On the \(\star\)-Sylvester Equation \(AX \pm X^*B^* = C\)

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Abstract

We consider the solution of the \(\star\)-Sylvester equation \(AX \pm X^*B^* = C\), for \(\star = T, H\) and \(A, B, \in \mathbb{C}^{m \times n}\), and some related linear matrix equations \((AXB^* \pm X^* = C, AXB^* \pm CX^*D^* = E, AX \pm X^*A^* = C, AX \pm YB = C, AXB \pm CYD = E, AXA^* \pm BYB^* = C\) and \(AXB \pm (AXB)^* = C\). Solvability conditions and stable numerical methods are considered, in terms of the (generalized and periodic) Schur, QR and (generalized) singular value decompositions. We emphasize on the square cases where \(m = n\) but the rectangular cases will be considered. The \(\star\)-sylvester equation is important in the solution of some generalized algebraic Riccati equations by Newton’s method.

Key words: error analysis, least squares, linear matrix equation, Lyapunov equation, palindromic eigenvalue problem, QR decomposition, generalized algebraic Riccati equation, Schur decomposition, singular value decomposition, solvability, Sylvester equation

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1 Introduction

In [3], the Lyapunov-like linear matrix equation

\[ A^*X + X^*A = B \], \quad A, X \in \mathbb{C}^{m \times n} \ (m \neq n) \]

with \((\cdot)^* = (\cdot)^T\) was considered using generalized inverses. Applications occur in Hamiltonian mechanics. At the end of [3], the more general Sylvester-like equation

\[ A^*X + X^*C = B \], \quad A, C, X \in \mathbb{C}^{m \times n} \ (m \neq n) \]

was proposed without solution. The equation (with \(* = T\)) was studied, again using generalized inverses, in [9,13]. However in [13], the necessary and sufficient conditions for solvability may be too complex for most applications. The formula for \(X\) for the special case, assuming \(m = n\), \(B^T = B\) and the invertibility of \(A \pm C^T\), may not be numerically stable or efficient. In [9], some necessary or sufficient conditions for solvability were derived. A (seemingly wrong) formula for \(X\) in terms of generalized inverse was also proposed (see Section 2.2 for more details on the approach taken in [9]).

In this paper, the (numerical) solution of this \(*\)-Sylvester equation (with \(* = T, H\); the latter indicating the complex conjugate transpose), as well as some related equations, will be studied. Our tools include the (generalized and periodic) Schur, (generalized) singular value and QR decompositions [10]. We are mainly interested in the square cases when \(m = n\) but the rectangular cases will be considered briefly.

Our interest in the \(*\)-Sylvester equation originates from the solution of the \(*\)-Riccati equation

\[ XAX^* + XB + CX^* + D = 0 \]

from an application related to the palindromic eigenvalue problem [5] (where eigenvalues appears in reciprocal pairs \(\lambda\) and \(\lambda^{-*}\)). The solution of the \(*\)-Riccati equation is difficult and the application of Newton’s method is an obvious possibility. The solution of the \(*\)-Sylvester equation is required in the Newton iterative process. Interestingly, the \(*\)-Sylvester and \(*\)-Lyapunov equations behave very differently from the ordinary Sylvester and Lyapunov equations. For example, from Theorem 2.1 below, the \(*\)-Sylvester equation is uniquely solvable if and only if the generalized spectrum \(\sigma(A, B)\) (the eigenvalues of the matrix pencil \(A - \lambda B\)) does not contain \(\lambda\) and \(\lambda^{-*}\) simultaneously, some sort of ‘palindromic’ requirement. For more detail of this application, see Appendix I.

\footnote{Not being palindromic, with “anti-palindromic” already describing something different.}
Another application of the $\ast$-Sylvester equation involves the generalized algebraic Riccati equations (GARE) in [4,15], whose solutions by Newton’s method require the solution of a coupled set of two T-Lyapunov equations, which is equivalent to a T-Sylvester equation, as described in Section 2.3. See Appendix II for more detail for this application.

The paper is organized as follows. After this introduction, Section 2 considers the $\ast$-Sylvester equation, in terms of its solvability, the proposed algorithms and the associated error analysis. We are mainly interested in the square case when $m = n$ but the rectangular case will be discussed briefly in Section 2.4. Section 3 contains several small illustrative examples. Section 4 considers some relatives of the $\ast$-Sylvester equation — $AXB \pm X \ast B \ast = C$, $AXB \ast \pm CX \ast D \ast = E$, the $\ast$-Lyapunov equation $AX \pm X \ast A \ast = C$, $AX \pm YB = C$, $AXB \pm CYD = E$, $AXA \ast \pm BYB \ast = C$ and $AXB \pm (AXB) \ast = C$. We conclude in Section 5 before describing two applications in the Appendices.

2 $\ast$-Sylvester Equation

Consider the $\ast$-Sylvester equation

$$AX \pm X \ast B \ast = C, \quad A, B, X \in \mathbb{C}^{n \times n}. \tag{2.1}$$

This includes the special cases of the T-Sylvester equation when $\ast = T$ and the H-Sylvester equation when $\ast = H$.

With the Kronecker product, (2.1) can be written as

$$\mathcal{P} \text{vec}(X) = \text{vec}(C), \quad \mathcal{P} \equiv I \otimes A \pm (B \otimes I)E, \tag{2.2}$$

where vec$(X)$ stacks the columns of $X$ onto a column vector and $E$ is the permutation matrix which maps vec$(X)$ into vec$(X^T)$. The matrix operator on the left-hand-side of (2.2) is $n^2 \times n^2$ and the application of Gaussian elimination and the like will be inefficient. In addition, the approach ignores the structure of the original problem, introducing errors to the solution process unnecessarily. We shall consider the eigenvalues of $\mathcal{P}$ later in Section 2.2.

Another approach will be to transform (2.1) by some unitary $P$ and $Q$, so that (2.1) becomes, for $\ast = T$:

$$PAQ \cdot \overline{Q}^T XP^T \pm PX^T \overline{Q} \cdot Q^T BP^T = PCP^T, \tag{2.3}$$

or, for $\ast = H$:

$$PAQ \cdot Q^H XP^H \pm PX^H Q \cdot Q^H BP^H = PCP^H. \tag{2.4}$$
Note that minimum residual and minimum norm solutions are possible with the unitary \( P \) and \( Q \). Let \((Q^H A^H P^H, Q^H B^H P^H)\) be in (upper-triangular) generalized Schur form (by QZ algorithm [10]). The transformed equations in (2.3) and (2.4) then have the form

\[
\begin{bmatrix}
  a_{11} & 0^T \\
a_{21} A_{22}
\end{bmatrix}
\begin{bmatrix}
x_{11} & x_{12}^* \\
x_{21} X_{22}
\end{bmatrix}
\pm
\begin{bmatrix}
x_{11}^* & x_{21}^* \\
x_{12} X_{22}^*
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{21}^* \\
0 & B_{22}^*
\end{bmatrix}
= \begin{bmatrix}
c_{11} & c_{12}^* \\
c_{21} C_{22}
\end{bmatrix}.
\]

(2.5)

Multiply the matrices out, we have

\[
\begin{align*}
a_{11} x_{11} \pm b_{11}^* x_{11}^* &= c_{11}, \quad (2.6) \\
 a_{11} x_{12}^* \pm x_{21}^* B_{22} &= c_{12}^* \mp x_{11}^* b_{21}^*, \quad (2.7) \\
 A_{22} x_{21} \pm b_{11}^* x_{12} &= c_{21} - x_{11} a_{21}, \quad (2.8) \\
 A_{22} x_{22} \pm X_{22}^* B_{22} &= \tilde{C}_{22} \equiv C_{22} - a_{21} x_{12}^* \mp x_{12} b_{21}. \quad (2.9)
\end{align*}
\]

From (2.6) for \( * = T \), we have

\[
(a_{11} \pm b_{11}) x_{11} = c_{11}. \quad (2.10)
\]

Let \( \lambda_1 \equiv a_{11}/b_{11} \in \sigma(A, B) \). The solvability condition of the above equation is

\[
a_{11} \pm b_{11} \neq 0 \Leftrightarrow \lambda_1 \neq \mp 1. \quad (2.11)
\]

This solvability condition (2.11) will be superseded by the more general solvability condition (2.19) later.

From (2.6) when \( * = H \), we have

\[
a_{11} x_{11} \pm b_{11}^* x_{11} = c_{11}. \quad (2.12)
\]

Let \( x_{11} \equiv x_r + i x_i, \ a_{11} \equiv a_r + i a_i, \ b_{11} \equiv b_r + i b_i \) and \( c_{11} \equiv c_r + i c_i \). The above equation becomes

\[
(a_r + i a_i)(x_r + i x_i) \pm (b_r - i b_i)(x_r - i x_i) = c_r + i c_i
\]

or

\[
a_r x_r - a_i x_i \pm b_r x_r \mp b_i x_i = c_r, \quad a_r x_i + a_i x_r \mp b_r x_i \pm b_i x_r = c_i.
\]

These imply

\[
\begin{bmatrix}
a_r \pm b_r & -a_i \mp b_i \\
a_i \mp b_i & a_r \pm b_r
\end{bmatrix}
\begin{bmatrix}
x_r \\
x_i
\end{bmatrix}
= \begin{bmatrix}
c_r \\
c_i
\end{bmatrix}.
\]

(2.13)

Let \( \lambda_1 = a_{11}/b_{11} \in \sigma(A, B) \). The determinant of the matrix operator in (2.13):

\[
d = (a_r^2 - b_r^2) - (b_i^2 - a_i^2) = |a_{11}|^2 - |b_{11}|^2 \neq 0 \Leftrightarrow |\lambda_{11}| \neq 1. \quad (2.14)
\]

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requiring that no eigenvalue \( \lambda \in \sigma(A, B) \) lies on the unit circle. Again, this condition (2.14) will be superseded by the more general solvability condition (2.19) later.

Another way to solve (2.12) is to write it together with its complex conjugate in the composite form

\[
\begin{pmatrix}
  a_{11} & \pm b_{11}^* \\
  \pm b_{11} & a_{11}^*
\end{pmatrix}
\begin{pmatrix}
  x_{11} \\
  x_{11}^*
\end{pmatrix} =
\begin{pmatrix}
  c_{11} \\
  c_{11}^*
\end{pmatrix}
\]

which produces the equivalent formula

\[x_{11} = \frac{a_{11}^* c_{11} \mp b_{11}^* c_{11}^*}{|a_{11}|^2 - |b_{11}|^2}.
\]

From (2.7) and (2.8), we obtain

\[
\begin{pmatrix}
  a_{11}^* I & \pm b_{11}^* I \\
  \pm b_{11}^* I & A_{22}
\end{pmatrix}
\begin{pmatrix}
  x_{12} \\
  x_{21}
\end{pmatrix} =
\begin{pmatrix}
  \tilde{c}_{12} \\
  \tilde{c}_{21}
\end{pmatrix} \equiv \begin{pmatrix}
  c_{12} \\
  c_{21}
\end{pmatrix} + x_{11} \begin{pmatrix}
  \mp b_{21} \\
  -a_{21}
\end{pmatrix}.
\]

With \( a_{11} = b_{11} = 0 \), \( x_{11} \) will be undetermined. However, \((A, B)\) then forms a singular pencil, \( \sigma(A, B) = \mathbb{C} \) and this case will be excluded by (2.19). If \( a_{11} \neq 0 \), (2.15) is then equivalent to

\[
\begin{pmatrix}
  a_{11}^* I & \pm B_{22} \\
  0 & A_{22} - \frac{b_{11}^*}{a_{11}} B_{22}
\end{pmatrix}
\begin{pmatrix}
  x_{12} \\
  x_{21}
\end{pmatrix} =
\begin{pmatrix}
  \tilde{c}_{12} \\
  \tilde{c}_{21}
\end{pmatrix} \equiv \begin{pmatrix}
  c_{12} \\
  c_{21}
\end{pmatrix} + x_{11} \begin{pmatrix}
  \mp b_{11} \\
  a_{11}^*
\end{pmatrix} \begin{pmatrix}
  \tilde{c}_{12} \\
  \tilde{c}_{21}
\end{pmatrix}.
\]

The solvability condition of (2.15) and (2.16) is

\[\det \tilde{A}_{22} \neq 0, \quad \tilde{A}_{22} \equiv A_{22} - \frac{b_{11}^*}{a_{11}} B_{22}\]

or that \( \lambda \) and \( \lambda^* \) cannot be in \( \sigma(A, B) \) together. Note that \( \tilde{A}_{22} \) is still lower-triangular, just like \( A \) or \( B \).

If \( b_{11} \neq 0 \), (2.15) is equivalent to

\[
\begin{pmatrix}
  0 & B_{22} - \frac{a_{11}^*}{b_{11}} A_{22} \\
  b_{11}^* I & \pm A_{22}
\end{pmatrix}
\begin{pmatrix}
  x_{12} \\
  x_{21}
\end{pmatrix} =
\begin{pmatrix}
  \tilde{c}_{12} \\
  \pm \tilde{c}_{21}
\end{pmatrix} \equiv \begin{pmatrix}
  \pm \frac{a_{11}^*}{b_{11}} \tilde{c}_{12} \\
  \pm \tilde{c}_{21}
\end{pmatrix}.
\]

with an identical solvability condition (2.19).

Lastly, (2.9) is of the same form as (2.1) but of smaller size.
Remark 2.1 Interestingly, for the ordinary Sylvester equation \(AX - XB = C\), numerical solution will be possible when \((A, B)\) is transformed into quasi-triangular/triangular form (not necessarily both of the same type) or the cheaper quasi-triangular/Hessenberg form. It is not the case for (2.1) and the \(\times\) somehow alters the behaviour of the equation greatly.

Remark 2.2 We can arrange the above solution process into a large quasi-triangular linear system. This enables us to apply the error analysis of triangular linear systems to proposed Algorithms \(SSylvester\) and \(TSylvester_R\) in Section 2.2. Because \(x_{11}\) can be solved via a scalar or \(2 \times 2\) system and \(X_{22}\) can be treated recursively, we only need to consider the solution of (2.15) for \(x_{12}\) and \(x_{21}\). The equation has the form, for some right-hand-side \(R_1\):

\[
\begin{pmatrix}
  r_{11} & r_{12} & & \cdot \\
  s_{11} & s_{12} & & \cdot \\
  & \cdot & \cdot & \cdot \\
  r_{21} & r_{22} & & \cdot \\
  s_{21} & s_{22} & & \cdot \\
& \cdot & \cdot & \cdot \\
\end{pmatrix}
\begin{pmatrix}
  z_{r1} \\
  z_{s1} \\
  \vdots \\
  z_{r2} \\
  z_{s2} \\
\end{pmatrix}
= R_1.
\]  

(2.18)

This is equivalent to a series of \(2 \times 2\) systems, for known right-hand-sides \(R_r, R_s, \cdots\):

\[
M_r z_r = R_r, \quad M_s z_s = R_s, \quad \cdots,
\]

where

\[
M_r \equiv [r_{ij}], \quad M_s \equiv [s_{ij}], \quad \cdots; \quad z_r \equiv [z_{r1}, z_{r2}]^T, \quad z_s \equiv [z_{s1}, z_{s2}]^T, \quad \cdots.
\]

Consequently, (2.18) is a quasi-lower-triangular linear system with at most \(2 \times 2\) diagonal blocks. By implication, so is (2.5). This comment still holds when \(a_{11}\) and \(b_{11}\) are replaced by \(2 \times 2\) blocks, as in Section 2.1. In that case, the diagonal blocks in the corresponding quasi-triangular matrix will be at most \(4 \times 4\).

We summarize the solvability condition for (2.1) in the following theorem:

Theorem 2.1 The \(\times\)-Sylvester equation (2.1):

\[
AX \pm X^*B^* = C, \quad A, B \in \mathbb{C}^{n \times n}
\]

is uniquely solvable if and only if the condition:

\[
\lambda \in \sigma(A, B) \Rightarrow \lambda^{-\times} \notin \sigma(A, B)
\]

(2.19)
is satisfied. Here, the convention that 0 and ∞ are mutually reciprocal is followed.

The process in this subsection is summarized below: (with BS denoting back-substitution)

**Algorithm SSylvester**

(For the unique solution of $AX ± X^*B^* = C$; $A, B, C, X ∈ C^{n×n}$.)

- Compute the lower-triangular generalized Schur form $(PAQ, PBQ)$ using QZ.
- Store $(PAQ, PBQ, PCP^*)$ in $(A, B, C)$.
- Solve (2.10) for $⋆ = T$, or (2.13) for $⋆ = H$; if fail, exit.
- If $|a_{11}| |b_{11}|$ or $n = 1$ or $|a_{11}|^2 + |b_{11}|^2 ≤$ tolerance, exit.
- If $|a_{11}| ≥ |b_{11}|$, then
  - if $\tilde{A}_{22} ≡ A_{22} - \frac{b_{11}}{a_{11}}B_{22}$ has any negligible diagonal elements, then exit.
  - Else compute $x_{21} = \tilde{A}_{22}^{-1} \tilde{c}_{21}$ by BS, $x_{12} = (\tilde{c}_{12} + B_{22}x_{21})/a_{11}^*$ (c.f. (2.16)).
- Else if $\tilde{B}_{22} ≡ B_{22} - \frac{a_{11}}{b_{11}}A_{22}$ has any negligible diagonal elements, then exit.
  - Else compute $x_{21} = \tilde{B}_{22}^{-1} \tilde{c}_{12}$ by BS, $x_{12} = (±\tilde{c}_{21} + A_{22}x_{21})/b_{11}^*$ (c.f. (2.17)).

Apply Algorithm TSylvester to $A_{22}X_{22} ± X_{22}^*B_{22}^* = \tilde{C}_{22}$, $n ← n - 1$.

Output $X ← QXP$ for $⋆ = T$, or $X ← QXP$ for $⋆ = H$.

**End of algorithm**

Let the operation count of the Algorithm SSylvester, in addition to the $66n^3$ complex flops for the QZ procedure [10] for the generalized Schur decomposition of $(A, B)$, be $f(n)$ complex flops, mainly involves the solution of (2.9) and (2.16) or (2.17). This involves forming and inverting $\tilde{A}_{22}$ or $\tilde{B}_{22}$ ($n^2$ flops), computing $x_{12}$ ($\frac{1}{2}n^2$ flops) and forming $\tilde{C}_{22}$ ($2n^2$). Thus $f(n) ≈ f(n - 1) + \frac{7}{2}n^2$, ignoring $O(n)$ terms. This implies that $f(n) ≈ \frac{7}{6}n^3$ and the total operation count for Algorithm SSylvester is $67\frac{1}{6}n^3$ complex flops, ignoring $O(n^2)$ terms.

From the above analysis and Theorem 2.1, the condition of (2.1) will be bad if the separation $λ_iλ^*_i - 1$ is narrow (or when the assumption for unique solvability is nearly violated). The same conclusion can also be drawn from the analogous analysis in Section 2.1 below. For error analysis, see Section 2.2 for more detail.

More generally, note that (2.1) can be transformed by some invertible (not necessarily unitary) $P$ and $Q$ to

$$PAQ · Q^{-1}XP^T ± PX^TQ^{-T} · Q^TPT = PCPT$$

and

$$PAQ · Q^{-1}XP^H ± PX^HQ^{-H} · Q^HB^H = PCPH$$

similar to (2.3) and (2.4), respectively. With $(PAQ, PBQ)$ in the Kronecker canonical form, similar algorithms and theoretical results to those in this Section can be derived. Of course, such transformation will not be stable numerically and will be of little practical value.
2.1 The real case or divide-and-conquer

When \( A, B \) and \( C \) are all real, the solution \( X \), judging from (2.2), will be real. To guarantee a real solution \( X \), the real Schur form [10] for \((A, B)\) has to be used. The transformed equations in (2.3) or (2.4) have the form

\[
\begin{bmatrix}
A_{11} & 0^T \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
X_{11} & X_{12}^* \\
X_{21} & X_{22}
\end{bmatrix}
\pm
\begin{bmatrix}
X_{11}^* & X_{21}^* \\
X_{12}^* & 0
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{21}^* \\
0 & B_{22}^*
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12}^* \\
C_{21} & C_{22}
\end{bmatrix},
\]

where \( A_{11} \) and \( B_{11} \) may be \( 1 \times 1 \) or \( 2 \times 2 \). The former case will be trivial as in (2.6) and the latter can be handled using the Kronecker product. The theory leading to the conditions in (2.11) and (2.19) from the complex Schur form still holds. We shall assume that \( A_{11} \) and \( B_{11} \) are not scalar in the rest of this subsection.

Again, the Kronecker product can be applied to (2.21) and (2.22). A better approach is to consider (2.21)* and (2.22):

\[
\pm B_{22}X_{21} + X_{12}A_{11}^* = \tilde{C}_{12}, \quad A_{22}X_{21} \pm X_{12}B_{11}^* = \tilde{C}_{21}.
\]

A linear combination of these equations will be

\[
(\beta A_{22} \pm \alpha B_{22})X_{21} + X_{12}(\alpha A_{11}^* \pm \beta B_{11}^*) = \alpha \tilde{C}_{12} + \beta \tilde{C}_{21}.
\]

Assume regularity of the pencil \((A, B)\), there exists real \( \alpha \) and \( \beta \) such that \( \alpha A_{11}^* \pm \beta B_{11}^* \) is nonsingular (or well-conditioned). We then have

\[
X_{12} = -(\beta A_{22} \pm \alpha B_{22})X_{21}(\alpha A_{11}^* \pm \beta B_{11}^*)^{-1} + \tilde{C}_{12}
\]

with

\[
\tilde{C}_{12} = (\alpha \tilde{C}_{12} + \beta \tilde{C}_{21})(\alpha A_{11}^* \pm \beta B_{11}^*)^{-1}.
\]

Substitute \( X_{12} \) in (2.25) into (2.22), we have a generalized Sylvester equation [6] for \( X_{21} \):

\[
A_{22}X_{21} - (\alpha B_{22} \pm \beta A_{22})X_{21}(\alpha A_{11}^* \pm \beta B_{11}^*)^{-1}B_{11}^* = \tilde{C}_{21} \mp \tilde{C}_{12}B_{11}^*.
\]

From [6], (2.26) is uniquely solvable when there is no intersection of the spectra \( \sigma(A_{22}, \alpha B_{22} \pm \beta A_{22}) \) and \( \sigma(B_{11}^*, \alpha A_{11}^* \pm \beta B_{11}^*) \). Let \((A_{11}, B_{11})\) and \((A_{22}, B_{22})\)
be transformed into generalized Schur forms with diagonal elements \((\alpha_i, \beta_i)\) and \((\alpha_j, \beta_j)\) respectively. For \(\alpha \neq 0\), the solvability condition is
\[
\frac{\alpha_j}{\alpha \beta_j \pm \beta \alpha_j} \neq \frac{\beta_j^*}{\alpha^* \alpha_j \pm \beta^* \beta_j} \iff \alpha^*_i \alpha_j \neq \beta^*_i \beta_j,
\]

exactly condition (2.19). The same conclusion is reached when \(\alpha = 0\), which implies that \(B_{11}\) is invertible, and \(X_{12}\) in (2.25) should then be substituted into the \(*\) of (2.21) to produce a similar generalized Sylvester equation for \(X_{21}\):
\[
B_{22}X_{21} - A_{22}X_{21}B_{11}^{-*}A_{11}^* = \pm \tilde{C}_{12} + \tilde{C}_{12}A_{11}^*.
\]
(2.27)

Also \(X_{12}\) is retrievable from (2.25) in a numerical stable and efficient manner. Note that the matrix operators \(\alpha A_{11}^* \pm \beta B_{11}^*\) in (2.25) is block-upper triangular with \((\alpha, \beta)\) controlling its condition. In (2.26), \(A_{22}\) and \(B_{22}\) are block-lower triangular with \((\alpha A_{11}^* \pm \beta B_{11}^*)^{-1}B_{11}\) being at most \(2 \times 2\), enabling \(X_{21}\) to be easily calculated as in the generalized Stewart-Bartel algorithm in [6]. (For illustration, let us consider (2.27). With \(B_{22}\) and \(A_{22}\) being lower-triangular and \(B_{11}^{-*}A_{11}^*\) upper-triangular, the first row and column of \(X_{21}\) can be computed easily, leaving a smaller but similar system. This can then be solved recursively and similarly.) A slightly more efficient alternative will be to consider the rows of (2.26) consecutively from the top, solving a \(2 \times 2\) system for each row of \(X_{21}\). Equation (2.27) can be solved analogously, also one row at a time.

We can then solve recursively (2.23), a smaller equation similar to (2.9).

Lastly, the procedure discussed in this subsection can applied as a divide-and-conquer strategy, with \(A_{11}\) and \(B_{11}\) being \([n^2_2]\) \times \([n^2_2]\). After transforming (2.1) using the (real) Schur form of \((A, B)\), the resulting equation can be split up in the middle, with the sizes of \(A_{11}\) and \(A_{22}\) roughly equal. Subsequent systems in terms of \((A_{ii}, B_{ii})\) \((i = 1, 2)\) can then be treated recursively in the same divide-and-conquer fashion, yielding a more efficient version of our algorithm. We summarize the procedure in this subsection in the following algorithms, with the subscripts “R” and “DC” for “Real” and “Divide-and-Conquer” respectively. We shall present only the complex version of the divide-and-conquer algorithm. The real version, using the generalized real Schur form instead of the generalized Schur form, can be constructed similarly.

**Algorithm TSylvester**

(For the unique solution of \(AX \pm X^T B^T = C; A, B, C, X \in \mathbb{R}^{n \times n}\).)

**Compute** the quasi-lower-triangular generalized real Schur form \((PAQ, PBQ)\) using QZ.

**Store** \((PAQ, PBQ, PCPT)\) in \((A, B, C)\).

**Solve** (2.6) for \(X_{11}\); if fail, exit.

If last block reached with \(n = 1, 2\), exit.

If \(A_{11}\) and \(B_{11}\) are scalar, solve (2.21) and (2.22) for \(X_{12}\) and \(X_{21}\) as in **Algorithm TSylvester**; if fail, exit.
else solve (2.26) or (2.27) with appropriate $\alpha, \beta$ for $X_{21}$ row-wise, using Gaussian elimination on the $2 \times 2$ systems; if fail, exit.

Retrieve $X_{12}$ from (2.25).

Apply Algorithm $\text{TSylvester}_R$ to $A_{22}X_{22} \pm X_{22}^T B_{22}^T = \tilde{C}_{22}$; $n \leftarrow n - 1, n - 2$.

Output $X \leftarrow QXP$.

End of algorithm

The operation count of Algorithm $\text{SSylvester}_R$ is approximately equal to $67\frac{1}{6}$ complex flops, similar to Algorithm $\text{SSylvester}$ and overwhelmed by the initial QZ process.

Algorithm $\text{SSylvester}_{\text{DC}}$

(For the unique solution of $AX \pm X^T B^T = C$; $A, B, C, X \in \mathbb{C}^{n \times n}$.)

Compute the lower-triangular generalized Schur form $(PAQ, PBQ)$ by QZ.

Store $(PAQ, PBQ, PC^T)$ in $(A, B, C)$.

If $n = 1$, $x_{11} = c_{11}/(a_{11} + b_{11})$; if fail, exit.

Choose $n_1 \approx n_2$ such that $A_{ii} \in \mathbb{C}^{n_i \times n_i}$; ($i = 1, 2$), with $n = n_1 + n_2$.

Apply Algorithm $\text{SSylvester}_{\text{DC}}$ to $A_{11}X_{11} \pm X_{11}^T B_{11}^T = C_{11}$, $n \leftarrow n_1$.

If fail, exit.

If $A_{11}$ and $B_{11}$ are scalar, solve (2.21) and (2.22) as in Algorithm $\text{TSylvester}$; if fail, exit;

else solve (2.26) or (2.27) with appropriate $\alpha, \beta$ for $X_{21}$ as in [6] if fail, exit.

Retrieve $X_{12}$ from (2.25).

Apply Algorithm $\text{SSylvester}_{\text{DC}}$ to $A_{22}X_{22} \pm X_{22}^T B_{22}^T = \tilde{C}_{22}$, $n \leftarrow n_2$.

Output $X \leftarrow QXP$.

End of algorithm

The operation count of Algorithm $\text{SSylvester}_{\text{DC}}$ has more matrix-matrix operations, and will be more efficient on certain processors. Another variant algorithm will be to have $A_{11}$ and $B_{11}$ $(n - 1) \times (n - 1)$, with similar efficiency but requiring more memory than the other algorithms.

Remark 2.3 Similar to Remark 2.2, Algorithms $\text{TSylvester}_R$ and $\text{SSylvester}_{\text{DC}}$ are equivalent to solving quasi-lower-triangular linear systems after the initial QZ step. The equations for the scalar (or $2 \times 2$) $X_{11}$ can be written as a $2 \times 2$ (or $8 \times 8$) linear system for the real and imaginary parts of the elements of $X_{11}$. For $X_{12}$ and $X_{21}$, expanding (2.21) and (2.22) using the Kronecker product yields a linear system with matrix operator

\[
\begin{bmatrix}
I_{n-2} \otimes A_{11} & B_{22} \otimes I_2 \\
I_{n-2} \otimes B_{11} & A_{22} \otimes I_2
\end{bmatrix}.
\]
Assuming without loss of generality that \( \star = T \). The matrix has the same form as the one in (2.18), except the elements may be \( 2 \times 2 \) blocks, producing a series of \( 4 \times 4 \) linear systems. Similar arguments as those in Remark 2.2 thus follows.

2.2 Error analysis

We shall discuss the condition and error associated with Algorithms TSylvester and TSylvester\(_R\), following the development in [11, Chapter 16] and [12].

Condition

The condition of (2.1) is obviously identical to that of (2.2). However, \( E \) reshuffles the columns of \( B \otimes I \), making the analysis of the matrix operator \( \mathcal{P} \) difficult. We shall investigate the eigenvalues of \( \mathcal{P} \), collaborating Theorem 2.1. First consider the trivial example when \( n = 2 \), \( A = [a_{ij}] \) and \( B = [b_{ij}] \), we have

\[
\mathcal{P} = \begin{bmatrix}
    a_{11} \pm b_{11}^* & a_{12} \pm b_{12}^* \\
    \pm b_{21}^* & a_{11} & \pm b_{22}^* & a_{12} \\
    a_{21} & \pm b_{11}^* & a_{22} & \pm b_{12}^* \\
    a_{21} & \pm b_{21}^* & a_{22} & \pm b_{22}^* 
\end{bmatrix}.
\]

To make things easier, let \((A, B)\) be in lower-triangular generalized Schur form after some QZ procedure. We then have \( a_{12}, b_{12} = 0 \) and the eigenvalues of the corresponding \( \tilde{\mathcal{P}} \) are \( a_{ii} \pm b_{ii}^* \) (\( i = 1, 2 \)) and those of the middle block \( W_{12} \) where

\[
W_{ij} \equiv \begin{bmatrix}
    a_{ii} \pm b_{jj}^* \\
    \pm b_{ii}^* & a_{jj}
\end{bmatrix}.
\]

The characteristic polynomial of \( W_{ij} \), identical to that for \( W_{ji} \), is \( \lambda^2 - (a_{ii} + a_{jj})\lambda + \det W_{ij} = a_{ii}a_{jj} - b_{ii}^*b_{jj}^* \), and the eigenvalues are

\[
\lambda_{W_{ij}} = \frac{1}{2} \left[ a_{ii} + a_{jj} \pm \sqrt{(a_{ii} - a_{jj})^2 + 4b_{ii}^*b_{jj}^*} \right].
\]

Note that some \( \lambda_{W_{ij}} \) or \( \det W_{ij} = 0 \) if and only if (2.19) in Theorem 2.1 is violated. For larger values of \( n \), the equivalence of our algorithms and quasi-triangular linear sytems (after the initial QZ step), as mentioned in Remarks 2.2 and 2.3, means that \( \tilde{\mathcal{P}} \) in (2.2) is quasi-triangular with the correct permutation of the variables and equations. The ordering considers the first diagonal element \( x_{11} \), then the first components in \( x_{12} \) and \( x_{21} \), and then their second components etc. until exhaustion, and then recursively ordering
$X_{22}$ in the same fashion. From (2.18), the eigenvalues of $\tilde{P}$ thus consists of $a_{ii} \pm b_{ii}^*$ (n of them, for $i = 1, \ldots, n$) and the eigenvalues $\lambda_{W_{ij}}$ ($nC_2$ of them, for $j > i$ and $i, j = 1, \ldots, n$). Notice that, as expected, there are exactly $2(nC_2) + n = n^2$ eigenvalues for $\tilde{P}$. It is conceptually simple but operationally tedious to reorder $\tilde{P}$ to show this result even for $n = 3$ and that will be left as an exercise.

As for the eigenvalues of $P$, we have, for $\star = T$:

$$E(P \otimes Q^H)\text{vec}(X) = E\text{vec}(Q^HXP^T) = \text{vec}(PX^TQ) = (Q^H \otimes P)E\text{vec}(X).$$

As the above holds for any $X$, we have

$$E(P \otimes Q^H) = (Q^H \otimes P)E. \quad (2.28)$$

(See [1, Section 7.4, result (xii)] for this result and [1, Chapter 7] for other results on the Kronecker product). Then the transformed equation (2.3) implies

$$\tilde{P}\text{vec}(Q^HXP^T) = \text{vec}(PCP^T) = (P \otimes P)\text{vec}(C)$$

where $\tilde{P} \equiv I \otimes (PAQ) \pm [(PBQ) \otimes I]E$. With the help of (2.28) and compare with (2.2), we obtain

$$P = (P^H \otimes P^H)\tilde{P}(P \otimes Q^H) \quad (\text{for } \star = T)$$

or

$$P = (P^T \otimes P^H)\tilde{P}(P \otimes Q^H) \quad (\text{similarly, for } \star = H).$$

As the transformations between $P$ and $\tilde{P}$ are unitary, the same singularity behaviour is shared. However, there is no clear relationship between the spectra of $P$ and $\tilde{P}$. This is expected as the QZ transformation of $(A,B)$ merely changes (2.1) to (2.3) or (2.4) without disturbing the solution $X$. However, this does not induce a similarity relation between $P$ and $\tilde{P}$.

**Residual**

As indicated in Remarks 2.2 and 2.3, Algorithms SSylvester and SSylvester$_R$ can be arranged into quasi-triangular linear systems. We can then apply the error analysis for triangular linear systems in [11, Theorem 8.5] to obtain

$$\|R\|_F \equiv \|C - (AX \pm X^*B^*)\|_F \leq c_nu\|A\|_F + \|B\|_F\|\tilde{X}\|_F \quad (2.29)$$

for a computed solution $\tilde{X}$ from our algorithms, where $c_n$ is a constant dependent on $n$ and $u$ is the unit round-off (typically $O(10^{-16})$), when the condition of the $2 \times 2$, $4 \times 4$ or $8 \times 8$ linear systems in (2.26), (2.27) and Remarks 2.2 and 2.3 are not bad. Note that the QZ transformation of $(A,B)$ is backward stable, similar to the QR process in [11, Equation 16.9]. Consequently, the
relative residual is bounded by a modest multiple of the unit round-off $u$. See the collaborating numerical examples in Section 3.

**Backward error**

Like ordinary Sylvester equations, the numerical solution of (2.1) is not backward stable in general. Similar to [11, §16.2], we can define the normwise relative backward error of an approximate solution $Y$ by

$$
\eta(Y) \equiv \min \{ \epsilon : (A + \delta A)Y \pm Y^* (B + \delta B)^* = C + \delta C, \\
\|\delta A\|_F \leq \epsilon \alpha, \|\delta B\|_F \leq \epsilon \beta, \|\delta C\|_F \leq \epsilon \gamma \} 
$$

with $\alpha \equiv \|A\|_F$, $\beta \equiv \|B\|_F$ and $\gamma \equiv \|C\|_F$. With $Y = U \Sigma V^H$ in singular value decomposition (SVD) [10], the $Y^*$ terms do not affect the analysis in [11, §16.2]. With $\Sigma = \text{diag}\{\sigma_1, \cdots, \sigma_n\}$, it can be shown that

$$
\eta(Y) \leq \mu \frac{\|R\|_F}{(\alpha + \beta)\|Y\|_F + \gamma},
$$

(2.30)

where

$$
\mu \equiv \frac{(\alpha + \beta)\|Y\|_F + \gamma}{[(\alpha^2 + \beta^2)\sigma_n^2 + \gamma^2]^{1/2}}, \quad R \equiv \delta AY \pm Y^* \delta B^* - \delta C.
$$

Consequently, $\eta(Y)$ can be large when $Y$ is ill-conditioned, and a small residual $R$ does not always imply a small backward error $\eta(Y)$. This phenomenon has been observed in Example 3.3, where $Y$ is ill-conditioned. However, from our experience, severely backward unstable $\ast$-Sylvester equations are rare and have to be artificially constructed. This suggests that our algorithms may well be conditionally backward stable. Similar to the Sylvester equation [11, §16.2], we do not know the conditions under which a $\ast$-Sylvester equation has a well-conditioned solution.

**Perturbation and practical error bounds**

For perturbation, the usual results for linear systems apply. In terms of the $\ast$-Sylvester equation (2.1), consider the perturbed equation

$$(A + \delta A)(X + \delta X) \pm (X + \delta X)^* (B + \delta B)^* = C + \delta C$$

with “$\delta$” denoting perturbations. Define the $\ast$-Sylvester operator

$$S(X) \equiv AX \pm X^* B^*.$$

We then obtain

$$S(\delta X) = \delta C - \delta AX \mp X^* \delta B^* - \delta A \delta X \mp \delta X^* \delta B^*.$$
Application of norm gives rise to
\[ \|\delta X\| \leq \|S^{-1}\| \{ \|\delta C\| + (\|\delta A\| + \|\delta B\|)(\|X\| + \|\delta X\|) \}. \]

When \(\|\delta S\| \equiv \|\delta A\| + \|\delta B\|\) is small enough so that \(1 \geq \|S^{-1}\| \cdot \|\delta S\|\), we can rearrange the above result to
\[ \frac{\|\delta X\|}{\|X\|} \leq \frac{\|S^{-1}\|}{1 - \|S^{-1}\| \cdot \|\delta S\|} \left( \|\delta C\| + \|\delta S\| \right). \]

With \(\|C\| = \|S(X)\| \leq \|S\| \cdot \|X\|\) and the condition number \(\kappa(S) \equiv \|S\| \cdot \|S^{-1}\|\), we arrive at the standard perturbation result
\[ \frac{\|\delta X\|}{\|X\|} \leq \frac{\kappa(S)}{1 - \kappa(S) \cdot \|\delta S\|/\|S\|} \left( \|\delta C\| + \|\delta S\| \right). \]

Thus the relative error in \(X\) is controlled by those in \(A\), \(B\) and \(C\), magnified by the condition number \(\kappa(S)\).

As indicated in [11, §16.4], practical error bounds can be estimated, just like for other linear matrix equations. Several applications of the solution algorithm will be required. More work has to be done along this direction.

### 2.3 An alternative formulation

We can consider the sum/difference of (2.1) and its \(\star\), producing
\[ (A \pm B)X \pm X^\star (A \pm B)^\star = C \pm C^\star. \]  

(2.31)

The pair of equations represent the symmetric (or Hermitian) and skew-symmetric (or Hermitian) parts of (2.1) and can be solved using the generalized Schur form of \((A + B, A - B)\). Identical solvability condition as (2.19) can be derived. In terms of the eigenvalues \(\tilde{\lambda}_i \in \sigma(A + B, A - B)\), (2.1) and (2.31) are uniquely solvability if and only if \(\tilde{\lambda}_i + \tilde{\lambda}_j \neq 0\), with \(\tilde{\lambda}_i = (\lambda_i + 1)/(\lambda_i - 1)\) for some \(\lambda_i \in \sigma(A, B)\). It is easy to see that mapping between \((A, B)\) and \((A + B, A - B)\) corresponds to some (inverse) Cayley transformations.

In [9], a formula for the solution \(X\) of (2.1) (for \(\star = T\) and the “+” case) was derived using the first equation in (2.31) only, throwing away the information in the second equation. We cannot see how the formula can be correct using only half the information of (2.1) in the first half of (2.31). In the extreme case with \(A = -B\), the first equation in (2.1) will be degenerate and the solution \(X\) will be totally free. Anyway, \(X\) is a solution of (2.1) if and only if it is also a solution of (2.31), but a solution of half of (2.31) in general does not satisfy (2.1).
2.4 Rectangular $A, B$

The equation with rectangular $A, B$ has been studied in [13] and an explicit formula using generalized inverse has been produced only for very special cases.

Using the generalized Kronecker canonical form [7,8], we have

$$P(A - \lambda B)Q = \begin{bmatrix}
A_r - \lambda B_r & * & * \\
0 & A_{reg} - \lambda B_{reg} & * \\
0 & 0 & A_l - \lambda B_l
\end{bmatrix},$$

where $P$ and $Q$ are unitary with the subscripts $r$ and $l$ indicating the singular parts and $\text{reg}$ the square regular part of the pencil $A - \lambda B$. With $X$ and $C$ partitioned appropriately, the solution of (2.1) with rectangular $A, B$ can then be investigated. We shall not pursue this more general problem further in this paper.

We consider the efficient and numerically stable solution of the “rectangular” $\star$-Sylvester equation an unsolved problem.

3 Numerical Examples

In this section, we apply Algorithm SSylvester (denoted by ASS) and the Kronecker product approach in (2.2) (denoted by KRP) to some examples for illustrative and comparative purposes. All computations were performed in MATLAB/version 7.5 on a PC with an Intel Pentium-IV 4.3GHZ processor and 3GB main memory, using IEEE double-precision.

Example 3.1 We choose $\hat{A}, \hat{B} \in \mathbb{R}^{n \times n}$ to be real lower-triangular matrices with given diagonal elements (specified by $a, b \in \mathbb{R}^n$) and random strictly lower-triangular elements. They are then reshuffled by the orthogonal matrices $Q, Z \in \mathbb{R}^{n \times n}$ to form $(A,B) = (Q\hat{A}Z, Q\hat{B}Z)$. In MATLAB [14] commands, we have $\hat{A} = \text{tril}(\text{randn}(n),-1) + \text{diag}(a)$, $\hat{B} = \text{tril}(\text{randn}(n),-1) + \text{diag}(b)$ and $C = \text{randn}(n)$. To guarantee condition (2.19), let $b = \text{randn}(n,1), a = 2b$. In Table 3.1, we list the CPU time ratios of the ASS and the KRP approaches as well as the corresponding residuals and their ratios, with increasing dimensions $n = 16, 20, 25, 30, 35, 40$. Note that the operation counts for the SSA and KRP methods are approximately $6n^3$ and $2n^6$ flops respectively (the latter for the LU decomposition of the $n^2 \times n^2$ matrix in (2.2)). The results in Table 3.1 show that the advantage of ASS over KRP in CPU time grows rapidly as $n$
increases, as predicted by the operation counts. Even with better management of sparsity or parallelism, the $O(n^6)$ operation count makes the KRP approach uncompetitive even for moderate size $n$. The residuals from ASS is also better than that from KRP, as (2.2) is solved by Gaussian elimination in an unstructured way. See the other examples for more comparison of the residuals of ASS and KRP.

Table 3.1: Results for Example 3.1

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_{\text{KRP}}$</th>
<th>Res(ASS)</th>
<th>Res(KRP)</th>
<th>Res(KRP)/Res(ASS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1.00e+00</td>
<td>1.8527e-17</td>
<td>2.1490e-17</td>
<td>1.16</td>
</tr>
<tr>
<td>25</td>
<td>1.31e+01</td>
<td>2.3065e-17</td>
<td>2.8686e-17</td>
<td>1.24</td>
</tr>
<tr>
<td>30</td>
<td>2.61e+01</td>
<td>3.1126e-18</td>
<td>5.7367e-18</td>
<td>2.20</td>
</tr>
<tr>
<td>35</td>
<td>6.48e+01</td>
<td>7.0992e-18</td>
<td>1.2392e-17</td>
<td>1.75</td>
</tr>
<tr>
<td>40</td>
<td>1.05e+02</td>
<td>1.7654e-18</td>
<td>6.4930e-18</td>
<td>3.68</td>
</tr>
</tbody>
</table>

Example 3.2 We let $a = [\alpha + \epsilon, \beta]$, $b = [\beta, \alpha]^\top$, where $\alpha, \beta$ are two randomly numbers greater than 1, with the spectral set $\sigma(A, B) = \{\alpha + \epsilon, \beta\}$, and $|\lambda_1 \lambda_2 - 1| = \frac{\epsilon}{\alpha}$. Judging from (2.19), (2.1) has worsening condition when $\epsilon$ decreases. We report a comparison of absolute residuals for the ASS and KRP approaches for $\epsilon = 10^{-1}, 10^{-3}, 10^{-5}, 10^{-7}$ and $10^{-9}$ in Table 3.2. The results show that if (2.2) is solved by Gaussian elimination, its residual will be larger than that for ASS especially for smaller $\epsilon$. Note that the size of $X$ (the last column in Table 3.2) reflects partially the condition of (2.1), as indicated in (2.29). The residuals will be worsen for large values of $\|X\|_F$, with the quotient of $\text{res}(\text{ASS})$ and $\|X\|$ approximately equal to the unit round-off $u$. The KRP approach copes less well than the ASS approach for an ill-conditioned problem.

Table 3.2: Results for Example 3.2

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Res(ASS)</th>
<th>Res(KRP)</th>
<th>Res(KRP)/Res(ASS)</th>
<th>$O(|X|)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0e-1</td>
<td>2.0673e-15</td>
<td>2.4547e-15</td>
<td>1.19</td>
<td>$10^1$</td>
</tr>
<tr>
<td>1.0e-3</td>
<td>8.6726e-13</td>
<td>4.3279e-13</td>
<td>0.50</td>
<td>$10^3$</td>
</tr>
<tr>
<td>1.0e-5</td>
<td>2.3447e-12</td>
<td>2.4063e-12</td>
<td>1.03</td>
<td>$10^3$</td>
</tr>
<tr>
<td>1.0e-7</td>
<td>5.9628e-10</td>
<td>1.1786e-09</td>
<td>1.98</td>
<td>$10^6$</td>
</tr>
<tr>
<td>1.0e-9</td>
<td>5.8632e-08</td>
<td>3.4069e-07</td>
<td>5.81</td>
<td>$10^8$</td>
</tr>
</tbody>
</table>
Example 3.3 Let $Q \in \mathbb{R}^{n \times n}$ be orthogonal and the exact solution be $X_e$, where

$$X_e = Q^T \begin{bmatrix} 10^{-m} & 0 \\ 0 & 10^m \end{bmatrix} Q, \quad A = \begin{bmatrix} \text{randn} & 0 \\ \text{randn} & 10^{-m} \end{bmatrix} Q, \quad B = \begin{bmatrix} \text{randn} & 0 \\ \text{randn} & 2 \times 10^{-m} \end{bmatrix} Q$$

and $C = AX_e + X_e^T B^T$. Solving the corresponding T-Sylvester equation by Algorithm SSylvester produces the results in Table 3.3, using symbols from Section 2.2.

Table 3.3: Results for Example 3.3

| $m$  | Res(ASS) | RRes(ASS) | $|X_{ASS} - X_e| / |X_e|$ | $O(\|X\|)$ | $\mu$ | $\eta(X_{ASS})$ |
|------|----------|-----------|---------------------------|----------|------|----------------|
| 0    | 1.0129e-16 | $10^{-16}$ | 2.6624e-16 | $10^0$ | 3.2440e+00 | 2.7169e-16 |
| 2    | 1.5268e-14 | $10^{-16}$ | 2.0519e-15 | $10^2$ | 9.7188e+01 | 5.8991e-15 |
| 4    | 2.4170e-12 | $10^{-16}$ | 5.0599e-13 | $10^4$ | 7.3715e+03 | 1.0410e-12 |
| 6    | 1.6955e-10 | $10^{-16}$ | 2.4933e-11 | $10^6$ | 9.0423e+05 | 6.8488e-11 |
| 8    | 3.7545e-09 | $10^{-17}$ | 2.7786e-09 | $10^8$ | 8.4485e+07 | 1.2658e-09 |

The column for the backward error $\eta(Y)$ (estimated using the bound (2.30)) confirms that our algorithm is not numerically backward stable. The problem is increasingly ill-conditioned for increasing values of $m$ and the large values of $\mu$ worsen the backward errors $\eta(Y)$, although the relative residuals $R\text{Res}(\text{ASS}) = \text{Res}(\text{ASS}) / \|X\|$ are of machine accuracy. On the other hand, from our experience, badly behaved examples are rare and have to be artificially constructed.

4 Related Equations

4.1 Generalized $\star$-Sylvester equation I

Consider the more general version of the $\star$-Sylvester equation (2.1):

$$AXB^* \pm X^* = C \quad (4.1)$$

with $A, B^*, X^* \in \mathbb{C}^{m \times n}$ and $m \neq n$. The generalized Kronecker canonical form [7,8] for $(A, B^*)$ may be used to analyze and solve the equation. We shall not pursue this line of attack further.

For $A, B, C \in \mathbb{C}^{n \times n}$, the equation is equivalent to the $\star$-Sylvester equation in Section 2 when either $A$ or $B$ is nonsingular. In general, consider the periodic
Schur or PQZ decomposition [2] for $B^H A^H$ so that $(Q^H A^H P^H, PB^H Q)$ is in upper triangular form.

Consider the transformed equation, for $\star = H$:

$$PAQ \cdot Q^H XP^H \cdot PB^H Q \pm PX^H Q = PCQ,$$

or for $\star = T$:

$$PAQ \cdot Q^H XP^T \cdot PB^T Q \pm PX^T Q = PCQ.$$

The case when $(A, B)$ are real and $\star = T$ with a real PQZ decomposition is similar but will be ignored here.

With $(Q^H A^H P^H, PB^H Q)$ or $(Q^H A^H P^H, PB^T Q)$ being upper-triangular, the transformed equations look like

$$\begin{bmatrix}
a_{11} & 0^T \\
a_{21} & A_{22}
\end{bmatrix} \begin{bmatrix} x_{11} & x_{12}^* \\
x_{21} & X_{22}^*
\end{bmatrix} \begin{bmatrix} b_{11}^* & b_{12}^* \\
0 & B_{22}^*
\end{bmatrix} \pm \begin{bmatrix} x_{11}^* & x_{21}^* \\
x_{12} & X_{22}^*
\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12}^* \\
c_{21} & C_{22}
\end{bmatrix}.$$

We then have

$$a_{11}b_{11}^* x_{11} \pm x_{11}^* = c_{11}, \quad (4.2)$$
$$a_{11}x_{12}^* B_{22}^* \pm x_{21}^* = c_{12}^* - a_{11}x_{11}^* b_{12}^*, \quad (4.3)$$
$$b_{11}^* A_{22} x_{21} \pm x_{12} = c_{21} - b_{11}^* x_{11} a_{21}, \quad (4.4)$$
$$A_{22} X_{22}^* B_{22}^* \pm X_{22}^* = C_{22} - x_{11} a_{21} b_{12}^* - A_{22} x_{21} b_{12}^* - a_{21} x_{12}^* B_{22}^*. \quad (4.5)$$

Let $\lambda_i \equiv a_{ii} b_{ii} \in \sigma(AB)$ be the $i$th eigenvalue of $AB$. Detail analysis shows the solvability condition

$$\lambda \in \sigma(AB) \Rightarrow \lambda^{-\star} \notin \sigma(AB). \quad (4.6)$$

Algorithms can easily be constructed from (4.2)–(4.5) but will be ignored here.

### 4.2 Generalized $\star$-Sylvester equation II

Consider the more general version of the $\star$-Sylvester equation (2.1) and (4.1):

$$AXB^\star \pm CX^\star D^\star = E \quad (4.7)$$

with the complex matrices $A^\star$ and $C^\star$, $B^\star$ and $D^\star$, $B$ and $C$, and $A$ and $D$ possessing the same number of columns. This is a more general equation than the rectangular $\star$-Sylvester equation in Section 2.4. It is also a special case of the equation in section 4.5. We do not know how to tackle this equation.
For $A, B, C, D, E \in \mathbb{C}^{n \times n}$, the equation is equivalent to the $\star$-Sylvester equation in Section 2, when $A$ and $D$ (or $B$ and $C$) are nonsingular. In general, we can transform the equation to, for $\star = H$:

$$PAR \cdot R^H X S \cdot S^H B^H Q \pm PCS \cdot S^H X^H R \cdot R^H D^H Q = PEQ,$$

or, for $\star = T$:

$$PAR \cdot R^T X S \cdot S^H B^T Q \pm PCS \cdot S^T X^T R \cdot R^H D^T Q = PEQ.$$

These equations have the form

$$\tilde{A}X\tilde{B}^* \pm \tilde{C}X^*\tilde{D}^* = \tilde{E}.$$

The transformation can be realized using the periodic Schur or PQZ decomposition [2] for $B^{-1}DA^{-1}C$ (or other similar formations), where $P, Q, R$ and $S$ are unitary, and $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ are (quasi-)lower-triangular (with diagonal elements $\alpha_i, \beta_i, \gamma_i$ and $\delta_i$, respectively). Consequently, similar solution procedure as in Section 2 applies, with both minimum norm and minimum residual solutions feasible. The transformed equations give rise to equations in the form, for $i, j = 1, \ldots, n$:

$$\begin{bmatrix}
\alpha_i\beta_j \pm \gamma_i\delta_j \\
\pm\gamma_j\delta_i & \alpha_j\beta_i
\end{bmatrix}
\begin{bmatrix}
x_{ij} \\
x_{ji}
\end{bmatrix}
= \begin{bmatrix}
\tilde{e}_{ij} \\
\tilde{e}_{ji}
\end{bmatrix}$$

for some known $\tilde{e}_{ij}$ and $\tilde{e}_{ji}$, when $x_{ij}$ are solved in the correct order. The equation will then be uniquely solvable if and only if $\alpha_i\alpha_j\beta_i\beta_j \neq \gamma_i\gamma_j\delta_i\delta_j$ ($i, j = 1, \ldots, n$), a condition more general than but similar to (2.19).

### 4.3 $\star$-Lyapunov equation

Consider the $\star$-Lyapunov equation

$$AX \pm X^* A^* = C, \quad A \in \mathbb{C}^{n \times n}.$$

With unitary $P$ and $Q$, the equation can be transformed to, for $\star = T$:

$$PAQ \cdot Q^H X P^T \pm PX^T Q \cdot Q^T A^T P^T = PC P^T,$$

or, for $\star = H$:

$$PAQ \cdot Q^H X P^H \pm PX^H Q \cdot Q^H A^H P^H = PC P^H.$$

Note that the unitary transformation of $A$ allows for minimum norm or residual solutions of the equations. We can choose $P$ and $Q$ from the SVD of $A$. 

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This is more suited to the case when \( A \) is rectangular and this line of attack will be pursued later. For a square \( A \), we can choose \( Q = P^H \) using the Schur decomposition of \( A \), solving the equation in a similar fashion as in Section 2. The transformed equations have the form

\[
\begin{bmatrix}
  a_{11} & 0^T \\
  a_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
  x_{11} & x_{12}^* \\
  x_{21} & X_{22}
\end{bmatrix}
\pm
\begin{bmatrix}
  x_{11}^* & x_{21}^* \\
  x_{12} & X_{22}^*
\end{bmatrix}
\begin{bmatrix}
  a_{11}^* & a_{21}^* \\
  0 & A_{22}^*
\end{bmatrix}
= \begin{bmatrix}
  c_{11} & c_{12}^* \\
  \pm c_{12} & C_{22}
\end{bmatrix}.
\]

Multiply the matrices out, we have

\[
a_{11}x_{11} \pm a_{11}^*x_{11}^* = c_{11}, \quad \text{(4.10)}
\]
\[
a_{11}x_{12}^* \pm x_{21}^*a_{21}^* = c_{12}^* \mp x_{11}a_{21}, \quad \text{(4.11)}
\]
\[
A_{22}X_{22} \pm X_{22}^*a_{21}^* = \tilde{C}_{22} \equiv C_{22} - a_{21}x_{12} \mp x_{12}a_{21}^*, \quad \text{(4.12)}
\]

Because of the (anti-)symmetry of the \( \star \)-Lyapunov equation, we only need to consider the above three equations, with the fourth containing redundant information.

For \( \star = T \), \( x_{11} \) is free for the “−” case, requiring \( c_{11} = 0 \) for consistency. For the “+” case, \( x_{11} = \frac{c_{11}}{a_{11}} \) when the eigenvalue \( \lambda_1 = a_{11} \in \sigma(A) \) is nonzero. For \( \star = H \), we have the underdetermined equation \( \Re(a_{11}x_{11}) = c_{11} \) (for the “+” case) or \( \Im(a_{11}x_{11}) = 0 \) (for the “−” case). For \( x_{12} \) and \( x_{21} \), we have the equation

\[
\begin{bmatrix}
  a_{11}^*I & A_{22}
\end{bmatrix}
\begin{bmatrix}
  x_{12} \\
  x_{21}
\end{bmatrix} = \tilde{c}_{12}
\]

which is underdetermined when \( \tilde{c}_{12} \) is in the span of \( [a_{11}^*I, A_{22}] \) (always holds if \( A \) is nonsingular).

The equation for \( X_{22} \) is smaller but similar to the original \( \star \)-Lyapunov equation.

### 4.3.1 Symmetric/Hermitian solution

With the transformed equations, for \( \star = T \):

\[
PAP^H \cdot PXP^T \pm PX^TP^T \cdot P^T A^TP^T = PCP^T,
\]

or for \( \star = H \):

\[
PAP^H \cdot PXP^H \pm PX^HP^H \cdot P^T A^HP^H = PCP^H.
\]

We can impose the (anti-)symmetry constraint \( X^\star = \pm X \). Equations (4.10)–(4.12) then imply similar equations for \( x_{11} \) and \( X_{22} \) as in the non-symmetric.
(or Hermitian) case. For \( x_{12} = x_{21} \) (and similarly for the anti-symmetric or Hermitian case), we have
\[
(a_{11}^* I \pm A_{22}) x_{12} = \tilde{c}_{12},
\]
retrieving the solvability condition for the ordinary Sylvester/Lyapunov equation. This requires the eigenvalues \( \lambda_j \) and \( \lambda_j \) of \( A \) to satisfy \( \lambda_j^* \pm \lambda_j \neq 0 \). When \( i = j \) and \( \star = T \), this indicates that we cannot have zero eigenvalues for the “+” case and the “−” case gives rise to an undetermined \( x_{11} \), with \( c_{11} = 0 \) automatically from the anti-symmetry of \( C \). When \( i = j \) and \( \star = H \), no eigenvalue \( \lambda_i \) can be purely imaginary/real. Note that \( x_{11} \) is underdetermined and so are all the diagonal elements of \( X \).

4.3.2 Rectangular \( A \)

The T-Lyapunov equation with rectangular \( A \) has been studied in [3] using generalized inverse (which can only be realized using the SVD). Please consult [3] for solvability conditions and the formula for the general solution. Here we construct the solution, and implicitly derive the solvability conditions, using the SVD. In the next subsection, the cheaper QR decomposition [10] is used instead to derive the same solution.

When \( A \) is rectangular, the SVD of \( A \):
\[
A = UDV^H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^H
\]
gives rise to the transformed T-Lyapunov equation:
\[
UDV^H X \pm X^T \nabla D^T U^T = C \iff D(V^H XU) \pm (U^H X^T \nabla) D = U^H C U,
\]
or the transformed H-Lyapunov equation:
\[
UDV^H X \pm X^H V D^T U^H = C \iff D(V^H XU) \pm (U^H X^H V) D = U^H C U.
\]
We then have
\[
\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \pm \begin{bmatrix} X_{11}^* & X_{21}^* \\ X_{12}^* & X_{22}^* \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ \pm C_{12}^* & C_{22} \end{bmatrix}
\]
\[\text{(4.14)}\]
or
\[
\Sigma X_{11} \pm X_{11}^* \Sigma = C_{11} , \quad \Sigma X_{12} = C_{12} , \quad X_{21}, X_{22} = \text{free}
\]
requiring \( C_{22} = 0 \) for consistency. With \( \sigma_k \) being the singular values of \( A \), the first equation has the form
\[
\sigma_i x_{ij} \pm \sigma_j x_{ji}^* = c_{ij}.
\]
For $i \neq j$, we can solve these equations in the least squares sense:

\[
\begin{bmatrix}
  x_{ij} \\
  x_{ji}^\star
\end{bmatrix} = \frac{c_{ij}}{\sigma_i^2 + \sigma_j^2} \begin{bmatrix}
  \sigma_i \\
  \pm \sigma_j
\end{bmatrix},
\]

or let $x_{ji} (j > i)$ be free and express $x_{ij}$ in terms of $x_{ji}$:

\[
x_{ij} = \frac{c_{ij} \mp \sigma_j x_{ji}^\star}{\sigma_i}.
\]

For $i = j$, we have

\[
\sigma_i(x_{ii} \pm x_{ii}^\star) = c_{ii}.
\]

When $\star = T$, $x_{ii} = \frac{c_{ii}}{2\sigma_i}$ for the “+” case, or $x_{ii}$ is free requiring $c_{ii} = 0$ (from the anti-symmetry of $C$) for consistency for the “-” case. When $\star = H$, $\Re(x_{ii}) = \frac{c_{ii}}{2\sigma_i}$ with $\Im(x_{ii})$ free for the “+” case, or $\Im(x_{ii}) = \frac{c_{ii}}{2\sigma_i}$ with $\Re(x_{ii})$ free for the “-” case.

Note that minimum norm and minimum residual solutions are feasible from the above formulation.

Applying the formula in [3] with $A$ in SVD, we obtain

\[
\tilde{X} \equiv \begin{bmatrix}
  X_{11} & X_{12} \\
  X_{21} & X_{22}
\end{bmatrix} = \begin{bmatrix}
  \frac{1}{2} \Sigma^{-1} C_{11} + Z_{11} \Sigma \Sigma^{-1} C_{12} \\
  Y_{21} & Y_{22}
\end{bmatrix}, \tag{4.15}
\]

where $Y_{21}$ and $Y_{22}$ are arbitrary and $Z_{11} = \mp Z_{11}^\star$. The solutions are identical except the (underdetermined) calculations involving $X_{11}$ is handled differently in [3] by the choice of $Z_{11}$. For a general $A$, we have to choose an arbitrary $Z$ such that

\[
(P_2^T Z P_2)^T = \mp P_2^T Z P_2, \tag{4.16}
\]

where $P_2 = A^T G$ with $G$ satisfying $A^T G A = A^T$. To choose $Z$ using the SVD in (4.13), we have $P_2 = V_1 V_1^H$ and (4.16) becomes

\[
\tilde{V}_1 V_1^T (Z^T \pm Z) V_1 V_1^H = 0 \iff V_1^T \tilde{V}(\tilde{Z}^T \pm \tilde{Z}) V^H V_1 = 0, \quad \tilde{Z} \equiv V^T Z V = \begin{bmatrix}
  Z_{11} & Z_{12} \\
  Z_{21} & Z_{22}
\end{bmatrix}
\]

implying the same condition for $Z_{11} (= \mp Z_{11}^\star)$ as in (4.15). Consequently, we might as well use the SVD of $A$ to solve the T-Lyspuniov equation as in (4.14).
4.3.3 QR

The SVD in Section 4.3.2 can be replaced by the cheaper but equally effective QR decomposition. Let

\[ A = QR\Pi = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \Pi \]

for some nonsingular \( R_{11} \) and permutation matrix \( \Pi \). The transformed equation is, for \( \star = T \):

\[ R(\Pi XQ) \pm (\Pi XQ)^T R^T = Q^H CQ, \]

or, for \( \star = H \):

\[ R(\Pi XQ) \pm (\Pi XQ)^H R^H = Q^H CQ. \]

These have the form

\[
\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \pm \begin{bmatrix} X_{11}^* & X_{21}^* \\ X_{12}^* & X_{22}^* \end{bmatrix} \begin{bmatrix} R_{11}^* & 0 \\ R_{12}^* & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ \pm C_{12} & C_{22} \end{bmatrix}.
\]

Then we have

\[ R_{11}X_{11} \pm X_{11}^* R_{11}^* = C_{11} - R_{12}X_{21} \mp X_{21}^* R_{12}^* , \quad R_{11}X_{12} = C_{12} - R_{12}X_{22}. \]

with \( X_{21} \) and \( X_{22} \) free. We can obtain \( X_{12} \) from the second equation and then retrieve \( X_{11} \) from the first. The first equation can be solved using the techniques in Section 4.2.

Alternatively, let \( A^* = QR\Pi \), then we have \( R^*(Q^* X) \pm (Q^* X) R = \Pi^T C\Pi \) and similar procedures follow. Minimum norm and minimum residual solutions are feasible from the above formulation.

Note that the solution of the \( \star \)-Lyapunov equation, with more symmetry, is easier than that of the Lyapunov equation, which requires the more expensive Schur decomposition.

4.4 \( AX \pm YB = C \)

From here on, we shall consider some more distant relatives of (2.1). We shall consider some obvious transformation of the equations, using SVD or GSVD [10] without going into the detail.

For our particular equation, \( A, Y \) and \( C \) (or \( X, B \) and \( C \)) have the same number of rows (or columns), with the products \( AX \) and \( YB \) compatible.
With the SVDs of the arbitrary A and B, we have

\[ AX \pm YB = C \iff U_A \Sigma_A V_A^H X \pm Y U_B \Sigma_B V_B^H = C \]

\[ \iff \Sigma_A \tilde{X} \pm \tilde{Y} \Sigma_B = \tilde{C} , \quad \tilde{X} \equiv V_A^H X V_B , \quad \tilde{Y} \equiv U_A^H Y U_B , \quad \tilde{C} \equiv U_A^H C V_B . \]

The solution X and Y can then be considered via \( \tilde{X} \) and \( \tilde{Y} \) from the above simplified transformed equation. Minimum norm and minimum residual solutions are feasible from the unitary transformations of X and Y.

4.5 \( AXB \pm CYD = E \)

We have equal numbers of rows for A and C, or columns for B and D. With the generalized singular value decompositions (GSVD) [10]:

\[
\begin{bmatrix}
A^H \\
C^H
\end{bmatrix} =
\begin{bmatrix}
U_1 \Sigma_A \\
V_1 \Sigma_C
\end{bmatrix} Z_1 ,
\begin{bmatrix}
B \\
D
\end{bmatrix} =
\begin{bmatrix}
U_2 \Sigma_B \\
V_2 \Sigma_D
\end{bmatrix} Z_2 ,
\]

we have

\[ AXB \pm CYD = E \iff Z_1^H \Sigma_A^H U_1^H \cdot X \cdot U_2 \Sigma_B Z_2 \pm Z_1^H \Sigma_C^H V_1^H \cdot Y \cdot V_2 \Sigma_D Z_2 = E . \]

We then have the transformed equation:

\[ \Sigma_A^H \tilde{X} \Sigma_B \pm \Sigma_C^H \tilde{Y} \Sigma_D = \tilde{E} \]

with

\[ \tilde{X} \equiv U_1^H X U_2 , \quad \tilde{Y} \equiv V_1^H Y V_2 , \quad \tilde{E} \equiv Z_1^H E Z_2^{-1} . \]

Minimum norms solution, but not minimum residual solutions, are feasible because \( Z_i \) are invertible but not unitary as \( U_i \) and \( V_i \).

4.6 \( AXA^* \pm BYB^* = C \)

For A, B, C, X, Y, \( \in \mathbb{C}^{n \times n} \), we can transform our equation with unitary P and Q to, for \( \star = H \):

\[ PAQ \cdot Q^H X Q \cdot Q^H A^H P^H \pm PBQ \cdot Q^H Y Q \cdot Q^H B^H P^H = PCP^H , \]

or, for \( \star = T \):

\[ PAQ \cdot Q^H X \overline{Q} \cdot Q^T A^T P^T \pm PBQ \cdot Q^H Y \overline{Q} \cdot Q^T B^T P^T = PCP^T . \]

The transformed equations have the form

\[ A\overline{X}A^* \pm B\overline{Y}B^* = \overline{C} \]
with $\tilde{X} \equiv Q^H X Q$ and $\tilde{Y} \equiv Q^H Y Q$ (or $\tilde{X} \equiv Q^H X \overline{Q}$ and $\tilde{Y} \equiv Q^H Y \overline{Q}$), and

$$\tilde{A} \equiv PAQ, \quad \tilde{B} = PBQ, \quad \tilde{C} \equiv PCP^*.$$  

With $(PAQ, PBQ)$ in lower-triangular generalized Schur form, $\tilde{X}$ and $\tilde{Y}$ can be computed as before. It is possible to impose the additional constraint that $X$ and $Y$ are (anti-)symmetric/Hermitian. Note that minimum norm and minimum residual solution are feasible.

When $A$ and $B$ are rectangular, we can use the GSVD:

$$
\begin{bmatrix}
A^H \\
B^H
\end{bmatrix} =
\begin{bmatrix}
U \Sigma_A \\
V \Sigma_B
\end{bmatrix} \tilde{Z}.
$$

The transformed equation has the form, for $\star = H$:

$$Z^H \Sigma_A^T U^H \cdot X \cdot U \Sigma_A Z \pm Z^H \Sigma_B^H V^H \cdot Y \cdot V \Sigma_B Z = C,$$

or, for $\star = T$:

$$Z^H \Sigma_A^T U^H \cdot X \cdot \overline{U} \Sigma_A Z \pm Z^H \Sigma_B^H V^H \cdot Y \cdot \overline{V} \Sigma_B Z = C.$$

These equations further reduce to

$$
\Sigma_A^T \tilde{X} \Sigma_A \pm \Sigma_B^T \tilde{Y} \Sigma_B = \tilde{C}
$$

with $\tilde{C} \equiv Z^{-H} C Z^{-1}$ and

$$\tilde{X} \equiv U^H X U \quad \text{or} \quad \tilde{X} \equiv U^H X \overline{U}, \quad \tilde{Y} \equiv V^H Y V \quad \text{or} \quad \tilde{Y} \equiv V^H Y \overline{V}.$$  

Consequently, only minimum norm solutions are feasible and additional (anti-)symmetry/Hermitian conditions on $X$ and $Y$ can be imposed.

4.7 $AXB \pm (AXB)^* = C$

With $A, B \in \mathbb{C}^{n \times n}$, choose unitary $P$ and $Q$ to transform the equation to

$$PAQ \cdot Q^H X Q \cdot Q^H BP^H \pm PB^H Q \cdot Q^H X^H Q \cdot Q^H A^H P^H = PCP^H$$

or

$$PAQ \cdot Q^H X \overline{Q} \cdot Q^T BP^T \pm PB^T Q \cdot Q^H X^T \overline{Q} \cdot Q^T A^T P^T = PCP^T.$$  

We can choose $(\tilde{A}, \tilde{B}^*) \equiv (PAQ, PB^* Q)$ in lower-triangular generalized Schur form, the transformed equations have the form

$$\tilde{A} \tilde{X} \tilde{B} \pm (\tilde{A} \tilde{X} \tilde{B})^* = \tilde{C}$$

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with \( \tilde{C} \equiv P C P^* \) and \( \tilde{X} \equiv Q^H X Q \) (or \( \tilde{X} \equiv Q^H X \overline{Q} \)). Solution \( X \) can then be computed as before and both minimum norm and minimum residual solutions are feasible. Additional (anti-)symmetry/Hermitian constraint on \( X \) can also be imposed.

5 Conclusions

We have considered the solution of the \( \star \)-Sylvester equation which has not been fully investigated before. For the square case, solvability conditions have been derived and algorithms have been proposed. Preliminary numerical results shows that the algorithms behave promisingly. The rectangular case and some related equations, especially the \( \star \)-Lyapunov equation, have also been considered.

It is interesting and exciting that the \( \star \) above the second \( X \) in (2.1) makes the equation behave very differently. The solvability condition in terms of non-intersecton of the spectra \( \sigma(A) \) and \( \sigma(B) \), for the ordinary Sylvester equation \( AX \pm XB = C \), is shifted to (2.19) for the generalized spectrum \( \sigma(A, B) \). In addition, (2.1) looks like a Sylvester equation associated with continuous-time but (2.19) is satisfied when \( \sigma(A, B) \) in totally inside the unit circle, hinting at a discrete-time type of stability behaviour.

For numerical solution, the varying levels of difficulty and complexity for various equations are also intriguing. In terms of increasing complexity, the \( \star \)-Lyapunov, Lyapunov, Sylvester, \( \star \)-Sylvester and generalized \( \star \)-Sylvester equations require, respectively, the QR, Schur, Schur-Hessenberg, generalized Schur and periodic Schur decompositions. The \( \star \) makes the Lyapunov equation easier (by creating more symmetry) yet forces the Sylvester equation the opposite direction.

On future work, we are interested in the solution of the \( \star \)-Riccati equation (Appendix I) and the generalized algebraic Riccati equations (Appendix II). Preliminary numerical results by Newton’s method are encouraging but also reveal several problems. The related work will be published elsewhere. Other possible future research problems include the estimation of practical error bounds and condition numbers, the conditions which guarantee good condition of \( X \) in (2.1), (4.1) and (4.7), more thorough numerical tests and a cheaper algorithm (preferably involving the Schur/Hessenberg decomposition of \( (A, B) \)) for (2.1), as well as the detailed analysis and numerical solution of the rectangular case of the \( \star \)-Sylvester equation and the other related equations in Section 4 and Appendix II, and the behaviour of the alternative sep
functions
\[
\text{sep}_2(A, B) = \min_{X \neq 0} \frac{\|AX - X^*B^*\|}{\|X\|}, \quad \text{sep}_3(A, B) = \min_{X \neq 0} \frac{\|AXB^* - X^*\|}{\|X\|}
\]
and
\[
\text{sep}_4\{(A, C); (B, D)\} = \min_{X \neq 0} \frac{\|AXB - CX^*D\|}{\|X\|}
\]
for \(A, B, C, D \in \mathbb{C}^{n \times n}\).

**Appendix I: Palindromic Linearization \(\lambda Z + Z^*\)**

An interesting application, for the \(*\)-Sylvester equation (2.1)
\[
AX \pm X^*B^* = C, \quad A, B, X \in \mathbb{C}^{n \times n}
\]
arises from the eigensolution of the palindromic linearization [5]
\[
(\lambda Z + Z^*)x = 0, \quad Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{2n \times 2n}.
\]
Applying congruence, we have
\[
\begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} (\lambda Z + Z^*) \begin{bmatrix} I_n & X^* \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \lambda A + A^* & \lambda(AX^* + B) + (XA + C)^* \\ \lambda(XA + C) + (AX^* + B)^* & \lambda \mathcal{R}(X) + \mathcal{R}(X)^* \end{bmatrix}
\]
with
\[
\mathcal{R}(X) \equiv XAX^* + XB + CX^* + D.
\]
If we can solve the \(*\)-Riccati equation
\[
\mathcal{R}(X) = 0
\]
the palindromic linearization can then be “square-rooted”. We then have to solve the generalized eigenvalue problem for the pencil \(\lambda(AX^* + B) + (XA + C)^*\), with the reciprocal eigenvalues in \(\lambda(XA + C) + (AX^* + B)^*\) obtained for free.
It is easy to show from the \( \ast \)-Riccati equation that its solution corresponds to the (stabilizing) deflating subspaces of \( \lambda Z + Z^\ast \) spanned by

\[
(S_1, S_2) \equiv \begin{pmatrix} X^\ast \\ I \\ -X \end{pmatrix}.
\]

It turns out that the palindromic symmetry in the problem leads to the orthogonality property \( S_1^\ast S_2 = 0 \), allowing the above congruence to annihilate the lower-right corner of the transformed pencil, thus square-rooting the problem.

Solving the \( \ast \)-Riccati equation is of course as difficult as the original eigenvalue problem of \( \lambda Z + Z^\ast \). The usual invariance/deflating subspace approach for Riccati equations leads back to the original difficult eigenvalue problem. The obvious application of Newton’s method lead to the iterative process

\[
\delta X_{k+1}(AX_k^* + B) + (X_kA + C)\delta X_{k+1}^* = -R(X_k)
\]

which is a \( \ast \)-Sylvester equation for \( \delta X_{k+1} \).

**Appendix II: Generalized Algebraic Riccati Equations**

In [4], the numerical solution of the following generalized algebraic Riccati equation (GARE) was investigated:

\[
A_a^T X_a + X_a^T A_a + (C_a^T J C_a - B_a J' B_a^T) - X_a^T B_a (J')^{-1} B_a^T X_a = 0 \quad \text{s.t.} \quad E_a^T X_a = X_a^T E_a
\]

where

\[
E_a = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad A_a = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, \quad C_a = \begin{bmatrix} C & D \\ E & 0 \end{bmatrix}
\]

and some \( E, A, B, C, D, E \) are \( \in \mathbb{R}^{n \times n} \), \( B, C, D \) are \( \in \mathbb{R}^{p \times m} \), \( J, J' \) are symmetric and nonsingular. Applying Newton’s method, each iterative step will involve the solution of the coupled set of two T-Lyapunov equations

\[
(\tilde{A} X + \tilde{X}^T \tilde{A}^T, \tilde{E} X - \tilde{X}^T \tilde{E}^T) = (\tilde{B}, \tilde{C})
\]

with some square \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{E} \) and \( \tilde{X} \) with \( \tilde{B} \) (and \( \tilde{C} \)) being (anti-)symmetric. This coupled set of equations is equivalent to a T-Sylvester equation, as described in Section 2.3. Numerical solution can be achieved through the equivalent T-Sylvester equation, or directly through the generalized Schur decomposition of \( (\tilde{A}, \tilde{E}) \).
There is also a similar GARE in [15]:

\[ A^T X + X^T A + C^T C + X^T B B^T X = 0 \quad \text{s.t.} \quad E^T X = X^T E \]

in the H_\infty control of the descriptor system

\[ E \dot{x} = Ax + Bu , \quad y = Cx , \]

where \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{p \times n} \). This GARE may also be solved similarly.

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