A semi-analytic method for solving rib-type waveguide problems

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A semi-analytic method for solving rib-type waveguide problems

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A new semi-analytic method for solving optical rib-type waveguide problems is presented. In the method, the cross-section of a rib-type waveguide is divided into several regions. In each region, the refractive index profile and field distribution are expanded into Fourier cosine series, and then are substituted in the wave equation. A second-order differential matrix equation is then derived for each region, with a closed-form solution obtainable. With the boundary conditions used, an eigenmode equation for the rib waveguide can be derived and solved numerically to give the modal indices. Here, the presented method is used to deal with two rib waveguides in different geometric dimensions and/or compositions, respectively. Computational results show that the presented method is quite efficient, in terms of CPU time, in finding the modal indices accurately. The relative error in computing the modal index with the method is about $10^{-5}$–$10^{-6}$.

Keywords: semi-analytic method; Fourier cosine series expansion; rib-type waveguides; eigenmode equation

1. Introduction

For more than a decade, considerable effort has been directed to computing the modes of optical rib waveguides, which form the important parts of photonic integrated circuits. Many kinds of numerical and semi-analytic methods were utilized for the computation of modal fields and the modal indices of rib-type waveguides. These include finite difference method [1–4], finite element method [5–8], beam propagation method [9–12] and many other semi-analytic methods [13–16]. Numerical methods based on finite element or finite difference basically discretize the transverse domain of an optical waveguide to induce an eigenvalue problem. A beam propagation algorithm was also introduced in the finite-difference discretization [10] or Fourier transform scheme [12] to determine the modal field and the modal index of a z-invariant structure. With the use of some
variational approaches in some semi-analytic methods, the modal index can be determined by solving an eigenmode equation, given a modal field expression at each transverse region of the rib waveguide. Although not quite accurate as the finite-difference-based beam propagation method, these semi-analytic approaches provide considerable efficiency in the computational time.

In this paper, we propose an exact semi-analytic approach to find the modal indices and modal fields of optical rib waveguides. In the approach, the cross-section of a rib waveguide is divided into several regions, in each of which both the refractive index profile and the field distribution are expressed as Fourier cosine series, respectively. In each region, a solution form of modal fields can be derived from a second-order differential matrix equation. Similarly to other existing semi-analytic methods, the proposed method is quite efficient in computational time compared with the time-consuming finite-difference-based beam propagation method. The accuracy of finding the modal index depends on the number of terms used in expanding the aforementioned refractive profile (as well as the modal field) into Fourier cosine series. It should be emphasized here that the proposed method is different from the discrete spectral index method [16], the generalized Fourier variational method [14,16] or the Fourier operator transform method [15,16]. In the discrete spectral index method, the field is expressed into a Fourier series in each region under the assumption of an effective rib waveguide. In the generalized Fourier variational method, the fields in the regions above and below the rib are represented by two Fourier series, while the field in the rib region is expressed as a combination of a discrete set of modes by considering this region as a slab waveguide. In the proposed method, the field in each region is expressed as a Fourier (cosine) series. Note that neither an effective rib waveguide nor mode expansion (for the rib region) is used by the proposed method. Furthermore, the index profile is expressed as a Fourier (cosine) series in each region and then substituted in the wave equation to obtain a corresponding matrix equation, from which an exact closed-form solution for the field can be found. This is different from the Rayleigh–Ritz method in which two-dimensional cosine series expansions were needed [17]. In Section 2, the theory of the proposed matrix method is outlined for the case of quasi-TE and quasi-TM modes. The computational results are presented in Section 3. Finally, Section 4 concludes this paper.

2. Theory
The geometric structure of the optical rib waveguide considered in this study is shown in Figure 1, where the width and height of the rib are denoted by \( w \) and \( h \), respectively, and the thickness of the slab is represented by \( d \). The refractive indices of the guiding region, substrate and cover in the structure are \( n_1 \), \( n_2 \) and \( n_3 \), respectively. Only quasi-TE and quasi-TM modes are discussed in this study.

2.1 Quasi-TE modes
The cross-section of the rib waveguide is divided into four regions, as shown in Figure 2(a), where the coordinates \( y_1 \) to \( y_3 \) represent, respectively, the boundaries between two corresponding adjacent regions. The coordinates \( x_1 \) and \( x_2 \) are the positions of the
two rib’s sidewalls. Clearly, for regions I (i.e. for $0 < y < y_1$), II ($y_1 < y < y_2$), and IV ($y_3 < y < y_4$), the refractive indices are $n_2$, $n_1$ and $n_3$ (all of which are constant), respectively; while for region III (i.e. for $y_2 < y < y_3$), the refractive index $n_0(x)$ follows the distribution

$$n_0^2(x) = n_1^2, \quad \text{for } x_1 < x < x_2,$$

$$= n_3^2, \quad \text{for } 0 < x < x_1 \text{ and } x_2 < x < x_3.$$  

(1)

Note that the function $n_0^2(x)$ can be extended into an even periodic function with a period of $2 \times 3$. Then we can have the Fourier cosine series expansion $n_0^2(x) = \sum_{n=0}^{N} a_n \cos n \Delta \omega x$ with $\Delta \omega = \pi / x_3$ and $N$ being large enough.

The field considered for quasi-TE modes is the magnetic field component in the $y$ axis direction, $H_y$. This field component satisfies the following wave equation

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} - \frac{1}{n^2} \frac{\partial n^2}{\partial x} \frac{\partial H_y}{\partial x} + (k_0^2 n^2 - \beta^2) H_y = 0,$$  

(2)
where $n$ is the refractive index (a function of $x$ and $y$) and $\beta$ is the propagation constant of a quasi-TE mode.

Equation (2) is solved for each of the four regions as indicated in Figure 2(a). Note that for regions I, II and IV, $n^2$ is constant and the term $(1/n^2)(\partial n^2/\partial x)$ is zero. Therefore, for these regions, Equation (2) reduces to

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + (k_0^2 n^2_i - \beta^2) H_y = 0,$$  \hspace{1cm} (3)

where the constant $n^2_i$ is equal to $n^2_2$, $n^2_1$ and $n^2_3$ for regions I, II and IV, respectively. In solving Equation (3) for each of the three regions, we use the Fourier cosine series expansion $H_y = \sum_{n=0}^N h_n^i(y) \cos n \Delta \omega x$, where the superscript $i$ denotes a specified region ($i = 1, 2$ and 4 for regions I, II and IV, respectively). Note that we have extended the rib waveguide structure such that one period extends from $-x_3$ to $x_3$ and each period includes two images of a rib waveguide. In this case all fields (including symmetric and anti-symmetric modes) are expanded in cosine series. For each region, Equation (3) then reduces to

$$- \sum_{n=0}^N h_n^i(n \Delta \omega)^2 \cos n \Delta \omega x + \sum_{n=0}^N \frac{\partial^2 h_n^i}{\partial y^2} \cos n \Delta \omega x + (k_0^2 n^2_i - \beta^2) \sum_{n=0}^N h_n^i \cos n \Delta \omega x = 0.$$ \hspace{1cm} (4)

By equating all the coefficients of $\cos n \Delta \omega x(n = 0, 1, 2, \ldots, N)$ in Equation (4) to zero, we obtain $N + 1$ differential equations that can be expressed in a matrix form. For region I, the matrix equation turns out to be

$$\frac{\partial^2 H_1}{\partial y^2} + (k_0^2 n^2_2 I - \beta^2 I) H_1 = 0.$$ \hspace{1cm} (5)

The matrix equations for regions II and IV are, respectively,

$$\frac{\partial^2 H_2}{\partial y^2} + (k_0^2 n^2_1 I - \beta^2 I) H_2 = 0,$$ \hspace{1cm} (6)

$$\frac{\partial^2 H_4}{\partial y^2} + (k_0^2 n^2_3 I - \beta^2 I) H_4 = 0.$$ \hspace{1cm} (7)

Here vectors $H_1$, $H_2$ and $H_4$ are defined by $H_1 = [h_0^1, h_1^1, h_2^1, \ldots, h_N^1]^T$, $H_2 = [h_0^2, h_1^2, h_2^2, \ldots, h_N^2]^T$ and $H_4 = [h_0^3, h_1^3, h_2^3, \ldots, h_N^3]^T$, respectively; $I$ is the identity matrix and $W$ is defined by

$$W = \begin{bmatrix}
0 & 0 & \cdots & \cdots & 0 \\
0 & (\Delta \omega)^2 & 0 & \cdots & \cdots \\
0 & 0 & (2\Delta \omega)^2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & (N-1)\Delta \omega)^2 \\
0 & \cdots & \cdots & \cdots & (N\Delta \omega)^2
\end{bmatrix}.$$ \hspace{1cm} (8)
To find \( H_y \) in region III, we note that \( n^2 \) in Equation (2) is not homogeneous and we then use the Fourier cosine series expansion of \( n^2 \), i.e. \( n^2 = \sum_{n=0}^{N} a_n \cos n\Delta \omega x \). We then put this series expansion and \( H_y = \sum_{n=0}^{N} h_n^2(y) \cos n\Delta \omega x \) into Equation (2) to derive a differential matrix equation, i.e.

\[
\frac{\partial^2 H_3}{\partial y^2} + (k_0^2A - W - P^{-1}Q - \beta^2I)H_3 = 0,
\]

(9)

where vector \( H_3 \) is given as \( H_3 = [h_0^3, h_1^3, h_2^3, \ldots, h_N^3]^T \); matrix \( A \) is defined by

\[
A = \frac{1}{2} \begin{bmatrix}
2a_0 & a_1 & a_2 & \cdots & a_{N-1} & a_N \\
2a_1 & 2a_0 + a_2 & a_1 + a_3 & \cdots & a_{N-2} + a_N & a_{N-1} + a_{N+1} \\
2a_2 & a_1 + a_3 & 2a_0 + a_4 & \cdots & a_{N-3} + a_{N+1} & a_{N-2} + a_{N+2} \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
2a_N & a_{N-1} + a_{N+1} & \cdots & \cdots & 2a_0 + a_{2(N-1)} & 2a_0 + a_{2N}
\end{bmatrix};
\]

(10)

matrix \( P \) is equal to \( 2A \); and matrix \( Q \) is given as

\[
Q = \begin{bmatrix}
0 & a_1 \Delta \omega & 2a_2 \Delta \omega & 3a_3 \Delta \omega & 4a_4 \Delta \omega & \cdots & Na_N \Delta \omega \\
2a_1 \Delta \omega & 2a_2 \Delta \omega & (-a_1 + 3a_2) \Delta \omega & (-2a_2 + 4a_3) \Delta \omega & (-3a_3 + 5a_4) \Delta \omega & \cdots & ([N + 1]a_{N+1} - (N - 1)a_{N-1}) \Delta \omega \\
4a_2 \Delta \omega & (a_1 + 3a_2) \Delta \omega & 4a_3 \Delta \omega & (-a_1 + 5a_2) \Delta \omega & (-2a_2 + 6a_3) \Delta \omega & \cdots & ([N + 2]a_{N+2} - (N - 2)a_{N-2}) \Delta \omega \\
6a_3 \Delta \omega & (2a_2 + 4a_3) \Delta \omega & (a_1 + 5a_2) \Delta \omega & 6a_4 \Delta \omega & (-a_1 + 7a_3) \Delta \omega & \cdots & ([N + 3]a_{N+3} - (N - 3)a_{N-3}) \Delta \omega \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
2Na_N \Delta \omega & \cdots & \cdots & \cdots & \cdots & \cdots & 2Na_{2N} \Delta \omega
\end{bmatrix}
\]

(11)

Explicit solutions to Equations (5), (6), (7) and (9) have the solution forms, respectively, as

\[
H_1(y) = \begin{bmatrix}
g_0 \exp[(\beta^2 - \kappa_0)^{1/2}(y - y_1)] \\
g_1 \exp[(\beta^2 - \kappa_1)^{1/2}(y - y_1)] \\
\vdots \\
g_N \exp[(\beta^2 - \kappa_N)^{1/2}(y - y_1)]
\end{bmatrix}, \quad H_2(y) = \begin{bmatrix}
b_0 \cos[(\gamma_0 - \beta^2)^{1/2}(y - y_1) - \phi_0] \\
b_1 \cos[(\gamma_1 - \beta^2)^{1/2}(y - y_1) - \phi_1] \\
\vdots \\
b_N \cos[(\gamma_N - \beta^2)^{1/2}(y - y_1) - \phi_N]
\end{bmatrix},
\]

\[
H_3(y) = \begin{bmatrix}
d_0 \exp[-(\beta^2 - \delta_0)^{1/2}(y - y_3)] \\
d_1 \exp[-(\beta^2 - \delta_1)^{1/2}(y - y_3)] \\
\vdots \\
d_N \exp[-(\beta^2 - \delta_N)^{1/2}(y - y_3)]
\end{bmatrix}, \quad H_4(y) = \sum_{n=0}^{N} Y_{\lambda n} \cos[(\alpha_n - \beta^2)^{1/2}(y - y_2) - \theta_n].
\]

(12)
Here in this subsection $\kappa, y_1, \lambda_i$, and $\delta_i$ ($i = 0, 1, 2, \ldots, N$) are the eigenvalues of matrices $(k_0^2 n_x^2 I - W)$, $(k_0^2 n_y^2 I - W)$, $(k_0^2 A - W - P^{-1}Q)$ and $(k_0^2 n_z^2 I - W)$, respectively; and $Y_i (i = 0, 1, 2, \ldots, N)$ is the eigenvector of matrix $(k_0^2 A - W - P^{-1}Q)$. The parameters $g_i$, $b_i$, $d_i$, $c_i$, and $\theta_i$ ($i = 0, 1, 2, \ldots, N$) are all undetermined. They should, however, satisfy the boundary conditions at the interfaces $y = y_1$, $y = y_2$ and $y = y_3$. The boundary conditions for quasi-TE modes state that $H_1$ and $\partial H_2/\partial y$ are continuous at the interfaces. That is, we have the following boundary conditions:

$$H_1(y_1) = H_2(y_1), \quad \frac{\partial H_1}{\partial y} \bigg|_{y_1} = \frac{\partial H_2}{\partial y} \bigg|_{y_1}, \quad H_2(y_2) = H_3(y_2), \quad \frac{\partial H_2}{\partial y} \bigg|_{y_2} = \frac{\partial H_3}{\partial y} \bigg|_{y_2}, \quad H_3(y_3) = H_4(y_3), \quad \frac{\partial H_3}{\partial y} \bigg|_{y_3} = \frac{\partial H_4}{\partial y} \bigg|_{y_3}. \quad (13)$$

Using the six boundary conditions above, we can derive two matrix identities as shown below.

$$K(\beta) \cdot X - L(\beta) \cdot Y = 0, \quad (14)$$
$$M(\beta) \cdot X + N(\beta) \cdot Y = 0. \quad (15)$$

Here vectors $X$ and $Y$ are defined as $X = [c_0 \cos \theta_0, c_1 \cos \theta_1, \ldots, c_N \cos \theta_N]^T$ and $Y = [c_0 \sin \theta_0, c_1 \sin \theta_1, \ldots, c_N \sin \theta_N]^T$, where the superscript $T$ denotes a transpose. Matrices $K, L, M$ and $N$ contain elements that depend on $\beta$.

To solve for $\beta$, we rewrite Equations (14) and (15) as

$$\begin{bmatrix} K(\beta) & -L(\beta) \\ M(\beta) & N(\beta) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (16)$$

Since there should exist a nontrivial solution for Equation (16), the following equation must hold.

$$\det \left\{ \begin{bmatrix} K(\beta) & -L(\beta) \\ M(\beta) & N(\beta) \end{bmatrix} \right\} = 0, \quad (17)$$

where $\det(.)$ represents the determinant of a matrix. The equation in (17) is thus used to determine the modal index (which is equal to $\beta/k_0$). A Newton–Raphson algorithm [18] is quite efficient in solving Equation (17) and is used in this study. Once $\beta$ is found, all the vectors $H_1, H_2, H_3$ and $H_4$ (see Equation (12)) can be determined. All the parameters $g_i$, $b_i$, $d_i$, $c_i$, and $\theta_i$ (except one, which is assumed to be arbitrary) appearing in Equation (12) can be determined by using the boundary conditions stated in Equation (13). The field distribution of a quasi-TE mode can then be found.

### 2.2 Quasi-TM modes

To analyze quasi-TM modes, we divide the cross-section of the rib waveguide into three regions as shown in Figure 2(b). The coordinates $x_1$ and $x_2$ represent, respectively, the boundaries of two adjacent regions (e.g. regions I and II as well as regions II and III).
The refractive index and the magnetic field $H_x$ in each region are then expanded into Fourier cosine series, as denoted below.

\[
(n^2, H_x) = \left( \sum_{j=0}^{N} a_j \cos j \Delta \omega y, \sum_{j=0}^{N} h^1_j(x) \cos j \Delta \omega y \right), \quad \text{for region I,}
\]

\[
= \left( \sum_{j=0}^{N} b_j \cos j \Delta \omega y, \sum_{j=0}^{N} h^2_j(x) \cos j \Delta \omega y \right), \quad \text{for region II,}
\]

\[
= \left( \sum_{j=0}^{N} c_j \cos j \Delta \omega y, \sum_{j=0}^{N} h^3_j(x) \cos j \Delta \omega y \right), \quad \text{for region III. (18)}
\]

Here, $\Delta \omega$ is given as $\Delta \omega = \pi/y_4$, and $N$ is a large enough number. Note that the numbers 1, 2 and 3 as superscripts in the above expression denote coefficients of the series expansions for the field $H_x$ in different regions. It should be noted that the notations for parameters and matrices in this subsection are self-explanatory and one should not be confused with those that have been used in the previous subsection.

The wave equation

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} - \frac{1}{n^2} \frac{\partial^2 H_x}{\partial y^2} + (k_0^2 n^2 - \beta^2) H_x = 0
\]

is solved separately for each region. After substituting the Fourier cosine series expansions of $n^2$ and $H_x$ in Equation (19) for each region, we then obtain a set of second-order differential equations for each region. Expressed in matrix forms, the three sets of differential equations are

\[
\frac{d^2 H_1}{dx^2} + (k_0^2 U - W - S_1^{-1} T_1 - \beta^2) H_1 = 0,
\]

\[
\frac{d^2 H_2}{dx^2} + (k_0^2 V - W - P^{-1} Q - \beta^2) H_2 = 0,
\]

\[
\frac{d^2 H_3}{dx^2} + (k_0^2 Y - W - S_2^{-1} T_2 - \beta^2) H_3 = 0.
\]

Here vector $H_i (i = 1, 2, 3)$ is given as $H_i = [h^i_0, h^i_1, h^i_2, \ldots, h^i_N]^T$; matrices $U$, $V$, $Y$, $P$, $Q$, $S_1$, $S_2$, $T_1$ and $T_2$ are all constant matrices with their definitions given in Appendix 1.

Explicit solutions to Equations (20)–(22) can be found, respectively, as

\[
H_1(x) = \sum_{j=0}^{N} P_j \delta_j \exp[(\beta^2 - \kappa_j)^{1/2}(x - x_1)],
\]

\[
H_2(x) = \sum_{j=0}^{N} y_j \delta_j \exp[(\lambda_j - \beta^2)^{1/2}(x - x_1) - \theta_j],
\]

\[
H_3(x) = \sum_{j=0}^{N} R_j \delta_j \exp[-(\beta^2 - \delta_j)^{1/2}(x - x_2)].
\]

Here $\kappa_j(P_j), \lambda_j(y_j)$ and $\delta_j(R_j) (j = 0, 1, 2, \ldots, N)$ are the eigenvalues (eigenvectors) of $(k_0^2 U - W - S_1^{-1} T_1)$, $(k_0^2 V - W - P^{-1} Q)$ and $(k_0^2 Y - W - S_2^{-1} T_2)$, respectively.
The parameters $\theta_j, d_j, e_j$ and $f_j (j = 0, 1, 2, \ldots, N)$ are to be determined by the following boundary conditions at $x = x_1$ and $x = x_2$:

\[
H_1(x_1) = H_2(x_1), \quad \frac{\partial H_1}{\partial x} \bigg|_{x_1} = \frac{\partial H_2}{\partial x} \bigg|_{x_1}
\]

\[
H_2(x_2) = H_3(x_2), \quad \frac{\partial H_2}{\partial x} \bigg|_{x_2} = \frac{\partial H_3}{\partial x} \bigg|_{x_2}.
\]

Following a similar procedure as for TE modes, the modal indices of the quasi-TM modes can be found by solving the equation

\[
\det(G) = 0.
\]

Here the elements of the matrix $G$ are functions of $\beta$, which can be found by using a Newton–Raphson algorithm [18].

### 3. Numerical results

Here we present numerical results for two examples of rib waveguides. The refractive indices and structural parameters are given in Table 1. The single-mode waveguides in structures 1 and 2 were, respectively, studied by [4] and [16]. The modal indices of these two waveguides calculated by [4,16] are also given here for a comparison with the numerical results obtained using the commercial software BeamPROP and the proposed method. Note that the commercial software can find both types of semi-vectorial mode solutions (i.e. quasi-TE and quasi-TM solutions). We ran the software BeamPROP and the proposed method with the same PC.

Table 2 shows the effective index $n_{\text{eff}}$ calculated by the proposed method, the commercial software BeamPROP and the method of [4], for the quasi-TE mode of the rib waveguide of structure 1 (single-mode waveguide). In the proposed method, the number of terms in the Fourier cosine series expansion (i.e. $N$) varies from 15 to 41, and these $n_{\text{eff}}$ converge to 3.388646. It is evident that the calculated $n_{\text{eff}}$ is almost the same as those obtained by BeamPROP and [4]. The CPU time spent by the proposed method is only 0.38 s for $N = 41$, while BeamPROP needs much more time. Here, the result obtained by the widely-used commercial software BeamPROP with the finest discretization shown in the table is considered as a benchmark. The relative error caused by the proposed method is thus about $3 \times 10^{-6}$.

The rib waveguide of structure 2 was studied by a semi-analytic method (i.e. Fourier operator transform method) outlined in [16]. The proposed method is also used to
calculate the modal index $n_{\text{eff}}$ of this single-mode waveguide. From Table 3 we can see the modal indices calculated by the proposed method, the software BeamPROP and the method of [16] for the quasi-TE mode of the rib waveguide of structure 2.

Table 2. Comparison of modal indices calculated by the proposed method, the commercial software BeamPROP and the method of [4], for the quasi-TE mode of the rib waveguide of structure 1.

<table>
<thead>
<tr>
<th>The proposed method</th>
<th>Beam PROP</th>
<th>Ref. (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$n_{\text{eff}}$</td>
<td>CPU (s)</td>
</tr>
<tr>
<td>15</td>
<td>3.397819</td>
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<td>18</td>
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<tr>
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<td>0.33</td>
</tr>
<tr>
<td>41</td>
<td>3.388646</td>
<td>0.38</td>
</tr>
</tbody>
</table>

Table 3. Comparison of modal indices calculated by the proposed method, the commercial software BeamPROP and the method of [16], for the quasi-TE mode of the rib waveguide of structure 2.

<table>
<thead>
<tr>
<th>The proposed method</th>
<th>Beam PROP</th>
<th>Ref. [16]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$n_{\text{eff}}$</td>
<td>CPU (s)</td>
</tr>
<tr>
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<td>1.75</td>
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<td>2.52</td>
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</tbody>
</table>

calculate the modal index $n_{\text{eff}}$ of this single-mode waveguide. From Table 3 we can see the modal indices calculated by the proposed method, the software BeamPROP and [16] for the quasi-TE mode. It is clear that the calculated values of $n_{\text{eff}}$ for increasing $N$ converge to a value quite close to that obtained by BeamPROP. Meanwhile we can see that the $n_{\text{eff}}$ obtained by [16] for this waveguide is relatively inaccurate, noting that the result obtained by BeamPROP is considered as a benchmark. Note that the three semi-analytic methods described in [16] (i.e. the discrete spectral index method, the generalized Fourier variational method and the Fourier operator transform method) all yielded an inaccurate result, compared with the proposed method. The proposed method produces the results with an error of $3 \times 10^{-6}$ for the quasi-TE mode. The CPU times spent by the proposed method is only 2.52 s. However, BeamPROP would spend much more CPU time, as indicated in the table.

Table 4 shows the calculated $n_{\text{eff}}$ for the quasi-TM mode of the structure-1 waveguide. We can see that the proposed method produces almost the same result as those obtained by BeamPROP and [4] with $N = 32$ for the quasi-TM mode. Again, note that the result obtained by BeamPROP with the finest discretization shown in the tables can be considered as a benchmark. The relative error caused by the proposed method is thus
about $3 \times 10^{-6}$. The calculated field distribution of the quasi-TM mode is shown by the contour drawn in Figure 3(a). For comparison, the field contour calculated by BeamPROP is shown in Figure 3(b). Both results are in good agreement. The proposed method is also used to calculate the modal index $n_{\text{eff}}$ of the structure-2 waveguide. From Table 5 we can see the modal indices calculated by the proposed method, the software BeamPROP and [16] for the quasi-TM mode. The calculated values of $n_{\text{eff}}$ for increasing $N$ converge to a value quite close to that obtained by BeamPROP. Meanwhile we can see that the $n_{\text{eff}}$ obtained by [16] for this waveguide is relatively inaccurate, noting that the result obtained by BeamPROP is considered as a benchmark. The proposed method produces the result with an error of $1 \times 10^{-6}$ (for $N=36$) for the quasi-TM mode. The CPU time spent by the proposed method is only 0.41 s for the quasi-TM mode, much less than that needed by BeamPROP. Although the proposed method can give an accurate result, something is worth noting. The number of terms used for the calculations is not large, indicating an efficiency in modeling the piecewise constant refractive index profile. Figure 4(a) shows the reconstructed refractive index profile in the $y$ direction for region I for the case of Table 5 (with use of $N=36$). With an increasing $N$, the reconstructed index profile would show more fidelity to the real one, as indicated by Figure 4(b) and (c). The figure shows better index profiles reconstructed with $N=200$ and 300, respectively. The proposed method can use different numbers of terms to model, respectively, the refractive index and the field, for example a larger $N$ for the refractive index. However, the presented results show that there is no need to use a larger $N$ for modeling the piecewise constant refractive index. That is, the same $N$ can be used to model both the piecewise constant refractive index and the field for obtaining an adequately accurate estimation of the mode index.

Note that the computation time for the method in [16] was estimated to be about 180 s on using a standard 486 computer. The results shown in this paper were all obtained by using a more advanced computer. However, we have re-run our program by using a 486 computer. We found that the computation times for the case of Table 3 are 2.14 s (for $N=20$), 12.95 s (for $N=45$) and 37.55 s (for $N=68$), and that the computation times for the case of Table 5 are 0.76 s (for $N=20$), 1.48 s (for $N=28$) and 3.41 s (for $N=36$).
Figure 3. Field distribution of the quasi-TM mode for the waveguide of structure 1 calculated by the proposed method (a) and the software BeamPROP (b). The contour levels are at 10% intervals of the maximum field.

Table 5. Comparison of modal indices calculated by the proposed method, the commercial software BeamPROP and the method of [16], for the quasi-TM mode of the rib waveguide of structure 2.

<table>
<thead>
<tr>
<th>N</th>
<th>n_{eff}</th>
<th>CPU (s)</th>
<th>Beam PROP</th>
<th>Ref. [16]</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.403005</td>
<td>0.10</td>
<td>Grid sizes (μm)</td>
<td>n_{eff}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Δx = 0.1, Δy = 0.05, Δz = 0.5</td>
<td>3.401987</td>
</tr>
<tr>
<td>24</td>
<td>3.401796</td>
<td>0.20</td>
<td>Δx = 0.05, Δy = 0.025, Δz = 0.5</td>
<td>3.401555</td>
</tr>
<tr>
<td>28</td>
<td>3.401953</td>
<td>0.22</td>
<td>Δx = 0.025, Δy = 0.0125, Δz = 0.25</td>
<td>3.401542</td>
</tr>
<tr>
<td>30</td>
<td>3.401882</td>
<td>0.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>3.401574</td>
<td>0.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>3.401539</td>
<td>0.41</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 4. Reconstructed refractive index profile in the $y$ direction for region I (defined by Figure 2(b)) for the case of Table 5. The vertical axis represents $n^2$. (a) is for $N = 36$, (b) is for $N = 200$ and (c) is for $N = 300$. 
4. Conclusion

We have proposed a new semi-analytic method to compute the eigenmodes of optical rib waveguides. In the proposed method, the cross-section of a rib waveguide is divided into several regions. The wave equation is then treated separately in each region. By expanding the modal field as well as the index profile into a Fourier cosine series, a second-order differential matrix equation corresponding to the wave equation in each region is derived. An analytic solution form for the modal field in each region can then be readily found. By matching the boundary conditions at the interfaces between two adjacent regions, we obtain an eigenmode equation to be solved for modal indices. To demonstrate the efficiency of the proposed method, two examples (which were previously studied) are given in this study. Numerical results show that the modal index can be found quite accurately and furthermore the CPU time spent in computation is much less than that when using the commercial software BeamPROP.

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References

Appendix 1

\[ P = \begin{bmatrix}
2b_0 & b_1 & b_2 & b_3 & \cdots & b_{N-1} & b_N \\
2b_1 & 2b_0 + b_2 & b_1 + b_3 & b_2 + b_4 & \cdots & b_{N-2} + b_N & b_{N-1} + b_{N+1} \\
2b_2 & b_1 + b_3 & 2b_0 + b_4 & b_1 + b_5 & \cdots & b_{N-3} + b_{N+1} & b_{N-2} + b_{N+2} \\
2b_3 & b_2 + b_4 & b_1 + b_5 & 2b_0 + b_6 & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
2b_N & b_{N-1} + b_{N+1} & \cdot & \cdot & \cdots & 2b_0 + b_{2(N-1)} & 2b_0 + b_{2N}
\end{bmatrix}, \quad (29) \]

\[ Q = \begin{bmatrix}
0 & b_1(\Delta \omega)^2 & b_2(2\Delta \omega)^2 & b_3(3\Delta \omega)^2 & b_4(4\Delta \omega)^2 & \cdots & b_N(N\Delta \omega)^2 \\
0 & 2b_1\Delta \omega^2 & (b_1 + b_3)2\Delta \omega^2 & (2b_2 + 4b_13\Delta \omega^2 & (3b_3 + 5b_2)4\Delta \omega^2 & \cdots & ((N+1)b_{N+1} + (N-1)b_{N-1})N\Delta \omega^2 \\
0 & (-b_1 + 3b_3)\Delta \omega^2 & 4b_2\Delta \omega^2 & (b_1 + 5b_3)3\Delta \omega^2 & (2b_2 + 6b_1)4\Delta \omega^2 & \cdots & ((N+2)b_{N+2} + (N-2)b_{N-2})N\Delta \omega^2 \\
0 & (-2b_2 + 4b_4)\Delta \omega^2 & (-b_1 + 5b_3)2\Delta \omega^2 & 6b_2\Delta \omega^2 & (b_1 + 7b_3)3\Delta \omega^2 & \cdots & ((N+3)b_{N+3} + (N-3)b_{N-3})N\Delta \omega^2 \\
0 & (-3b_3 + 5b_5)\Delta \omega^2 & (-2b_2 + 6b_3)2\Delta \omega^2 & (-b_1 + 7b_3)3\Delta \omega^2 & 8b_3\Delta \omega^2 & \cdots & 2Nb_{2N}N\Delta \omega^2 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdots & \cdot & 2Nb_{2N}N\Delta \omega^2
\end{bmatrix}, \quad (30) \]

\[ V = \frac{1}{2} \begin{bmatrix}
2b_0 & b_1 & b_2 & b_3 & \cdots & b_{N-1} & b_N \\
2b_1 & 2b_0 + b_2 & b_1 + b_3 & b_2 + b_4 & \cdots & b_{N-2} + b_N & b_{N-1} + b_{N+1} \\
2b_2 & b_1 + b_3 & 2b_0 + b_4 & b_1 + b_5 & \cdots & b_{N-3} + b_{N+1} & b_{N-2} + b_{N+2} \\
2b_3 & b_2 + b_4 & b_1 + b_5 & 2b_0 + b_6 & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
2b_N & b_{N-1} + b_{N+1} & \cdot & \cdot & \cdots & 2b_0 + b_{2(N-1)} & 2b_0 + b_{2N}
\end{bmatrix}, \quad (31) \]

\[ U = \frac{1}{2} \begin{bmatrix}
2a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_N \\
2a_1 & 2a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & \cdots & a_{N-2} + a_N & a_{N-1} + a_{N+1} \\
2a_2 & a_1 + a_3 & 2a_0 + a_4 & a_1 + a_5 & \cdots & a_{N-3} + a_{N+1} & a_{N-2} + a_{N+2} \\
2a_3 & a_2 + a_4 & a_1 + a_5 & 2a_0 + a_6 & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
2a_N & a_{N-1} + a_{N+1} & \cdot & \cdot & \cdots & 2a_0 + a_{2(N-1)} & 2a_0 + a_{2N}
\end{bmatrix}, \quad (32) \]
\[
Y = \frac{1}{2} \begin{bmatrix}
2c_0 & c_1 & c_2 & c_3 & \cdots & c_{N-1} & c_N \\
2c_1 & 2c_0 + c_2 & c_1 + c_3 & c_2 + c_4 & \cdots & c_{N-2} + c_N & c_{N-1} + c_{N+1} \\
2c_2 & c_1 + c_3 & 2c_0 + c_4 & c_1 + c_5 & \cdots & c_{N-3} + c_{N+1} & c_{N-2} + c_{N+2} \\
2c_3 & c_2 + c_4 & c_1 + c_5 & 2c_0 + c_6 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2c_N & c_{N-1} + c_{N+1} & \cdots & \cdots & 2c_0 + c_{2(N-1)} & \cdots & 2c_0 + c_{2N} \\
\end{bmatrix} 
\]

(33)

\[
S_1 = 2U, 
\]

(34)

\[
S_2 = 2Y, 
\]

(35)

\[
T_1 = \begin{bmatrix}
0 & a_1(\Delta \omega)^2 & a_2(2\Delta \omega)^2 & a_3(3\Delta \omega)^2 & a_4(4\Delta \omega)^2 & \cdots & a_N(N\Delta \omega)^2 \\
0 & 2a_2\Delta \omega^2 & (a_1 + a_2)2\Delta \omega^2 & (2a_2 + 4a_3)3\Delta \omega^2 & (3a_3 + 5a_4)4\Delta \omega^2 & \cdots & [(N + 1)a_{N+1} + (N - 1)a_{N-1}]N\Delta \omega^2 \\
0 & (-a_1 + 3a_2)\Delta \omega^2 & 4a_1\Delta \omega^2 & (a_1 + 5a_2)3\Delta \omega^2 & (2a_2 + 6a_3)4\Delta \omega^2 & \cdots & [(N + 2)a_{N+2} + (N - 2)a_{N-2}]N\Delta \omega^2 \\
0 & (-2a_2 + 4a_3)\Delta \omega^2 & (-a_1 + 5a_2)2\Delta \omega^2 & 6a_1\Delta \omega^2 & (a_1 + 7a_2)3\Delta \omega^2 & \cdots & [(N + 3)a_{N+3} + (N - 3)a_{N-3}]N\Delta \omega^2 \\
0 & (-3a_2 + 5a_3)\Delta \omega^2 & (-2a_2 + 6a_3)2\Delta \omega^2 & (-a_1 + 7a_2)3\Delta \omega^2 & 8a_1\Delta \omega^2 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 2Na_{2N}N\Delta \omega^2 \\
\end{bmatrix} 
\]

(36)

\[
T_2 = \begin{bmatrix}
0 & c_1(\Delta \omega)^2 & c_2(2\Delta \omega)^2 & c_3(3\Delta \omega)^2 & c_4(4\Delta \omega)^2 & \cdots & c_N(N\Delta \omega)^2 \\
0 & 2c_2\Delta \omega^2 & (c_1 + c_2)2\Delta \omega^2 & (2c_2 + 4c_3)3\Delta \omega^2 & (3c_3 + 5c_4)4\Delta \omega^2 & \cdots & [(N + 1)c_{N+1} + (N - 1)c_{N-1}]N\Delta \omega^2 \\
0 & (-c_1 + 3c_2)\Delta \omega^2 & 4c_1\Delta \omega^2 & (c_1 + 5c_2)3\Delta \omega^2 & (2c_2 + 6c_3)4\Delta \omega^2 & \cdots & [(N + 2)c_{N+2} + (N - 2)c_{N-2}]N\Delta \omega^2 \\
0 & (-2c_2 + 4c_3)\Delta \omega^2 & (-c_1 + 5c_2)2\Delta \omega^2 & 6c_1\Delta \omega^2 & (c_1 + 7c_2)3\Delta \omega^2 & \cdots & [(N + 3)c_{N+3} + (N - 3)c_{N-3}]N\Delta \omega^2 \\
0 & (-3c_2 + 5c_3)\Delta \omega^2 & (-2c_2 + 6c_3)2\Delta \omega^2 & (-c_1 + 7c_2)3\Delta \omega^2 & 8c_1\Delta \omega^2 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 2Nc_{2N}N\Delta \omega^2 \\
\end{bmatrix} 
\]

(37)