Structured doubling algorithms for weakly stabilizing Hermitian solutions of algebraic Riccati equations

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Abstract

In this paper, we propose structured doubling algorithms for the computation of the weakly stabilizing Hermitian solutions of the continuous- and discrete-time algebraic Riccati equations, respectively. Assume that the partial multiplicities of purely imaginary and unimodular eigenvalues (if any) of the associated Hamiltonian and symplectic pencil, respectively, are all even and the C/DARE and the dual C/DARE have weakly stabilizing Hermitian solutions with property (P). Under these assumptions, we prove that if these structured doubling algorithms do not break down, then they converge to the desired Hermitian solutions globally and linearly. Numerical experiments show that the structured doubling algorithms perform efficiently and reliably.

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1. Introduction

In this paper, we investigate the structured doubling algorithms for the computation of the weakly stabilizing Hermitian solution $X$ to
(I) the continuous-time algebraic Riccati equation (CARE):

\[- XG + XAH + XA + H = 0, \tag{1.1}\]

or

(II) the discrete-time algebraic Riccati equation (DARE):

\[X = XH(I + GX)^{-1}A + H, \tag{1.2}\]

where \(A, G, H \in \mathbb{C}^{n \times n}\) with \(G = GH, H = H^H\) and \(I \equiv I_n\) is the identity matrix of order \(n\).

Eqs. (1.1) and (1.2) arise frequently in the pursuit of the “weakly” stabilizing controllers of continuous- and discrete-time \(H_\infty\)-optimal control systems, respectively [13,15,17,25]. In addition, several applications in Wiener filtering theory [47], network synthesis [3] and Moser–Veselov equations [9,40] also involve the Hermitian solution of CAREs.

We consider the \(2n \times 2n\) Hamiltonian matrix \(H\) associated with the CARE:

\[H = \begin{bmatrix} A & -G \\ -H & -A^H \end{bmatrix}, \tag{1.3}\]

which satisfies

\[HJ = -JH^H, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}; \tag{1.4}\]

and consider the \(2n \times 2n\) symplectic pair \((M, L)\) (or symplectic pencil \(M - \lambda L\)) associated with the DARE:

\[M = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \quad L = \begin{bmatrix} I & G \\ 0 & A^H \end{bmatrix}, \tag{1.5}\]

which satisfies

\[MJ = LJL^H. \tag{1.6}\]

The special symplectic pair \((M, L)\) of the form in (1.5) is referred to as a standard symplectic form (SSF).

Note that [30,42] \(\lambda \in \sigma(H)\) if and only if \(-\bar{\lambda} \in \sigma(H)\), and \(\lambda \in \sigma(M, L)\) if and only if \(1/\bar{\lambda} \in \sigma(M, L)\). Here \(\sigma(H)\) and \(\sigma(M, L)\) denote the spectrums of \(H\) and \(M, L\), respectively. It is well-known that (e.g. [30,32,42]) the CARE (1.1) has a weakly stabilizing Hermitian solution \(X\) if and only if

\[
\begin{bmatrix} A & -G \\ -H & -A^H \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Phi, \quad \Phi \in \mathbb{C}^{n \times n}, \tag{1.7}\]

where \(\sigma(\Phi) \subseteq \mathbb{R}_-\) (the closed left half plane); the DARE (1.2) has a weakly stabilizing Hermitian solution \(X\) if and only if

\[
\begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Phi, \quad \Phi \in \mathbb{C}^{n \times n}. \tag{1.8}\]

where \(\sigma(\Phi) \subseteq \mathbb{D}_+\) (the closed unit disk) and \((I + GX)\) is invertible. The particular invariant subspaces spanned by \([I, X^T]^T\) in (1.7) and (1.8) are usually referred to as stable Lagrangian subspaces.

**Definition 1.1.** A subspace \(\mathcal{U} \subseteq \mathbb{C}^{2n}\) with dimension \(n\) is called an \(H\)-stable Lagrangian subspace, if \(\mathcal{U}\) satisfies that (i) \(H\mathcal{U} \subseteq \mathcal{U}\), (ii) \(\mathcal{U}\) is isotropic; i.e., \(x^HJy = 0\), for all \(x, y \in \mathcal{U}\); and (iii) \(\text{Re}(\lambda(H|_\mathcal{U})) \leq 0\). Here \(\lambda(H|_\mathcal{U})\) denotes an eigenvalue of \(H\) restricted to \(\mathcal{U}\).
Definition 1.2. A subspace $V \subseteq \mathbb{C}^{2n}$ with dimension $n$ is called an $(M, L)$-stable Lagrangian subspace, if (i) $V$ is invariant under $(M, L)$; i.e., there is a subspace $W$ such that $MV, LV \subseteq W$ [44, pp. 303–305]; (ii) $V$ is isotropic and (iii) $|\lambda((M, L)|_V)| \leq 1$. Here $\lambda((M, L)|_V)$ denotes an eigenvalue of $(M, L)$ restricted to $V$.

Unfortunately, an $H$-stable Lagrangian subspace and an $(M, L)$-stable Lagrangian subspace do not always exist, when some purely imaginary eigenvalues of $H$ and some unimodular eigenvalues of $(M, L)$ have odd partial multiplicities, respectively. Counterexamples can be found in [41]. To guarantee the existence of the $H$- and $(M, L)$-stable Lagrangian subspaces and the weakly stabilizing Hermitian solutions of CAREs and DAREs, we assume that $H$ in (1.3) and $(M, L)$ in (1.5), respectively, satisfy the conditions:

(A1) The partial multiplicities (the sizes of Jordan blocks) of $H$ associated with the purely imaginary eigenvalues (if any) are all even.

(A2) The partial multiplicities of $(M, L)$ associated with the unimodular eigenvalues (if any) are all even.

Under these assumptions, equivalence statements for the existence of weakly stabilizing Hermitian solutions of CAREs and DAREs have first been given by [18,29,28], respectively. In order to enhance the uniqueness of the weakly stabilizing Hermitian solution in some sense, we, further, give the following definitions.

Definition 1.3. Assume that (A1) holds. The CARE (1.1) is said to have a weakly stabilizing Hermitian solution with property $(P)$, if the matrix $\Phi$ in (1.7) (i.e., $\Phi \equiv A - GX$) satisfies that $\sigma(\Phi) \subseteq \mathbb{R}_-\text{ and each purely imaginary eigenvalue has a half of the partial multiplicity of } H\text{ corresponding to the same eigenvalue.}$ (For example, $H$ has a purely imaginary eigenvalue with partial multiplicity $(2, 4, 6)$ i.e., the jordan blocks are of size 2, 4 and 6, respectively, then $\Phi$ has the same eigenvalue with the partial multiplicity $(1, 2, 3)$.)

Definition 1.4. Assume that (A2) holds. The DARE (1.2) is said to have a weakly stabilizing Hermitian solution with property $(P)$, if the matrix $\Phi$ in (1.8) (i.e., $\Phi \equiv (I + GX)^{-1}A$) satisfies that $\sigma(\Phi) \subseteq \mathbb{R}_+\text{ and each unimodular eigenvalue has a half of the partial multiplicity of } (M, L)\text{ corresponding to the same eigenvalue.}$

For the continuous-time case, a well-known backward stable algorithm care [30] computes a stabilizing Hermitian solution $X$ for the CARE by applying the $QR$ algorithm with reordering [43] to $H$. Unfortunately, the $QR$ algorithm preserves neither the Hamiltonian structure nor the associated splitting of eigenvalues. When $H$ in (1.3) has no purely imaginary eigenvalues, a strongly stable method has been proposed by [10] for computing the Hamiltonian Schur form of $H$, and therefore, the $H$-stable Lagrangian subspace. Efficient structured doubling algorithms (incorporating an appropriate Cayley transform) [11,27] and the matrix sign function methods [5,8,14,16] have been developed to compute the unique positive semidefinite solution of CARE (1.1). When $H$ in (1.3) satisfies Assumption (A1), an eigenvector deflation technique proposed by [13] guarantees that the eigenvalues appear with the correct pairing. This is certainly an advantage over the $QR$ algorithm, but the method ignores most of the structure of the problem during computation. A structured algorithm proposed by [1] only using symplectic orthogonal transformations, computes the $H$-stable Lagrangian subspace. But there are numerical difficulties
in the convergence of the deflation steps when purely imaginary eigenvalues occur [38, p. 143]. To avoid the numerical difficulties mentioned above, another stable and structured algorithm has been developed in [33], preprocessing to deflate all purely imaginary eigenvalues. When $\mathcal{H}$ satisfies Assumption (A1) with partial multiplicities equal to two, an efficient Newton’s method has been developed in [21] for solving the CAREs with global and linear convergence.

For the discrete-time case, a well-known backward stable algorithm dare [37,42,48] computes a stabilizing Hermitian solution $X$ for DAREs by applying the QZ algorithm with reordering to $(\mathcal{M}, \mathcal{L})$. Unfortunately, this algorithm does not take into account the symplectic structure of $(\mathcal{M}, \mathcal{L})$. Non-structure-preserving iterative processes spoil the symplectic structure, causing the algorithms to fail or lose accuracy in adverse circumstances. When $(\mathcal{M}, \mathcal{L})$ in (1.5) has no unimodular eigenvalues, an efficient doubling algorithm was firstly derived in [2] based on an acceleration scheme of the fixed point iteration for (1.2). Using different approaches, quadratic convergence of doubling algorithms has been shown in [26,35]. On the other hand, based on the viewpoint of the inverse-free iteration [4,36], a matrix disk function method (MDFM) [6,7] and a structure-preserving doubling algorithm (SDA) [12,24] have been developed for solving DAREs. The symplectic structure in the MDFM and the SSF form in the SDA are preserved at each iterative step. However, the symplectic structure in the MDFM is preserved only in exact arithmetic. When $(\mathcal{M}, \mathcal{L})$ in (1.5) satisfies Assumption (A2), a structured algorithm has been developed in [33], preprocessing to deflate all unimodular eigenvalues by determining the isotropic Jordan subbasis using the $S + S^{-1}$-transform of $\mathcal{M} - \lambda \mathcal{L}$ [31]. When $(\mathcal{M}, \mathcal{L})$ satisfies Assumption (A2) with partial multiplicities two, an efficient Newton-type method has been proposed by [20] to solve the DAREs with global and linear convergence.

As mentioned above, the MDFM and SDA have been proposed for solving DARE (1.2) with $\mathcal{M} - \lambda \mathcal{L}$ possessing no unimodular eigenvalues. To solve CARE (1.1) with $\mathcal{H}$ with no purely imaginary eigenvalues, the Hamiltonian matrix $\mathcal{H}$ is converted to a symplectic pencil $\hat{\mathcal{M}} - \lambda \hat{\mathcal{L}}$ in SSF by an appropriate Cayley transform and then the MDFM or the SDA algorithm can be applied. The main purpose of this paper is to apply the MDFM or SDA to solve CAREs and DAREs, where the associated $\mathcal{H}$ in (1.3) and $\mathcal{M} - \lambda \mathcal{L}$ in (1.5) satisfy Assumptions (A1) and (A2), respectively. Under these assumptions, we prove the globally linear convergence of the MDFM and SDA.

This paper is organized as follows. In Sections 2 and 3, we describe structured doubling algorithms, the D-MDFM/D-SDA and C-MDFM/C-SDA, for solving DAREs and CAREs, respectively. In Section 4, we prove that under Assumptions (A1) and (A2) structured doubling algorithms converge globally and linearly to the weakly stabilizing Hermitian solutions with property (P) of DAREs and CAREs, respectively. In Section 5, we test several numerical examples for illustrating the convergence behavior of the MDFM, SDA and Newton-type methods. Concluding remarks are given in Section 6.

Throughout this paper, we denote $A^H = \bar{A}^T$ the conjugate transpose of $A \in \mathbb{C}^{n \times n}$, $i = \sqrt{-1}$, $I \equiv I_n$ and $0 \equiv 0_n$ (the identity and zero matrices of order $n$, respectively). The vector $e_j$ the $j$th column of $I_n$, $\| \cdot \|$ a matrix norm, and $\sigma(A)$ and $\rho(A)$ the spectrum and the spectral radius of $A$, respectively. $\mathbb{R}_{-}$ and $\mathbb{O}_{1}$, respectively, denote the closed left half plane and the closed unit disk.

### 2. Structured doubling algorithms for DAREs

The matrix disk function method (MDFM) in [6,7] is developed to solve DARE (1.2) by using a swapping technique built on the $QR$ factorization. We refer to this step as a $QR$-swap.
For a given symplectic pair \((M, L)\), the QR-swap computes the \(QR\)-factorization of \([L^T, -M^T]^T\) from
\[
2 \begin{bmatrix} L \\ -M \end{bmatrix} = \begin{bmatrix} 2_{11} & 2_{12} \\ \ast & \ast \end{bmatrix} \begin{bmatrix} L \\ -M \end{bmatrix} = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix},
\]
where \(2 \in \mathbb{C}^{4n \times 4n}\) is unitary and \(\hat{R} \in \mathbb{C}^{2n \times 2n}\) is upper triangular. Let
\[
\hat{L} := L^*, \quad \hat{M} := M^*.
\]
It is easily seen that \((\hat{M}, \hat{L})\) is symplectic. From (2.1) and (2.2), \((\hat{M}, \hat{L})\) satisfies the doubling property:
\[
\hat{M}x = \hat{M} \hat{L}x = \lambda \hat{L}x = \lambda \hat{L}^2x,
\]
assuming that \(Mx = \lambda Lx\).

Algorithm 2.1. (D-MDFM for DAREs)

Input: \(A, G, H; \tau\) (a small tolerance);
Output: a weakly stabilizing Hermitian solution \(X\) to DARE.

Initialize: \(\hat{R} \leftarrow 0_{2n}, \hat{M} \leftarrow \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \hat{L} \leftarrow \begin{bmatrix} I & G \\ 0 & A^H \end{bmatrix}\);

Repeat: Compute the \(QR\)-factorization:
\[
\begin{bmatrix} 2_{11} & 2_{12} \\ \ast & \ast \end{bmatrix} \begin{bmatrix} L \\ -M \end{bmatrix} = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix};
\]
If \(\|\hat{R} - R\| \leq \tau \|\hat{R}\|\), Then solve the least squares problem for \(X\):
\[
-M(;, 1 : n) = M(;, n + 1 : 2n)X;
\]
Stop
Else
Set \(L \leftarrow L^*L, M \leftarrow M^*M, R \leftarrow \hat{R}\);
Go To Repeat
End If

The sequence \(\{(M_k, L_k)\}\) generated by Algorithm 2.1 satisfies the recursive formula
\[
M_{k+1} = M_{k}^*M_k, \quad L_{k+1} = L_{k}^*L_k,
\]
where
\[
\begin{bmatrix} 2_{11,k} & 2_{12,k} \\ \ast & \ast \end{bmatrix} \begin{bmatrix} L_k \\ -M_k \end{bmatrix} = \begin{bmatrix} \hat{R}_k \\ 0 \end{bmatrix}
\]
is the \(QR\)-factorization.

On the other hand, the SDA in [12] is developed to solve DARE (1.2) under conditions of stabilizability and detectability, by using a structured \(LU\) factorization instead of the \(QR\) factorization in (2.1). We refer to this step as a SLU-swap. As derived in [12], for \((M, L)\) in SSF (1.5), we construct
\[
M_* = \begin{bmatrix} A(I + GH)^{-1} & 0 \\ -A^H(I + HG)^{-1}H & I \end{bmatrix}, \quad L_* = \begin{bmatrix} I & AG(I + HG)^{-1} \\ 0 & A^H(I + HG)^{-1} \end{bmatrix}
\]
and consequently deduce that
\[
M_*L = L_*M.
\]
With 
\[ \hat{\mathcal{L}} = \mathcal{L} \star \mathcal{L} \] and 
\[ \hat{\mathcal{M}} = \mathcal{M} \star \mathcal{M} , \]
and apply the Sherman–Morrison–Woodbury formula, we obtain
\[ \hat{\mathcal{L}} = \begin{bmatrix} I & \hat{G} \\ 0 & \hat{A}^H \end{bmatrix} , \quad \hat{\mathcal{M}} = \begin{bmatrix} \hat{A} & 0 \\ -\hat{H} & I \end{bmatrix} , \] (2.7)
where
\[ \hat{A} = A(I + GH)^{-1} A , \] (2.8a)
\[ \hat{G} = G + AG(I + HG)^{-1} A^H , \] (2.8b)
\[ \hat{H} = H + A^H(I + HG)^{-1} HA . \] (2.8c)

Equations in (2.7) show that the newly derived matrix pair \((\hat{\mathcal{M}}, \hat{\mathcal{L}})\) is again in SSF form. From (2.6) and (2.7), \((\hat{\mathcal{M}}, \hat{\mathcal{L}})\) also satisfies the doubling property 
\[ \hat{\mathcal{M}} x = \lambda^2 \hat{\mathcal{L}} x . \]

Remark 2.1
(i) Equations in (2.8) have exactly the same form as the doubling algorithm (which has been first proposed and investigated by Anderson [2] and Kimura [26]). However, the original doubling algorithm was derived as an acceleration scheme for the fixed-point iteration from (1.2). Instead of producing the sequence \(\{X_k\}\), the doubling algorithm produces \(\{X_{2k}\}\). Furthermore, the convergence of the doubling algorithm was proven when \(A\) is nonsingular [2], and for \((A, G, H)\) which is reachable and detectable, or stabilizable and observable [26]. A stronger convergent result of the SDA algorithm under weaker conditions (stabilizability and detectability) can be found in [12,35].

(ii) The matrix \((I + GH)\) in (2.8) can possibly be singular in some step of SDA, thus \(\hat{A}, \hat{G}\) and \(\hat{H}\) in (2.8) do not exist and the SDA may break down. In our numerical experiments in Section 6, this happens only in the limiting case.

Algorithm 2.2. \((D\text{-SDA for DAREs})\)

Input: \(A, G, H; \tau \) (a small tolerance);
Output: a weakly stabilizing Hermitian solution \(X\) to DARE.
Repeat \(W \leftarrow I + GH\);
If \(W\) is singular, then break down.
Solve \(WV_1 = A, V_2 W = A\) for \(V_1\) and \(V_2\);
Set \(G \leftarrow G + V_2 GA^H\), \(\hat{H} \leftarrow H + A^H HV_1\), \(A \leftarrow AV_1\)
If \(\|\hat{H} - H\| \leq \tau \|\hat{H}\|\), then \(X \leftarrow \hat{H}\), Stop.
Set \(H \leftarrow \hat{H}\);
Goto Repeat

Remark 2.2. The linear systems \(WV_1 = A, V_2 W = A\) for \(V_1\) and \(V_2\) in Algorithm 2.2 can be solved by the \(LU\) factorization of \(W\) or the GSVD (generalized singular value decomposition) of \((W, A)\) and \((W^H, A^H)\), respectively. For the latter case, let
\[ \begin{align*}
    T_1 W U_a &= C_1 , & T_2 W^H U_b &= C_2 , \\
    T_1 A V_a &= S_1 , & T_2 A^H V_b &= S_2 
\end{align*} \]
be the GSVD of \((W, A)\) and \((W^H, A^H)\), respectively, where \(U_a, V_a, U_b, V_b\) are unitary, \(T_1, T_2\) are nonsingular, \(C_1, S_1, C_2, S_2\) are positive diagonal [19, p. 466]. Then \(V_1\) and \(V_2\) can be solved
by \( V_1 = U_aC_1^{-1}S_1V_a^H \), \( V_2 = V_bS_2C_2^{-1}U_b^H \). A detailed error analysis of D-SDA will be given in Appendix.

The sequence \( \{ (A_k, G_k, H_k) \} \) generated by Algorithm 2.2 satisfies the following recursive formula:

\[
A_{k+1} = A_k(I + G_k H_k)^{-1} A_k, \quad (2.9a)
\]
\[
G_{k+1} = G_k + A_k G_k (I + H_k G_k)^{-1} A_k^H, \quad (2.9b)
\]
\[
H_{k+1} = H_k + A_k^H H_k (I + G_k H_k)^{-1} A_k. \quad (2.9c)
\]

3. Structured doubling algorithm for CAREs

To solve CARE (1.1), a structured doubling algorithm was first proposed by Kimura [27] using a Cayley transformation. With an appropriate parameter \( \gamma > 0 \), the Hamiltonian matrix \( H \) in (1.3) can be transformed to a symplectic pair \((H, L) \equiv (H + \gamma I, H - \gamma I) [38,39] \), and then simplifies to a symplectic pair \((M_0, L_0)\) in the SSF form. Here

\[
M_0 = \begin{bmatrix}
A_0 & 0 \\
-H_0 & I
\end{bmatrix}, \quad L_0 = \begin{bmatrix}
I & G_0 \\
0 & A_0^H
\end{bmatrix}, \quad (3.1)
\]

with

\[
A_0 = I + 2\gamma (A_\gamma + G A_\gamma^{-H} H)^{-1}, \quad (3.2a)
\]
\[
G_0 = 2\gamma A_\gamma^{-1} G (A_\gamma^H + H A_\gamma^{-1} G)^{-1}, \quad (3.2b)
\]
\[
H_0 = 2\gamma (A_\gamma^H + H A_\gamma^{-1} G)^{-1} H A_\gamma^{-1}, \quad (3.2c)
\]

and \( A_\gamma \equiv A - \gamma I \). The DARE associated with the symplectic pair \((M_0, L_0)\) is

\[
X = A_0^H X (I + G_0 X)^{-1} A_0 + H_0
\]
on which Algorithm 2.1 or 2.2 can then be applied. For details on how a suitable \( \gamma \) is chosen for the Cayley transformation, see [11].

Algorithm 3.1. (C-MDFM/C-SDA for CAREs)

Input: \( A, G, H; \tau \) (a small tolerance);
Output: a stabilizing Hermitian solution \( X \) to CARE.

(I) Find an appropriate value \( \gamma > 0 \) so that \( A_\gamma \) and \( A_\gamma + G A_\gamma^{-H} H \) are well conditioned (see [11] for details).

(II) Initialize \( A_0 \leftarrow I + 2\gamma (A_\gamma + G A_\gamma^{-H} H)^{-1}, \)
\( G_0 \leftarrow 2\gamma A_\gamma^{-1} G (A_\gamma^H + H A_\gamma^{-1} G)^{-1}, \)
\( H_0 \leftarrow 2\gamma (A_\gamma^H + H A_\gamma^{-1} G)^{-1} H A_\gamma^{-1}; \)

(III) Call Algorithm 2.1 (D-MDFM) or 2.2 (D-SDA).

4. Convergence of structured doubling algorithms

Let

\[
M = \begin{bmatrix}
A & 0 \\
-H & I
\end{bmatrix}, \quad L = \begin{bmatrix}
I & G \\
0 & A^H
\end{bmatrix}, \quad (4.1)
\]
where $G = G^H$ and $H = H^H$. In the light of Definition 1.4, we assume that the matrix pair $(\mathcal{M}, \mathcal{L})$ is regular (i.e., $\det(\mathcal{M} - \lambda \mathcal{L}) \neq 0$) and satisfies Assumption (A2). In this section, we shall show that under these assumptions Algorithm 2.1 or 2.2 converges to a weakly stabilizing Hermitian solution with property $(P)$ for DARE (1.2). A similar proof can be applied to the convergence of Algorithm 3.1 to a weakly stabilizing Hermitian solution with property $(P)$ for the CARE when $\mathcal{H}$ satisfies Assumption (A1).

Denote the Jordan block of size $p$ corresponding to a unimodular eigenvalue $\omega \equiv e^{i\theta}$ by

$$J_{\omega,p} = \begin{bmatrix} \omega & 1 & 0 & \cdots & 0 \\ 0 & \omega & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \omega \end{bmatrix}_{p \times p}$$

(4.2)

We now quote or prove some useful lemmas. For example, the Jordan block $J_{\omega,p}$ to the power of $2^k$ can be explicitly evaluated.

**Lemma 4.1** ([19, pp. 557]). Let $J_{\omega,p}$ be given by (4.2). Then

$$J_{\omega,p}^{2^k} = \begin{bmatrix} \gamma_{1,k} & \gamma_{2,k} & \cdots & \gamma_{p,k} \\ 0 & \gamma_{1,k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \gamma_{1,k} \end{bmatrix},$$

(4.3a)

where

$$\gamma_{i,k} = \frac{1}{(i-1)!} \left. \frac{d^{i-1}}{dx^{i-1}} x^{2^k} \right|_{x=\omega} = \frac{2^k(2^k - 1) \cdots (2^k - i + 2)}{(i-1)!} \omega^{2^k-i+1}$$

(4.3b)

for $i = 1, \ldots, p$.

With $p = 2m$, let

$$\Gamma_{k,m} = \begin{bmatrix} \gamma_{m+1,k} & \gamma_{m+2,k} & \cdots & \gamma_{2m-1,k} & \gamma_{2m,k} \\ \gamma_{m,k} & \ddots & \ddots & \gamma_{2m-1,k} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \gamma_{3,k} & \ddots & \ddots & \gamma_{m+2,k} & \vdots \\ \gamma_{2,k} & \gamma_{3,k} & \cdots & \gamma_{m,k} & \gamma_{m+1,k} \end{bmatrix}_{m \times m}$$

$$\equiv J_{\omega,2m}^{2^k}(1:m, m+1:2m)$$

(4.4)

in which $\gamma_{i,k}$ are defined in (4.3b), for $i = 2, \ldots, 2m$. 
To show that $\Gamma_{k,m}$ is invertible, we first prove the following lemma.

**Lemma 4.2.** Let $2 \leq r \leq m$ and

$$F_r(m) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1r} \\ f_{21} & f_{22} & \cdots & f_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r1} & f_{r2} & \cdots & f_{rr} \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad (4.5)$$

where

$$f_{ij} = \begin{cases} 1 & \text{if } j = 1, \\ i + j - 2 \prod_{\nu=i}^{r} (m + r - \nu) & \text{if } 2 \leq j \leq r \end{cases}$$

for $i = 1, 2, \ldots, r$. Then

$$|\det(F_r(m))| = \prod_{v=1}^{r} (v - 1)!. \quad (4.6)$$

**Proof.** Since

$$\det(F_2(m)) = \begin{vmatrix} 1 & m+1 \\ 1 & m \end{vmatrix} = -1$$

(4.6) is true for $r = 2$. Suppose that

$$|\det(F_{r-1}(m))| = \prod_{v=1}^{r-1} (v - 1)!.$$ 

Eliminating the first to $(r - 1)$th entries in the first column of $F_r(m)$ by elementary row operations, we obtain

$$\begin{bmatrix} I_{r-2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{r-2} \end{bmatrix} F_r(m)$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{F}_{r-1}(m) \\ m, m(m-1), \ldots, m(m-1) \cdots (m-r+2) \end{bmatrix},$$

where $\hat{F}_{r-1}(m) \equiv [\hat{f}_{ij}] \in \mathbb{R}^{(r-1) \times (r-1)}$ with

$$\hat{f}_{ij} = \begin{cases} 1 & \text{if } j = 1, \\ i + j - 2 \prod_{\nu=i}^{r-1} [m + (r - 1) - \nu] & \text{if } 2 \leq j \leq r - 1 \end{cases}$$

for $i = 1, 2, \ldots, r - 1$. Using the factorization

$$\hat{F}_{r-1}(m) = F_{r-1}(m) \text{diag}\{1, 2, \cdots, r - 1\},$$

we have

$$|\det(F_r(m))| = (r - 1)!|\det(F_{r-1}(m))| = \prod_{v=1}^{r} (v - 1)!. $$

The proof is completed by mathematical induction. \[\square\]
Definition 4.1. Given $\ell \in \mathbb{Z}$. An $m \times m$ Toeplitz matrix $T = [t_{pq}]_{p,q=1}^m$ can be written in the form

$$T = [t_i]_{i=\ell}^{2m+\ell-2},$$

with $t_i = t_{pq}$, where $i = (m-1) - (p-q) + \ell$ and $p, q = 1, \ldots, m$.

Since $|\omega| = 1$, we assume w.l.o.g. that $\omega = 1$ in the following discussion for convenience.

Lemma 4.3. The Toeplitz matrix $\Gamma_{k,m}$ in (4.4) is nonsingular for sufficiently large $k$ and satisfies

$$\|\Gamma_{k,m}^{-1}\| = O(2^{-k}), \quad \|\Gamma_{\omega}^{2k} \Gamma_{k,m}^{-1} \Gamma_{\omega}^2\| = O(2^{-k}),$$

where $J_\omega \equiv J_{\omega,m}$ is defined in (4.2).

Proof. The Toeplitz matrix $T = \begin{bmatrix} 1_{m!} \end{bmatrix}_{i=1}^{2m-1}$ can be factorized by

$$T = \text{diag} \left\{ \frac{1}{(2m-1)!}, \ldots, \frac{1}{m!} \right\} F_m(m) \Pi,$$

where $\Pi = [e_\mu, \ldots, e_1]$ and $F_m(m)$ is given by (4.5) with $r = m$. From Lemma 4.2, $T$ is nonsingular. We write $\Gamma_{k,m}$ of (4.4) in the form

$$\Gamma_{k,m} = 2^k D_1 \tilde{T} D_2,$$

where $D_1 = \text{diag}(2^{(m-1)k}, \ldots, 2^k, 1)$, $D_2 = \text{diag}(1, 2^k, \ldots, 2^{(m-1)k})$ and $\tilde{T} = T + O(2^{-k})$. Similarly, $J_\omega^{2k} = D_2^{-1} \tilde{J}_1 D_2 = D_1 \tilde{T}_2 D_1^{-1}$, where $\tilde{J}_1 = I + O(2^{-k})$ and $\tilde{J}_2 = I + O(2^{-k})$. With these formulae, (4.7) obviously holds. □

For the unimodular eigenvalues $\omega_j = e^{i\theta_j}$ of $(\mathcal{M}, \mathcal{L})$ with an even partial multiplicity $p = 2m_j$, we have

$$J_{\omega_j,2m_j} = \begin{bmatrix} J_{\omega_j,m_j} & \Gamma_{1,m_j} \\ 0_{m_j} & J_{\omega_j,m_j} \end{bmatrix}, \quad \Gamma_{1,m_j} = e_{m_j}e_1^T$$

for $j = 1, \ldots, r$. From the symplectic Kronecker’s Theorem for $(\mathcal{M}, \mathcal{L})$ (see [32]), there exist a symplectic matrix $\widehat{\mathcal{F}}$ (i.e., $\widehat{\mathcal{F}}^H J \widehat{\mathcal{F}} = J$) and a nonsingular $\widehat{\mathcal{D}}$ such that

$$\widehat{\mathcal{D}} \mathcal{M} \widehat{\mathcal{F}} = \begin{bmatrix} J_s \oplus J_1 & 0_{\ell} \oplus \hat{T}_1 \\ 0_{\ell} \oplus J_1 & I_{\ell} \oplus J_{1,H} \end{bmatrix},$$

$$\widehat{\mathcal{D}} \mathcal{L} \widehat{\mathcal{F}} = \begin{bmatrix} I_{\ell} \oplus I_{\mu} & 0_{n} \\ 0_{n} & J_s^H \oplus I_{\mu} \end{bmatrix},$$

where $J_s \in \mathbb{C}^{\ell \times \ell}$ consists of asymptotically stable Jordan blocks (i.e., $\rho(J_s) < 1$),

$$J_1 = J_{\omega_1,m_1} \oplus \cdots \oplus J_{\omega_r,m_r}, \quad \hat{T}_1 = \hat{T}_{1,m_1} \oplus \cdots \oplus \hat{T}_{1,m_r},$$

with $\hat{T}_1,m_j = e_{m_j}e_j^T (j = 1, \ldots, r)$, $\ell = n - (m_1 + \cdots + m_r) \equiv n - \mu$ and $\oplus$ denotes the direct sum of matrices.

Based on the standard Weierstrass form and the special eigen-structure shown in (4.11), there exists a suitable nonsingular $W_j \in \mathbb{C}^{m_j \times m_j}$, for $j = 1, \ldots, r$, such that
$$\begin{bmatrix} I_{m_j} & 0 \\ 0 & W_j^{-1} \end{bmatrix} \begin{bmatrix} J_{\omega_j, m_j} & \hat{T}_{1, m_j} \\ 0 & J_{\omega_j, m_j}^{-1} \end{bmatrix} \begin{bmatrix} I_{m_j} & 0 \\ 0 & W_j \end{bmatrix} = \begin{bmatrix} J_{\omega_j, m_j} & \Gamma_{1, m_j} \\ 0 & J_{\omega_j, m_j} \end{bmatrix}. \quad (4.13)$$

Let
$$\mathcal{X} = \hat{\mathcal{Y}}(I_{n+\ell} \oplus W_1 \oplus \cdots \oplus W_r), \quad \mathcal{Y} = (I_{n+\ell} \oplus W_1^{-1} \oplus \cdots \oplus W_r^{-1}) \hat{\mathcal{Y}}. \quad (4.14)$$

Then from (4.13) and (4.14), the equations (4.11a) and (4.11b), respectively, become

$$\mathcal{M} \mathcal{Z} = \begin{bmatrix} J_s \oplus J_1 \oplus 0_\ell \oplus \Gamma_1 \\ 0_n \oplus I_\ell \oplus J_1 \end{bmatrix} \equiv J_{\mathcal{M}}, \quad (4.15a)$$

$$\mathcal{Q} \mathcal{L} \mathcal{Z} = \begin{bmatrix} I_n \oplus 0_n \oplus J_s \oplus I_{\mu} \\ 0_n \oplus J_s^H \oplus 0_n \oplus I_{\mu} \end{bmatrix} \equiv J_{\mathcal{L}}, \quad (4.15b)$$

where \(\Gamma_1 = \Gamma_{1, m_1} \oplus \cdots \oplus \Gamma_{1, m_j}\) with \(\Gamma_{1, m_j}\) being given in (4.10). Since \(J_{\mathcal{M}}\) and \(J_{\mathcal{L}}\) in (4.15) commute with each other and from (4.15), one can derive

$$\mathcal{M} \mathcal{Z} J_{\mathcal{L}} = \mathcal{Q}^{-1} J_{\mathcal{L}} J_{\mathcal{M}} = \mathcal{L} \mathcal{Z} J_{\mathcal{M}}. \quad (4.16)$$

From (4.11) and (4.14), it follows that \(\text{span} \{ \mathcal{Z}(\cdot, 1 : n) \}\) forms the unique stable Lagrangian subspace of \((\mathcal{M}, \mathcal{L})\) corresponding to \(J_s \oplus J_1\). On the other hand, if we interchange the roles of \(\mathcal{M}\) and \(\mathcal{L}\) in (4.15) and consider the symplectic pair \((\mathcal{L}, \mathcal{M})\), there are nonsingular matrices \(\mathcal{P}\) and \(\mathcal{Y}\) such that

$$\mathcal{P} \mathcal{L} \mathcal{Y} = \mathcal{J}_{\mathcal{M}}, \quad \mathcal{P} \mathcal{M} \mathcal{Y} = \mathcal{J}_{\mathcal{L}}. \quad (4.17)$$

Similar arguments also produce

$$\mathcal{L} \mathcal{Y} \mathcal{P} = \mathcal{M} \mathcal{Y} \mathcal{P}, \quad (4.18)$$

where \(\text{span} \{ \mathcal{Y}(\cdot, 1 : n) \}\) forms the unique stable Lagrangian subspace of \((\mathcal{L}, \mathcal{M})\) corresponding to \(J_s \oplus J_1\).

Let \((\mathcal{M}_k, \mathcal{L}_k)\) be the sequence of symplectic pairs generated by Algorithm 2.1 (see (2.4)), or \((\mathcal{M}_k, \mathcal{L}_k)\) be the sequence of symplectic pairs in SSF form with

$$\mathcal{M}_k = \begin{bmatrix} A_k & 0 \\ -H_k & I \end{bmatrix}, \quad \mathcal{L}_k = \begin{bmatrix} I & G_k \\ 0 & A_k^H \end{bmatrix} \quad (4.19)$$

generated by Algorithm 2.2 (see (2.9a)). With \(\mathcal{M}_0 = \mathcal{M}, \mathcal{L}_0 = \mathcal{L}\), from (4.16) as well as (2.1) and (2.2) or (2.6) and (2.7), it follows that

$$\mathcal{M}_1 \mathcal{Z} J_{\mathcal{L}}^2 = \mathcal{M}_* \mathcal{M}_0 \mathcal{Z} J_{\mathcal{L}}^2 = \mathcal{M}_* \mathcal{L}_0 \mathcal{Z} J_{\mathcal{M}} J_{\mathcal{L}} = \mathcal{L}_* \mathcal{M}_0 \mathcal{Z} J_{\mathcal{L}} J_{\mathcal{M}}. \quad (4.20)$$

By induction, we have

$$\mathcal{M}_k \mathcal{Z} J_{\mathcal{L}}^{2^k} = \mathcal{L}_k \mathcal{Z} J_{\mathcal{M}}^{2^k}. \quad (4.21)$$

By the result of Lemma 4.1 with \(p = 2m_j\) and the definitions of \(\Gamma_{k, m_j}, J_{\mathcal{M}}\) and \(J_{\mathcal{L}}\) in (4.4), (4.15a) and (4.15b), Eq. (4.21) can be rewritten as

$$\mathcal{M}_k \mathcal{Z} \begin{bmatrix} I_n & 0_n \\ 0_n & (J_s^H)^{2^k} \oplus I_{\mu} \end{bmatrix} = \mathcal{L}_k \mathcal{Z} \begin{bmatrix} J_s^{2^k} \oplus J_1^{2^k} & 0_\ell \oplus \Gamma_k \\ 0_n & I_\ell \oplus J_1^{2^k} \end{bmatrix}, \quad (4.22)$$
where
\[ \Gamma_k \equiv \Gamma_{k,m_1} \oplus \cdots \oplus \Gamma_{k,m_r} \tag{4.23} \]
with \( \Gamma_{k,m_j} \) being defined as in (4.4), for \( j = 1, \ldots, r \). Similarly, from (4.18) it also holds
\[ L_k \hat{J}_{\mathcal{L}}^{2k} = M_k \hat{J}_{\mathcal{L}}^{2k}. \tag{4.24} \]

**Lemma 4.4.** Let \( J_1 \) and \( \Gamma_k \) be defined in (4.12) and (4.23), respectively. Then \( \Gamma_k \) is invertible and satisfies
\[ \| \Gamma_k^{-1} J_1^{2k} \| = O(2^{-k}), \quad \| J_1^{2k} \Gamma_k^{-1} \| = O(2^{-k}). \tag{4.25} \]

**Proof.** By Lemma 4.3. \( \square \)

We now partition \( \mathcal{L} \) and \( \mathcal{Y} \) in (4.16) and (4.18):
\[ \mathcal{L} = \begin{bmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{bmatrix}, \tag{4.26} \]
where \( Z_i, Y_i \in \mathbb{C}^{n \times n} \), for \( i = 1, 2, 3, 4 \).

**Theorem 4.1.** Let \( (\mathcal{M}, \mathcal{L}) \) be given in (1.5) satisfying (A2). Suppose the corresponding DARE (1.2) has a weakly stabilizing Hermitian solution \( X \) with property (P). Then, the sequence \( \{ (\mathcal{M}_k, \mathcal{L}_k) \} \) (see (2.4)) generated by Algorithm 2.1 satisfies
\[ \| \mathcal{M}_k(\cdot; 1 : n) + \mathcal{M}_k(\cdot; n + 1 : 2n)X \| \leq O(\rho(J_2)2^k) + O(2^{-k}) \to 0, \quad \text{as } k \to \infty. \tag{4.27} \]

**Proof.** By the assumption it holds that \( (\mathcal{M}, \mathcal{L}) \) has a unique stable Lagrangian subspace of the form \( \text{span}([I, X^T]^T) \) satisfying (1.8), where \( \Phi \) is similar to \( J_s \oplus J_1 \) as in (4.11). From (4.16) and (4.26), we have \([I, X^T]^T \) and \([Z_1^T, Z_2^T]^T \) spanning the same Lagrangian subspace corresponding to \( J_s \oplus J_1 \). So, \( Z_1 \) is invertible and \( X = Z_2 Z_1^{-1} \).

To show (4.27), we partition \( \mathcal{M}_k, \mathcal{L}_k \) conformally with (4.26) into
\[ \mathcal{M}_k = \begin{bmatrix} M_{k,1} & M_{k,3} \\ M_{k,2} & M_{k,4} \end{bmatrix}, \quad \mathcal{L}_k = \begin{bmatrix} L_{k,1} & L_{k,3} \\ L_{k,2} & L_{k,4} \end{bmatrix}. \tag{4.28} \]

Substituting (4.26) and (4.28) into (4.22), we have
\[ M_{k,1} Z_1 + M_{k,3} Z_2 = L_{k,1} Z_1 (J_s^{2k} \oplus J_1^{2k}) + L_{k,3} Z_2 (J_s^{2k} \oplus J_1^{2k}), \tag{4.29a} \]
\[ (M_{k,1} Z_3 + M_{k,3} Z_4) ((J_s^H)^{2k} \oplus I_\mu) = L_{k,1} [Z_1 (0_\ell \oplus \Gamma_k) + Z_3 (I_\ell \oplus J_1^{2k})] + L_{k,3} [Z_2 (0_\ell \oplus \Gamma_k) + Z_4 (I_\ell \oplus J_1^{2k})], \tag{4.29b} \]
\[ M_{k,2} Z_1 + M_{k,4} Z_2 = L_{k,2} Z_1 (J_s^{2k} \oplus J_1^{2k}) + L_{k,4} Z_2 (J_s^{2k} \oplus J_1^{2k}), \tag{4.29c} \]
\[ (M_{k,2} Z_3 + M_{k,4} Z_4) ((J_s^H)^{2k} \oplus I_\mu) = L_{k,2} [Z_1 (0_\ell \oplus \Gamma_k) + Z_3 (I_\ell \oplus J_1^{2k})] + L_{k,4} [Z_2 (0_\ell \oplus \Gamma_k) + Z_4 (I_\ell \oplus J_1^{2k})]. \tag{4.29d} \]

Postmultiplying (4.29b) by \((0_\ell \oplus \Gamma_k^{-1} J_1^{2k}) Z_1^{-1}\) to eliminate \( L_{k,1} Z_1 (0_\ell \oplus J_1^{2k}) \) and \( L_{k,3} Z_2 (0_\ell \oplus J_1^{2k}) \) in (4.29a), we get
Similarly, from (4.29c) and (4.29d), we obtain
\begin{align*}
M_{k,2} + M_{k,4}X &= (M_{k,2}Z_3 + M_{k,4}Z_4)(0_\ell \oplus \Gamma_k^{-1}J_1^{2k})Z_1^{-1} \\
&\quad - (L_{k,2}Z_3 + L_{k,4}Z_4)(0_\ell \oplus J_1^{2k}\Gamma_k^{-1}J_1^{2k})Z_1^{-1} \\
&\quad + (L_{k,2}Z_1 + L_{k,4}Z_2)(J_2^{k} \oplus 0_\mu)Z_1^{-1}. \\
\end{align*}
(4.31)

By Algorithm 2.1 or 2.4, we deduce that \( L_k = L_{*,k-1}, M_k = M_{*,k-1, \mathcal{M}_{k-1}} \) with \( \|L_{*,k-1}\| \leq 1 \) and \( \|M_{*,k-1}\| \leq 1 \) for all \( k \). So we have \( \|L_k\| \leq \|L_0\|, \|M_k\| \leq \|M_0\| \) and therefore \( \|M_{k,i}\| \) and \( \|L_{k,i}\| \) are bounded, for \( i = 1, \ldots, 4 \). From (4.30) and (4.31) and Lemma 4.4, assertion (4.27) follows. \( \square \)

**Theorem 4.2.** Let \((\mathcal{M}, \mathcal{L})\) be given in (1.5) satisfying (A2). Suppose the corresponding DARE (1.2) and the dual DARE
\begin{align*}
Y &= AY(I + HY)^{-1}A^H + G \\
(4.32)
\end{align*}

have weakly stabilizing Hermitian solutions \( X \) and \( Y \) with property (P), respectively. If the sequence \((A_k, G_k, H_k)\) generated by Algorithm 2.2 (see (2.9)) is well-defined, then

(i) \( \|A_k\| \leq O(2^{-k}) \to 0 \), as \( k \to \infty \),

(ii) \( \|X - H_k\| \leq O(\rho(J_s^{2k}) + O(2^{-k}) \to 0 \), as \( k \to \infty \),

(iii) \( \|Y - G_k\| \leq O(\rho(J_s^{2k}) + O(2^{-k}) \to 0 \), as \( k \to \infty \),

(iv) \( I + G_kH_k \) approaches a singular matrix as \( k \to \infty \).

**Proof.** Applying the same argument in the proof of Theorem 4.1, it holds that \( Z_1^{-1} \) exists and \( X = Z_2Z_1^{-1} \). Substituting \((\mathcal{M}_k, \mathcal{L}_k)\) of (4.19) and \( \mathcal{L} \) of (4.26) into (4.22) and comparing both sides, we obtain
\begin{align*}
A_kZ_1 &= (Z_1 + G_kZ_2)(J_2^{k} \oplus J_1^{2k}), \\
A_kZ_3(J_2^{H}J_1^{2k} \oplus I_\mu) &= (Z_1 + G_kZ_2)(0_\ell \oplus \Gamma_k) + (Z_3 + G_kZ_4)(I_\ell \oplus J_1^{2k}), \\
-H_kZ_1 + Z_2 &= A_k^TZ_2(J_2^{k} \oplus J_1^{2k}), \\
(-H_kZ_3 + Z_4)(J_2^{H}J_1^{2k} \oplus I_\mu) &= A_k^TZ_2(0_\ell \oplus \Gamma_k) + A_k^TZ_4(I_\ell \oplus J_1^{2k}).
(4.33a)
(4.33b)
(4.33c)
(4.33d)
\end{align*}
Postmultiplying (4.33b) by \( (0_\ell \oplus \Gamma_k^{-1}J_1^{2k})Z_1^{-1} \) and using (4.33a), we have
\begin{align*}
A_k[I - Z_3(0_\ell \oplus \Gamma_k^{-1}J_1^{2k})Z_1^{-1}] &= (Z_1 + G_kZ_2)(J_2^{k} \oplus 0_\mu)Z_1^{-1} \\
&\quad - (Z_3 + G_kZ_4)(0_\ell \oplus J_1^{2k}\Gamma_k^{-1}J_1^{2k})Z_1^{-1}. \\
(4.34)
\end{align*}

On the other hand, Eq. (4.32) can be written into
\begin{align*}
\begin{bmatrix}
I & G \\
0 & A^H
\end{bmatrix}
\begin{bmatrix}
-Y \\
I
\end{bmatrix} &= \begin{bmatrix}
A & 0 \\
-H & I
\end{bmatrix}
\begin{bmatrix}
-Y \\
I
\end{bmatrix} \Phi. \\
(4.35)
\end{align*}
where $\Phi^T \tL_k \oplus \tI_1$. From (4.18) and (4.35) we see that $\text{span}(\{-Y^T, I^T\})$ and $\text{span}(Y^T_1, Y^T_2)$ form the unique stable Lagrangian subspace corresponding to $\tL_k \oplus \tI_1$, i.e., $Y_2^{-1}$ exists and $Y = -Y_1 Y_2^{-1}$.

Substituting $(\tL_k, \tM_k)$ of (4.19) and $Y$ of (4.26) into (4.24), we have

\begin{align}
Y_1 + G_k Y_2 &= A_k Y_1 (\tJ^2 k \oplus \tJ^2 k_1), \\
(Y_3 + G_k Y_4)((J^2 k_1) \oplus I_\mu) &= A_k Y_1 (0_\ell \oplus \tI_k) + A_k Y_3 (I_\ell \oplus \tJ^2 k_1).
\end{align}

(4.36a, 4.36b)

As above, postmultiplying (4.36b) by $(0_\ell \oplus \tI_k^{-1} \tJ^2 k_1 Y_2^{-1}$ and using (4.36a), we get

\begin{align}
-Y + G_k [I - Y_4 (0_\ell \oplus \tI_k^{-1} \tJ^2 k_1 Y_2^{-1}]
&= Y_3 (0_\ell \oplus \tI_k^{-1} \tJ^2 k_1 Y_2^{-1}) - A_k Y_3 (0_\ell \oplus \tJ^2 k \tI_k^{-1} \tJ^2 k_1 Y_2^{-1} + A_k Y_1 (\tJ^2 k_1 + I) Y_2^{-1}).
\end{align}

(4.37)

Then (4.37) can be rewritten into

\begin{align}
G_k ([I + O_n (2^{-k})) = Y + O_n (2^{-k}) + A_k (O_n (2^{-k}) + O_n (\rho (J_s))^{2k}),
\end{align}

(4.38)

where $O_n (2^{-k})$ and $O_n (\rho (J_s))^{2k}$ denotes some suitable $n \times n$ matrices with entries of the quantity $O(2^{-k})$ and $O(\rho (J_s))^{2k}$, respectively. Substituting $G_k$ in (4.38) into (4.34) and by Lemma 4.4 we get

\begin{align}
\|A_k\| \leq O(2^{-k}) + O(\rho (J_s))^{2k} \to 0
\end{align}

(4.39)

as $k \to \infty$. This proved (i).

Postmultiplying (4.33d) by $(0_\ell \oplus \tI_k^{-1} J^2 k_1 Z_1^{-1}$ and using (4.33c), we get

\begin{align}
-H_k[I - Z_3 (0_\ell \oplus \tI_k^{-1} J^2 k_1 Z_1^{-1})] + X
&= Z_4 (0_\ell \oplus \tI_k^{-1} J^2 k_1) Z_1^{-1} - A_k^T Z_4 (0_\ell \oplus J^2 k_1 \tI_k^{-1} J^2 k_1) Z_1^{-1} + A_k^T Z_2 (J^2 k_1 + I_\mu) Z_1^{-1}.
\end{align}

(4.40)

By Lemma 4.4 and (4.39), the matrix $X - H_k$ in (4.40) can be bounded by

\begin{align}
\|X - H_k\| \leq O(\rho (J_s))^{2k} + O(2^{-k}) \to 0
\end{align}

(4.41)

for $k \to \infty$. This proved (ii).

Similarly, by Lemma 4.4 and (4.39), the matrix $Y - G_k$ can be estimated by

\begin{align}
\|Y - G_k\| \leq O(\rho (J_s))^{2k} + O(2^{-k}) \to 0,
\end{align}

for $k \to \infty$. Therefore, it holds (iii).

Claim (iv): From (4.33a) and (4.33c), we have

\begin{align}
(A_k Z_1 = (I + G_k X) Z_1 (J^2 k_1 \oplus J^2 k_1)), \\
-A_k + X = A_k^T Z_2 (J^2 k_1 \oplus J^2 k_1) Z_1^{-1}.
\end{align}

(4.42, 4.43)

Multiplying (4.43) by $G_k$, we get

\begin{align}
-(I + G_k H_k) + (I + G_k X) = G_k A_k^T Z_2 (J^2 k_1 \oplus J^2 k_1) Z_1^{-1},
\end{align}

(4.44)
which implies that
\[
(I + G_k H_k)Z_1 - (I + G_k X)Z_1 = -G_k A_k^T Z_2 (J_s^{2k} \oplus J_1^{2k}).
\] (4.44)

Postmultiplying (4.44) by \((J_s^{2k} \oplus J_1^{2k})\) and using the result in (4.42), then
\[
(I + G_k H_k)Z_1 (J_s^{2k} \oplus J_1^{2k}) = A_k Z_1 - G_k A_k^T Z_2 (J_s^{2k} \oplus J_1^{2k}).
\] (4.45)

Postmultiply (4.45) by \((0_\ell \oplus J_1^{-2k})\) to get
\[
(I + G_k H_k)Z_1 \begin{bmatrix} 0 \\ I_\mu \end{bmatrix} = A_k Z_1 \begin{bmatrix} 0 \\ J_1^{-2k} \end{bmatrix} - G_k A_k^T Z_2 \begin{bmatrix} 0 \\ J_1^{2k} \end{bmatrix}.
\] (4.46)

The first column of (4.46) becomes
\[
(I + G_k H_k)Z_1 \begin{bmatrix} 0 \\ e_1 \end{bmatrix} = A_k Z_1 \begin{bmatrix} 0 \\ \omega_1^{-2k} e_1 \end{bmatrix} - G_k A_k^T Z_2 \begin{bmatrix} 0 \\ \omega_1^{2k} e_1 \end{bmatrix},
\]
which converges to zero vector as \(k \to \infty\), by using results of (i) and (iii). Therefore, \(I + G_k H_k\) converges to a singular matrix as \(k \to \infty\). \(\square\)

From (4.34), if \(\|G_k\| \leq O(2^k)\) for all \(k\), then the sequence \(\{A_k\}\) satisfies
\[
\|A_k\| \leq O(2^k) \cdot O(\rho(J_s)^{2^k}) + O(2^k) \cdot O(2^{-k}) = O(1).
\] (4.47)

By Lemma 4.4, (4.40) and (4.47), Inequality (4.41) holds which implies that the sequence \(\{H_k\}\) linearly converges to \(X\). We summarize this result in the following corollary.

**Corollary 4.1.** Let \((\mathcal{M}, \mathcal{L})\) be given in (1.5) satisfying (A2). Suppose the corresponding DARE (1.2) has weakly stabilizing Hermitian solution \(X\) with property (P), but the dual DARE (4.32) does not have. If the sequence \(\{(A_k, G_k, H_k)\}\) generated by Algorithm 2.2 is well-defined and \(\|G_k\| \leq O(2^k)\) for all \(k\), then the sequence \(\{H_k\}\) globally and linearly converges to \(X\).

Using a similar argument as in Theorems 4.1 and 4.2, we conclude the following theorem for CAREs.

**Theorem 4.3.** Let \(\mathcal{H}\) be given in (1.3) satisfying Assumption (A1). Suppose the corresponding CARE (1.1) has a weakly stabilizing Hermitian solution \(X\) with property (P).

(i) If \(\{(\mathcal{M}_k, \mathcal{L}_k)\}\) is generated by the C-MDFM in Algorithm 3.1, then (4.27) holds.

(ii) If, in addition, the dual CARE
\[
-YHY + AY + YA^H + G = 0
\] (4.48)
has a weakly stabilizing Hermitian solution \(Y\) with property (P), and the sequence \(\{(A_k, G_k, H_k)\}\) generated by C-SDA in Algorithm 3.1 is well-defined, then (i)–(iv) in Theorem 4.2 holds.

5. Numerical results

In this section, we test the SDA for DAREs and CAREs on six numerical examples, under the less restrictive Assumptions (A1) and (A2), to illustrate their convergence behavior. All computations were performed in MATLAB R2006a on a PC with an Intel Pentium-IV 3.4 GHz
processor and 2 GB main memory, using IEEE double-precision floating-point arithmetic ($\varepsilon \approx 2.22 \times 10^{-16}$). The operating system running the machine is Fedora Core 2 with Kernel 2.6.10-1.771-FC2.

The MATLAB commands “dare” and “care” fail to converge because the associated symplectic pencil and Hamiltonian matrix have eigenvalues on the unit circle and the imaginary axis, respectively. Furthermore, the strongly stable method [10] and matrix sign function methods [5,8,14,16] fail to converge because the associated Hamiltonian matrix has purely imaginary eigenvalues. Therefore, we only report on the SDA algorithms [11,12], MDFM [6,7] and Newton’s methods [20,21]. We summarize the flop counts for each iteration in the SDAs, MDFMs and NTMs in Table 1.

In Tables 2 and 3, data for various methods are listed in columns with obvious headings. The heading “D-SDA”, “D-MDFM” and “D-NTM” stand for Algorithm 2.2, Algorithm 2.1 and Newton’s method [20] applied to the DAREs, respectively. “C-SDA” and “C-MDFM” stand for Algorithm 3.1 while calling Algorithm 2.2 and 2.1, respectively, in Step (III). “C-NTM” stands for Newton’s method [21] applied to the CAREs.

We report the numbers of iterations by “ITs”, the total flops (= Flops \times ITs) by “TFs”, and the maximal error between the accurate and the approximate stable eigenvalues of $(I + G\tilde{X})^{-1}A$ or $A - G\tilde{X}$ by “Err”, where $\tilde{X}$ is an approximate solution to the DAREs or CARE. Let $\lambda_i$ be the exact stable eigenvalue and $\tilde{\lambda}_i$ be the corresponding approximate eigenvalue, then “Err” is defined by

$$\text{Err} = \max_{1 \leq i \leq n} |\lambda_i - \tilde{\lambda}_i|.$$
Table 3
Results for Examples 5.4, 5.5 and 5.6

<table>
<thead>
<tr>
<th></th>
<th>C-SDA</th>
<th>C-MDFM</th>
<th>C-NTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 5.4</td>
<td>CNRes 5.23 x 10^{-15}</td>
<td>1.43 x 10^{-13}</td>
<td>1.28 x 10^{-16}</td>
</tr>
<tr>
<td>ITs</td>
<td>5</td>
<td>27</td>
<td>14</td>
</tr>
<tr>
<td>TFs(n^3)</td>
<td>1912</td>
<td>3168</td>
<td>420</td>
</tr>
<tr>
<td>Err</td>
<td>6.71 x 10^{-8}</td>
<td>2.06 x 10^{-7}</td>
<td>5.27 x 10^{-9}</td>
</tr>
<tr>
<td>Example 5.5</td>
<td>CNRes 1.00 x 10^{-16}</td>
<td>3.25 x 10^{-9}</td>
<td>5.69 x 10^{-12}</td>
</tr>
<tr>
<td>ITs</td>
<td>9</td>
<td>17</td>
<td>36</td>
</tr>
<tr>
<td>TFs(n^3)</td>
<td>69</td>
<td>19942/7</td>
<td>1080</td>
</tr>
<tr>
<td>Err</td>
<td>3.27 x 10^{-8}</td>
<td>5.08 x 10^{-4}</td>
<td>1.80 x 10^{-5}</td>
</tr>
<tr>
<td>Example 5.6</td>
<td>CNRes 7.51 x 10^{-13}</td>
<td>6.07 x 10^{-9}</td>
<td>*</td>
</tr>
<tr>
<td>ITs</td>
<td>20</td>
<td>17</td>
<td>*</td>
</tr>
<tr>
<td>TFs(n^3)</td>
<td>153</td>
<td>1995</td>
<td>*</td>
</tr>
<tr>
<td>Err</td>
<td>6.56 x 10^{-2}</td>
<td>5.55 x 10^{-2}</td>
<td>*</td>
</tr>
</tbody>
</table>

The algorithm is terminated when the residual of the DARE or CARE cannot be reduced further.

5.1. Discrete-time algebraic Riccati equations

In this subsection, we report the numerical results of the D-SDA, D-MDFM and D-NTM. When the size p of J_ω,p in (4.2) is two, the theoretical rate of convergence for D-NTM is either quadratic or linear with rate 1/2 but quadratic convergence has not been observed in practice (see [20]). If p > 2, the convergence of the D-NTM is guaranteed but the rate of convergence is unknown. Furthermore, the initial matrix X_0 for the D-NTM is generated by choosing an initial matrix L_0 so that A_0 ≡ A - BL_0 is d-stable and taking X_0 to be the unique solution of the Stein equation

\[ X_0 - A_0^T X_0 A_0 = H + L_0^T R L_0, \]

where B and R satisfy G = BR^{-1}B^T.

In the following, we shall report three examples to illustrate the linear convergence of the D-SDA, D-MDFM and D-NTM with different values of p. In Example 5.1, p = 2 and G ≡ BB^T > 0 with (A, B) being d-stabilizable, i.e., if w^H A = μ w^H and w^H B = 0 with w ≠ 0, then |μ| < 1. In Example 5.2, p = 6 and G ≡ BB^T ≥ 0 with (A, B) d-stabilizable. Therefore, the convergence of the D-NTM for Examples 5.1 and 5.2 is guaranteed by choosing an initial matrix L_0 so that A_0 ≡ A - BL_0 is d-stable. In Example 5.3, p = 2 and G ≡ BB^T > 0 while (A, B) is not d-stabilizable, so that the convergence of D-NTM is not guaranteed.

For the residual of DAREs, we use the “normalized” residual (DNRes) formula

\[ DNRes \equiv \frac{||A^T \tilde{X}(I + G \tilde{X})^{-1} A + H - \tilde{X}||}{||\tilde{X}|| + ||A^T \tilde{X}(I + G \tilde{X})^{-1} A|| + ||H||}, \]

proposed in [7], where \( \tilde{X} \) is an approximate solution to the DARE. The numerical results from the D-SDA, D-MDFM and D-NTM for computing \( \tilde{X} \) are reported in Table 2.
Example 5.1 ([34, Example 2.2]). Let
\[
A = \begin{bmatrix} 0 & -\frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & 0 \end{bmatrix}, \quad H = -I_2,
\]
\[
G = 5 \left( \frac{\sqrt{5} - 1}{2} \right)^2 I_2 = BB^H > 0 \quad \text{with} \quad B = \left( \frac{5 - \sqrt{5}}{2} \right) I_2.
\]
Then the symplectic pencil \((\mathcal{M}, \mathcal{L})\) has eigenvalues \(\{1, -1\}\) with the partial multiplicities \(\{(2, 2)\}\) and \((A, B)\) is d-stabilizable.

Matrix \(L_0\) in the D-NTM is taken as a normally distributed random matrix with state \(\begin{bmatrix} 117982445 \\ 4147882577 \end{bmatrix}^T\). From Table 2, we see that the normalized residuals (or the backward error) for \(\tilde{X}\) from the D-SDA and D-NTM have 16 significant digits, attaining machine accuracy. It is only 12 significant digits from the D-MDFM.

Compared to the DNRes (which attains machine accuracy), the forward errors of stable eigenvalues have only eight and six significant digits for the D-SDA and D-MDFM, respectively. This is due to the poor separation between the \(d\)-stable and \(d\)-unstable subspectra of \((\mathcal{M}, \mathcal{L})\) [45,46].

Example 5.2. Let
\[
\mathcal{M}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & I_7 \end{bmatrix} \quad \text{and} \quad \mathcal{L}_0 = \begin{bmatrix} I_7 & G_0^T \\ 0 & A_0^T \end{bmatrix},
\]
with
\[
A_0 = \begin{bmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \oplus \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{2} & -1 \end{bmatrix} \oplus \left( -\frac{1}{3} I_2 \right)
\]
and
\[
G_0 = \begin{bmatrix} \frac{1}{32} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{2} \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{32} & \frac{1}{128} & \frac{1}{32} \\ \frac{1}{128} & \frac{1}{32} & \frac{1}{8} \\ \frac{1}{32} & \frac{1}{8} & \frac{1}{2} \end{bmatrix} \oplus \left( \frac{2}{9} I_2 \right).
\]
The symplectic pencil \((\mathcal{M}_0, \mathcal{L}_0)\) has eigenvalues \(-3, -\frac{1}{3}, -1\) with partial multiplicities \{(1, 1), (1, 1), (4, 6)\}. Take \(H_0 = -3.5 I_7\) so that \(I - G_0 H_0\) is nonsingular. Define a new symplectic pencil \((\mathcal{M}, \mathcal{L})\) by the equivalence transformation
\[
\mathcal{M} = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} \equiv \begin{bmatrix} (I - G_0 H_0)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0^T H_0 (I - G_0 H_0)^{-1} & 0 \\ 0 & I \end{bmatrix}
\]
\[
\times \begin{bmatrix} A_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -H_0 \end{bmatrix}
\]  \hspace{1cm} (5.1a)
and
\[
\mathcal{L} = \begin{bmatrix} I & G \\ 0 & A^T \end{bmatrix} \equiv \begin{bmatrix} (I - G_0 H_0)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A_0^T H_0 (I - G_0 H_0)^{-1} & I \end{bmatrix}
\]
\[
\times \begin{bmatrix} I & G_0 \\ 0 & A_0^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -H_0 & I \end{bmatrix}.
\]  \hspace{1cm} (5.1b)
One can check that \((\mathcal{M}, \mathcal{L})\) satisfies Assumption (A2), \(H\) is indefinite and \(G = BB^T \succeq 0\) with \((A, B)\) being \(d\)-stabilizable.

From Table 2, we see that the backward error for \(\tilde{X}\) by the D-SDA has 13 significant digits which is better than that from the D-MDFM and D-NTM which \(L_0\) is a normally distributed random diagonal matrix with \(\text{state} = [2355717396, 3700125409]^T\). The forward errors of stable eigenvalues by using these three methods equal to \(\sqrt{\varepsilon p^2}\) approximately.

**Example 5.3** ([20, Example 6.2]). Let
\[
\mathcal{M}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & I_8 \end{bmatrix} \quad \text{and} \quad \mathcal{L}_0 = \begin{bmatrix} I_8 & G_0 \\ 0 & A_0^T \end{bmatrix},
\]
with
\[
A_0 = \begin{bmatrix} -1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 1 & 1 \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}
\]
and
\[
G_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \cdots & \cdots \\ 1 & 1 \end{bmatrix}^T \in \mathbb{R}^{8 \times 8}.
\]
The symplectic pencil \((\mathcal{M}_0, \mathcal{L}_0)\) has eigenvalues \(\{2, \frac{1}{2}, -1, 1, -\frac{\sqrt{3}-1}{2}, -\frac{\sqrt{3}+1}{2}\}\) with partial multiplicities \(\{(3), (3), (2), (2, 2), (2), (2)\}\). Set the “state” in the MATLAB command “randn” to be \([3648486896, 1858934981]^T\) and \(H_0\) to be a diagonal matrix with normally distributed random diagonal elements so that \(I - G_0H_0\) is nonsingular. Using the same equivalence transformation in (5.1), we get a new DARE which \(G = BB^T \succeq 0\) but \((A, B)\) is not \(d\)-stabilizable.

Matrix \(L_0\) in the D-NTM is taken to be a normally distributed random diagonal matrix with \(\text{state} = [2271789144, 1397129797]^T\). Note that although the sequence generated by the D-NTM converges to \(\tilde{X}\), it is not the weakly stabilizing Hermitian solution of the DARE because \(2.0000708\) is an eigenvalue of \((I + G\tilde{X})^{-1}A\).

### 5.2. Continuous-time algebraic Riccati equations

In this section, we report the numerical comparison of the C-SDA, C-MDFM and C-NTM using three examples. Note that under assumption of c-stability of \((A, B)\), i.e., if \(w^H A = \mu w^H\) and \(w^H B = 0\) with \(w \neq 0\), then \(\text{Re}(\mu) < 0\), the convergence of the C-NTM is guaranteed if \(A - GX_0\) is stable. When \(p = 2\), the rate of convergence is linear with rate 1/2. However, the rate of convergence is unknown if \(p > 2\). Consequently, we give three examples to illustrate the numerical behavior. In Examples 5.4 and 5.5 with \(p = 2\) and 4, respectively, and under assumption of c-stability of \((A, B)\), all the rates of convergence of the C-SDA, C-MDFM and C-NTM are linear. In Example 5.6, where the maximal size of \(J_{\omega, p}\) is eight and \(G\) is symmetric indefinite, the rates of convergence for the C-SDA and C-MDFM are also linear but the sequence generated by the C-NTM diverges.
For the residuals of the CAREs, we use the “normalized” residual (CNRes) formula
\[
CNRes \equiv \frac{\| - \tilde{X}G\tilde{X} + A^T\tilde{X} + \tilde{X}A + H \|}{\| \tilde{X}G\tilde{X} \| + \| A^T\tilde{X} \| + \| \tilde{X}A \| + \| H \|}
\]
proposed in [11], where \( \tilde{X} \) is an approximate solution to the CAREs. The numerical results by using the C-SDA, C-MDFM and C-NTM for computing \( \tilde{X} \) are reported in Table 3.

**Example 5.4** ([21, Example 4.3]). Define
\[
\mathcal{H} = \begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix} = \begin{bmatrix} cI & sI \\ -sI & cI \end{bmatrix} \begin{bmatrix} A_0 & -G_0 \\ 0 & -A_0^T \end{bmatrix} \begin{bmatrix} cI & -sI \\ sI & cI \end{bmatrix},
\]
where \( c = -0.9764866252937641, s = \sqrt{1-c^2}, \)
\[
A_0 = 0_2 \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \oplus \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}
\]
and
\[
G_0 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & \ddots \\ \ddots & \ddots & 1 \\ 1 & 1 & 2 \end{bmatrix} \in \mathbb{R}^{8 \times 8}.
\]
Then \( \mathcal{H} \) has eigenvalues \( \{0, 1, -1, 2\iota, -2\iota\} \) with partial multiplicities \( \{(2, 2), (2, 2), (2, 2), (2, 2)\} \) and \( G = BB^T > 0 \) with \( (A, B) \) being c-stabilizable.

The initial matrix \( X_0 \) in the C-NTM is a normally distributed random diagonal matrix with state \( = [4042373946, 473476633]^T \) so that \( A - G X_0 \) is stable. We take \( \gamma = 2.2 \) in the Cayley transformation for the C-SDA and C-MDFM.

**Example 5.5.** Let \( A_0 \) and \( G_0 \) be \( 5 \times 5 \) real matrices defined by
\[
A_0 = \begin{bmatrix} U & I_2 & 0 \\ 0 & U & 0 \\ 0 & 0 & a \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & g \end{bmatrix}
\]
where \( a, g \) are parameters and \( U = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, \beta \neq 0 \). The matrix \( \begin{bmatrix} A_0 & -G_0 \\ 0 & -A_0^T \end{bmatrix} \) has nonzero eigenvalues \( a, -a \) and the purely imaginary eigenvalues \( \beta \iota, -\beta \iota \) have partial multiplicities \( \{(4, 4)\} \).

Define
\[
\begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix} = \begin{bmatrix} P^T & 0 \\ 0 & P^T \end{bmatrix} \begin{bmatrix} C & S \end{bmatrix} \begin{bmatrix} A_0 & -G_0 \\ 0 & -A_0^T \end{bmatrix} \begin{bmatrix} C & -S \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix},
\]
where \( P = I_5 - 2uu^T \) is a Householder matrix and
\[
C = \begin{bmatrix} I_4 & 0 \\ 0 & c \end{bmatrix}, \quad S = \begin{bmatrix} 0_4 & 0 \\ 0 & s \end{bmatrix}.
\]
with \( c > 0, s > 0 \) and \( c^2 + s^2 = 1 \). Taking
\[
a = 0.6781616521431886, \quad \beta = 5.513985806849778, \\
g = 43.14437852853182, \quad c = 0.3559455724227920,
\]
\[
u = \begin{bmatrix}
0.8210373788415466 \\
0.436544378807448 \\
0.1651283156254210 \\
0.3286639287858224 \\
0.6263991871530291 \times 10^{-2}
\end{bmatrix},
\]
we have that \( G = BB^T > 0 \) with \((A, B)\) being \( c \)-stabilizable and \( H \) is negative semidefinite.

Take the initial \( X_0 \) in the C-NTM to be a normally distributed random symmetric matrix with state \( \begin{bmatrix} 2885252095 \\ 1305289620 \end{bmatrix}^T \) so that \( A - GX_0 \) is stable and \( \gamma = 35 \) in the Cayley transformation. From Table 3, we see that the CNRes for \( \tilde{X} \) from the C-SDA attains machine accuracy. The C-MDFM and C-NTM give only 9 and 12 significant digits, respectively. Furthermore, the accuracy of stable eigenvalues from the C-SDA is better than that from the C-MDFM and C-NTM.

**Example 5.6 ([33]).** Let
\[
A_0 = 0 \oplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus (-I_2),
\]
\[
H_0 = (-1) \oplus \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus (-I_2).
\]
Construct a Hamiltonian matrix \( \mathcal{H} \):
\[
\mathcal{H} = \begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix}
\equiv \begin{bmatrix} I & V_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & V_1^{-1} \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ H_0 & -A_0^T \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_1^T \end{bmatrix} \begin{bmatrix} I & -V_2 \\ 0 & I \end{bmatrix},
\]
where
\[
V_1 = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 1 & -1 & 2 & \cdots \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 7 & -1 & 8 \\ 0 & \cdots & \cdots & 8 & 1 \end{bmatrix}.
\]
It is easily seen that \( \mathcal{H} \) has eigenvalue \{0\} with partial multiplicities \{(2, 4, 8)\}. Note that here \( G \) is symmetric indefinite. Thus \( c \)-stabilizability does not hold.
Take $\gamma = 10.5$ in the Cayley transformation and the initial $X_0$ to be a normally distributed random symmetric matrix with state $\mathbf{state} = [2540448925, 601278700]^T$ so that $A - GX_0$ is stable. From Table 3, we see that the normalized residual for $\tilde{X}$ by the C-SDA and C-MDFM has 13 and 9 significant digits, respectively, while the C-NTM is divergent. The CNRes by the C-SDA lost 3 significant digits compared to machine accuracy, because the highest partial multiplicity of the zero eigenvalue of $\mathcal{H}$ is eight, making it very sensitive to perturbation [45,46]. The forward errors of the stable eigenvalues from the C-SDA and C-MDFM are approximately $\sqrt{\varepsilon}$.

5.3. Comments

(i) In Examples 5.1–5.4 and 5.6, $A_k$, $G_k$ and $H_k$ converge linearly and $I + G_k H_k$ tends to a singular matrix, matching the results (i)–(iv) of Theorem 4.2. Furthermore, before the matrices $I + G_k H_k$ approach ill-conditioning, the sequence \{\$H_k\$\} has converged well to the solutions of the AREs. Here, we only show the results of the Frobenius norms of $A_k$, $G_{k+1} - G_k$ and $H_{k+1} - H_k$ and the condition number of $I + G_k H_k$ in each iteration of the D-SDA for Example 5.3 in Fig. 1a–d, respectively.

(ii) In Example 5.5, the Frobenius norms of $A_k$, $G_{k+1} - G_k$ and $H_{k+1} - H_k$ and the condition number of $I + G_k H_k$ in each iteration of the D-SDA are shown in Fig. 2a–d, respectively. In Fig. 2c, the convergence behavior of $H_k$ coincides with the result (ii) of Theorem 4.2. However, $\|A_k\|$ and $\|G_k\|$ seem to be divergent as $k$ is increasing. Furthermore, according to Fig. 2d, the matrix $I + G_k H_k$ is well-conditioned in each iteration. Numerically results of Example 5.5 do not match the results (i), (iii) and (iv) of Theorem 4.2 because a weakly stabilizing Hermitian solution with property (P) of the dual CARE for this example does not exist.

(iii) From Table 1, the flop count in each iterative step of the SDA is about 7% of that of MDFM. This is mainly due to the fact that the main step in the MDFM involves the $QR$-factorization of $[\mathcal{L}^T, -\mathcal{M}^T]^T$ and the formation of $\mathcal{Z} \in \mathbb{C}^{4n \times 4n}$, all in higher dimensions. The operations in the SDA are all in $\mathbb{C}^{n \times n}$ while keeping the SSF form. Moreover, Tables 2 and 3 show that

Fig. 1. The Frobenius norms of $A_k$, $G_{k+1} - G_k$ and $H_{k+1} - H_k$ and the condition number of $I + G_k H_k$ for Example 5.3.
the approximate solutions $\tilde{X}$ from the SDA are more accurate than those from the MDFM. These behaviors illustrate the importance of the SSF form.

(iv) Examples investigated here are all ill-conditioned because the associated symplectic pencils or Hamiltonian matrices have eigenvalues on the unit circle or the purely imaginary axis. However, the SDA algorithms solve them efficiently and accurately without failure and with less flops counts.

6. Concluding remarks

In this paper, we propose structured doubling algorithms for finding weakly stabilizing Hermitian solutions with property (P) for DAREs and CAREs. Under Assumption (A2) and the existence of weakly stabilizing Hermitian solutions with property (P) of the DARE and the dual DARE, respectively, we prove the global and linear convergence for the D-SDA algorithm if it does not break down. A similar convergence result for C-SDA is also shown. The advantage of structured doubling algorithms is evident in that the Hermitian solutions are obtained by the iterative process without any deflation preprocessing of unimodular eigenvalues. The MATLAB commands “care” and “dare” fail for the selected test examples, because the associated Hamiltonian matrix and symplectic pencil have eigenvalues on the imaginary axis and the unit circle, respectively. Nevertheless, the normalized residuals of desired Hermitian solutions of almost all tested examples computed by SDA algorithms are accurate to machine accuracy. Numerical experiments show that SDA algorithms converge to the desired Hermitian solutions efficiently and reliably.

Acknowledgements

The second author was invited to give a 45-min talk at the 14th ILAS conference. However, due to conflict of its schedule with ICIAM’07 Zürich, he was not able to give the talk in person. It is his pleasure and honor to still publish this paper in the proceedings. The authors thank the anonymous referees for their valuable comments and suggestions.
Appendix: Error analysis of D-SDA

We now give an error analysis of the computed matrices $A_{k+1}$, $G_{k+1}$ and $H_{k+1}$ in the D-SDA for one iterative step $k$. For convenience, we drop the index $k$ and consider equations in (2.8).

We use $fl(\cdot)$ to denote the computed floating point matrix. The quantity $u$ is the unit roundoff (or machine precision) and $c_m$ denotes a modest constant depending on a polynomial of $n$ with low degree. When $A$ and $B$ are $m \times n$ matrices, the matrix $B := |A|$ if $b_{ij} = |a_{ij}|$ and $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i, j$. The $\infty$-matrix norm is denoted by $\| \cdot \|_\infty$.

Consider Eq. (2.8a) $\hat{A} = A(I + GH)^{-1}A$.

We first compute $W = I + GH$. From [23, Chapter 3] we have

$$fl(W) \equiv \tilde{W} = W + \Delta W, \quad |\Delta W| \leq c_m u |G||H|. \quad (A.1)$$

Next, we compute $V = \tilde{W}^{-1}A$.

Case (i): If $\tilde{W}$ is well-defined, we solve $\tilde{W}V = A$ by using the $LU$ factorization $\tilde{W} = L\tilde{W} U\tilde{W}$. From [23, Chapter 11] it holds

$$fl(V) \equiv \tilde{V} = V + \Delta V = \tilde{W}^{-1}A + \Delta V, \quad (A.2a)$$

where

$$|\Delta V| \leq c_m u |\tilde{W}^{-1}||L\tilde{W}||U\tilde{W}||\tilde{W}^{-1}A|. \quad (A.2b)$$

Case (ii): If $\tilde{W}$ is ill-conditioned, we compute the GSVD of $\tilde{W}$ and $A$ [19, p. 466]

$$T_h \tilde{W} U_h = C_h \geq 0, \quad (A.3a)$$
$$T_h A V_h = S_h \geq 0, \quad (A.3b)$$

where $T_h$ is nonsingular, $U_h$ and $V_h$ are unitary, $C_h$ and $S_h$ are positive diagonal. The GSVD has a strongly backward numerical stability. We solve $\tilde{W}V = A$ using (A.3) and get

$$V = \tilde{W}^{-1}A = (U_h C_h^{-1})(S_h V_h^H). \quad (A.4)$$

Therefore, the computed $fl(V)$ is estimated by

$$fl(V) \equiv \tilde{V} = V + \Delta V = \tilde{W}^{-1}A + \Delta V, \quad (A.5a)$$

where

$$|\Delta V| \leq c_m u |\tilde{U}_h^{-1}H^{-1}|C_h^{-1}||C_h\tilde{U}_h^{-1}H||\tilde{W}^{-1}A| = c_m u |\tilde{U}_h^{-1}H||\tilde{U}_h^{-1}||\tilde{W}^{-1}A|, \quad (A.5b)$$

in which $\tilde{U}_h$ departs from $U_h$ by a small rounding-error perturbation.

Finally, we compute $\hat{A} = AV$. From [23, Chapter 3] follows that

$$fl(\hat{A}) = A\tilde{V} + \Delta(\hat{A}\tilde{V}), \quad |\Delta(\hat{A}\tilde{V})| \leq c_m u |A||\tilde{V}|. \quad (A.6)$$

From (A.1), (A.2), (A.5) and (A.6) we estimate the forward error bound for $\hat{A}$ by

$$fl(\hat{A}) = A\tilde{V} + \Delta(\hat{A}\tilde{V}) = A(\tilde{W}^{-1}A + \Delta V) + \Delta(\hat{A}\tilde{V})$$
$$= AW^{-1}A + (AW^{-1})\Delta W(W^{-1}A) + A\Delta V + \Delta(\hat{A}\tilde{V}) + O(u^2)$$
$$\equiv \hat{A} + \Delta\hat{A}, \quad (A.7)$$
where
\[ \| \Delta \hat{A} \|_\infty \leq c_m u \left( \| W^{-1} A \|_\infty^2 \| G \| H \|_\infty + (2\delta + 1) \| A \|_\infty \| \tilde{W}^{-1} A \|_\infty \right) + O(u^2) \] (A.8a)
in which
\[ \delta \leq \left\{ \frac{\| \tilde{W}^{-1} \|_\infty \| L \tilde{W} \| U \tilde{W} \|_\infty}{\| \tilde{U}_h^{-1} \|_\infty \| \tilde{U}_h H \|_\infty} \right\} \] (LU factorization),
\[ \left\{ \frac{\| \tilde{U}_h^{-1} \|_\infty \| \tilde{U}_h H \|_\infty}{\| \tilde{U}_h^{-1} \|_\infty \| \tilde{U}_h H \|_\infty} \right\} \] (GSVD).

In practice, the matrix \( \tilde{W} \) becomes very ill-conditioned, only when \( k \) is sufficiently large. In the light of Theorem 4.2 (i) and (iv) we see that
\[ \sigma_{\min}(\tilde{W}) \approx \| A \|_\infty \approx O(2^{-k}) + O(\rho(J_s)^{2k}) \rightarrow 0 \] (A.9)
as \( k \rightarrow \infty \). This implies that
\[ \| \tilde{W}^{-1} A \|_\infty \approx \| W^{-1} A \|_\infty \approx O(1) \] (A.10)
for \( k \) sufficiently large.

If \( \tilde{W} \) is well-conditioned, i.e., the sequence \( \{ (A_k, G_k, H_k) \} \) has not converged, then \( \| \tilde{W}^{-1} \|_\infty, \| \tilde{W}^{-1} A \|_\infty \) and \( \| W^{-1} A \|_\infty \) can not become too large. Hence, for both cases, the forward error bound (A.8) for \( \Delta \hat{A} \) should be relatively small.

Applying the similar argument as above, one can also derive
\[ fl(\hat{G}) = \hat{G} + \Delta \hat{G}, \quad fl(\hat{H}) = \hat{H} + \Delta \hat{H}, \] (A.11)
where the forward error bounds for \( \Delta \hat{G} \) and \( \Delta \hat{H} \) are of the same quantity as in (A.8). Hence, we show that one iterative step in (2.8) is numerically forward stable provided the condition (A.10) holds.

References