Vertex Coloring

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Chromatic number $\chi(G)$

Definition 0.1. (i) Let $G = (V, E)$ be a graph. A $k$-coloring of $G$ is a function $f : V(G) \rightarrow S$, for some set $S$ with cardinality $|S| = k$.

(ii) The elements of $S$ are called colors.

(iii) A $k$-coloring $f$ is proper if $f(u) \neq f(v)$ if $uv \in E$.

(iv) A graph is $k$-colorable if it has a proper $k$-coloring.

(v) The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable.
Examples

(i) $\chi(K_n) = n$.

(ii) $\chi(K_{m,n}) = 2$. (True for all bipartite graphs)

(iii) $\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

(iv) $\chi(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n \geq 2. \end{cases}$

Lemma 0.2. $\chi(H) \leq \chi(G)$ for any subgraph $H$ of $G$.

Proof. Clear. \qed
Examples

Figure. $\chi(G') \leq 4$ as shown; $\chi(G) \geq 4$ since $G$ contains $K_4$. 
Clique number and independent number

Definition 0.3. (i) A clique in $G$ is a complete subgraph in $G$.

(ii) The clique number $\omega(G)$ is the maximum size of a clique in $G$.

(iii) A coclique is a set of vertices with each pair of vertices not adjacent.

(iv) The independent number $\alpha(G)$ of $G$ is the maximum size of a coclique in $G$. 
Examples

Figure. $\omega(G) = 4$; $\alpha(G) = 3$ is the number of red vertices.
Two Lower bounds of $\chi(G)$

**Lemma 0.4.** Let $f : V(G) \rightarrow S$ be a proper coloring and $s \in S$. Then the graph induced on $f^{-1}(s)$ is a coclique.

*Proof.* Clear. $\square$

**Proposition 0.5.** $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

*Proof.* The first bound is from Lemma 0.2 and the second bound is from Lemma 0.4. $\square$
Cartesian Product of Graphs

Definition 0.6. Let $G$, $H$ be two graphs. The \textit{cartesian product} $G \square H$ of $G$ and $H$ is the graph with vertex set

$$V(G \square H) = V(G) \times V(H)$$

and $(u, v)(u', v') \in E(G \square H)$ if

$$u = u', vv' \in E(H), \text{ or } v = v', uu' \in E(G).$$

Example 0.7. $P_2 \square P_2 = C_4$.

Remark 0.8. $G$ and $H$ are subgraphs of $G \square H$. 
Theorem

Theorem 0.9. $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$.

Proof. By Remark 0.8 we have
$\chi(G \Box H) \geq \max\{\chi(G), \chi(H)\}$. Suppose
g : $V(G) \to \{1, 2, \ldots, \chi(G)\}$

and
$h : V(H) \to \{1, 2, \ldots, \chi(H)\}$

are proper colorings of $G$ and $H$ respectively. Set
$k = \max\{\chi(G), \chi(H)\}$ and define
$f : V(G \Box H) \to \{1, 2, \ldots, k\}$ by $f(u, v) = g(u) + h(v)$
modulo $k$. \qed
Continue Proof

Proof. We shall check $f$ is a proper coloring of $G \Box H$. Suppose $(u, v)(u', v') \in E(G \Box H)$. Then either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Say $u = u'$ and $vv' \in E(H)$. Then $g(u) = g(u')$ and $h(v) \neq h(v')$. Hence $f(u, v) - f(u', v') = g(u) + h(v) - (g(u') + h(v')) \neq 0$. This shows $f$ is a proper $k$-coloring. Then $\chi(G \Box H) \leq \max\{\chi(G), \chi(H)\}$. \qed