that

\[
E^1_k S_{kk} = 0 \quad \text{and} \quad E^1_{k+i} S_{k+i,k} = A^1_{k+i} S_{k+i-1,k}, \quad i = 1, \ldots, p - k,
\]

and therefore

(4.38)

\[
E^1 S_1 = 0.
\]

Moreover, a short calculation gives rise to

(4.39)

(4.40)

where \( \hat{A}_R^{(k,k)} = A_R^{(k,k)} S_{R(k)}^p = \begin{bmatrix} R^{1}_{k,k} S_{R(k)}^p \\ 0 \end{bmatrix}, k = 1, \ldots, p - 1, \) with \( R^{1}_{k,k} S_{R(k)}^p \) nonsingular. From (4.40), and the special structures of \( E^1_k \) and \( A^1_k, k = 1, \ldots, p, \) as in (4.24) and (4.25), similarly to the proof of (3.16) we can derive that

\[
\text{rank}[\tilde{E}^1, \hat{A}^1 S_1] = pn.
\]
This, together with $S_1 = S_\infty(\hat{E}_j^1)$, implies that (4.28) holds for $j = 1$. For $j = 2, \ldots, p$, (4.28) can also be proved in a similar way. Furthermore, by Corollary 2.4 the periodic matrix pairs $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ have $\gamma - (p-1)n$ finite eigenvalues, where

$$\gamma \equiv \gamma_j = \text{rank}(\hat{E}_j^1)$$

for any $j \in \{1, \ldots, p\}$. From (3.14), (3.15), (4.24), (4.25), and (4.16) one can easily see that $\gamma = r_j + \hat{l}_j + (p-1)n$; therefore, $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ have $r_j + \hat{l}_j = r_j + \hat{l}_j - \sum_{k=1}^{p-1} n_k^{i+1}$ finite eigenvalues for any $j \in \{1, \ldots, p\}$.

Remark. (i) From Theorem 4.3 we see that the number of finite eigenvalues of $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ is equal to $\hat{l}_1 + r_1$, where $r_1$ is any given nonnegative integer with $U \leq r_1 \leq L + m$. By Lemma 4.2 the integers $\{r_j, s_j\}_{j=1}^p$ defined by (4.16) are all nonnegative and satisfy $0 \leq r_j, s_j \leq m$ for $j = 1, \ldots, p$. If $r_1 = U$ and there is some $s_k = 0$, then at the time $k$ we can only use the proportional feedback control $u_k = F_kx_k$ to regularize the periodic systems. If $r_1 = L + m$ and there are some $s_k = 0$, then at the time $k$ we can only use the derivative feedback control $u_k = G_kx_k + 1$ to regularize the periodic systems.

(ii) For the case of $p = 1$, Theorem 4.3 can be simplified to the result that for any given integer $r_1$ with $0 \leq r_1 \leq m$, i.e., $\hat{l}_1 \leq r_1 + \hat{l}_1 \leq m + \hat{l}_1 = \text{rank}(\{E, B\})$ (here $\hat{l}_1 = \ell_1$), there exist matrices $F, G \in M(m, n)$ such that $(E + BG, A + BF)$ is regular and of index at most one, and $\text{rank}(E + BG) = r_1 + \hat{l}_1$ is the number of finite eigenvalues, which is equivalent to Theorem 6 of the main results of [6].

5. Pole assignment of periodic descriptor systems. In this section, we will study solvability for pole assignment problem of the resulting periodic regular descriptor systems in section 4. The problem of pole assignment is stated as follows.

**Problem I.** Given periodic matrix triples $\{(E_j, A_j, B_j)\}_{j=1}^p$ and a set

$$\mathcal{L} = \{ (\pi_{\alpha_i}, \pi_{\beta_j}), \ldots, (\pi_{\alpha_n}, \pi_{\beta_n}) \},$$

closed under conjugation, where $\{(\pi_{\alpha_i}, \pi_{\beta_i}) \in \mathbb{C}^2 \}$ for $i = 1, \ldots, n$, find $G_j, F_j \in M(m, n), j = 1, \ldots, p$, such that the periodic matrix pairs $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ are regular and have all eigenvalue pairs in $\mathcal{L}$.

In practice, the number of infinite poles to be prescribed is limited and it is not desirable to assign finite poles to infinite positions. Thus, we construct the periodic feedback matrices $G_j$ and $F_j, j = 1, \ldots, p$, such that the periodic closed-loop systems not only are regular and have the required finite poles, but also have index at most one. We have the following result for finite pole assignment.

**Theorem 5.1.** If the periodic matrix triples $\{(E_j, A_j, B_j)\}_{j=1}^p$ satisfy (C1) and (C2) and $U \leq L + m$, where $U, L$ are given by (4.17), then for any arbitrary set $\mathcal{L}$ of $\gamma$ self-conjugate finite poles $(\pi_{\alpha_i}, \pi_{\beta_i})$, $\pi_{\beta_i} \neq 0$, $i = 1, \ldots, f$, and $n - f$ infinite poles $(\pi_{\alpha_i}, 0), i = f + 1, \ldots, n$, where $U + \hat{l}_1 \leq f \leq L + m + \hat{l}_1$, there exist periodic feedback matrices $G_j$ and $F_j, j = 1, \ldots, p$ solving the pole assignment problem, Problem I, such that $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ are regular and of index at most one.

**Proof.** By Theorem 4.3 there exist $G_j$ and $F_j^j, j = 1, \ldots, p$, such that the periodic closed-loop systems $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ are regular and of index at most one, and

$$\text{rank}(\hat{E}(E_1 + B_1G_1, \ldots, E_p + B_pG_p; A_2 + B_2F_2, \ldots, A_p + B_pF_p)) = f + (p-1)n,$$

where $U + \hat{l}_1 \leq f \leq L + m + \hat{l}_1$. Moreover, by Lemma 2.7 the periodic matrix triples $\{(E_j + B_jG_j, A_j + B_jF_j)\}_{j=1}^p$ satisfy (C1) and (C2). From the results of [3] and
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[22], it follows that there exist $F^2_j, j = 1, \ldots, p$, which assign $f$ finite poles to these periodic systems while preserving precisely the remaining $n - f$ infinite poles invariant and such that the periodic closed-loop systems \{$(E_j + B_jG_j, A_j + B_jF_j)$\}_j=1^p, with $F_j = F^1_j + F^2_j, j = 1, \ldots, p$, are regular and of index at most one. □

Remark. As for Remark (ii) of Theorem 4.3, in the case $p = 1$, Theorem 5.1 can also be reduced to the result of Theorem 14 of [6].

6. Conclusion. In this paper, we construct derivative and proportional state feedback controls so that the periodic closed-loop systems not only are regular and have the required finite poles, but also have index at most one. This property ensures the solvability of the resulting periodic closed-loop systems of the dynamic-algebraic equation. For the time-invariant case our main theorems can be simplified to the main results of [6]. The construction procedures are based on orthogonal and elementary transformations which can be used to develop an algorithm implementing in a numerically efficient way.

In practice it is expected that the periodic regularizing closed-loop systems are well-conditioned in the sense that the reduction to the periodic canonical forms is computationally reliable. How to develop a computational algorithm which optimizes the conditioning of the periodic regularizing closed-loop systems is currently still under investigation.

REFERENCES