Distance-regular graphs, pseudo primitive idempotents, and the Terwilliger algebra

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Abstract

Let \( \Gamma \) denote a distance-regular graph with diameter \( D \geq 3 \), intersection numbers \( a_i, b_i, c_i \) and Bose-Mesner algebra \( \mathbf{M} \). For \( \theta \in \mathbb{C} \cup \infty \) we define a 1 dimensional subspace of \( \mathbf{M} \) which we call \( \mathbf{M}(\theta) \). If \( \theta \in \mathbb{C} \) then \( \mathbf{M}(\theta) \) consists of those \( Y \) in \( \mathbf{M} \) such that \( (A-\theta I)Y \in \mathbb{C}A_D \), where \( A \) (resp. \( A_D \)) is the adjacency matrix (resp. \( D \)th distance matrix) of \( \Gamma \). If \( \theta = \infty \) then \( \mathbf{M}(\theta) = \mathbb{C}A_D \). By a pseudo primitive idempotent for \( \theta \) we mean a nonzero element of \( \mathbf{M}(\theta) \). We use these as follows. Let \( X \) denote the vertex set of \( \Gamma \) and fix \( x \in X \). Let \( \mathbf{T} \) denote the subalgebra of \( \text{Mat}_X(\mathbb{C}) \) generated by \( A, E_1^*, E_2^*, \ldots, E_D^* \), where \( E_i^* \) denotes the projection onto the \( i \)th subconstituent of \( \Gamma \) with respect to \( x \). \( \mathbf{T} \) is called the Terwilliger algebra. Let \( W \) denote an irreducible \( \mathbf{T} \)-module. By the endpoint of \( W \) we mean \( \min\{i|E_i^*W \neq 0\} \). \( W \) is called thin whenever \( \text{dim}(E_i^*W) \leq 1 \) for \( 0 \leq i \leq D \). Let \( V = \mathbb{C}^X \) denote the standard \( \mathbf{T} \)-module. Fix \( 0 \neq v \in E_1^*V \) with \( v \) orthogonal to the all 1’s vector. We define \( (\mathbf{M};v) := \{ P \in \mathbf{M}|Pv \in E_2^*V \} \). We show the following are equivalent: (i) \( \text{dim}(\mathbf{M};v) \geq 2 \); (ii) \( v \) is contained in a thin irreducible \( \mathbf{T} \)-module with endpoint 1. Suppose (i), (ii) hold. We show \( (\mathbf{M};v) \) has a basis \( J, E \) where \( J \) has all entries 1 and \( E \) is defined as follows. Let \( W \) denote the \( \mathbf{T} \)-module which satisfies (ii). Observe \( E_1^*W \) is an eigenspace for \( E_1^*AE_1^* \); let \( \eta \) denote the corresponding eigenvalue. Define \( \tilde{\eta} = -1 - b_1(1+\eta)^{-1} \) if \( \eta \neq -1 \) and \( \tilde{\eta} = \infty \) if \( \eta = -1 \). Then \( E \) is a pseudo primitive idempotent for \( \tilde{\eta} \).

Keywords: distance-regular graph, pseudo primitive idempotent, subconstituent algebra, Terwilliger algebra.

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1 Introduction

Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$, intersection 
numbers $a_i, b_i, c_i$, Bose-Mesner algebra $M$ and path-length distance function $\partial$ (see section 2 for formal definitions). In order to state our main theorems we make a few comments. Let $X$ denote the vertex set of $\Gamma$. Let $V = \mathbb{C}^X$ denote 
the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are 
indexed by $X$ and whose entries are in $\mathbb{C}$. We endow $V$ with the Hermitean 
inner product $\langle \cdot, \cdot \rangle$ satisfying $\langle u, v \rangle = u^\dagger v$ for all $u, v \in V$. For each $y \in X$ let $\hat{y}$ denote the vector in $V$ with a 1 in the $y$ coordinate and 0 in all other 
coordinates. We observe $\{ \hat{y} | y \in X \}$ is an orthonormal basis for $V$. Fix $x \in X$. 
For $0 \leq i \leq D$ let $E_i^*$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ which has $yy$ 
entry 1 (resp. 0) whenever $\partial(x, y) = i$ (resp. $\partial(x, y) \neq i$). We observe $E_i^*$ 
acts on $V$ as the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$. For $0 \leq i \leq D$ define $s_i = \sum \hat{y}$, where the sum is over all vertices $y \in X$ 
such that $\partial(x, y) = i$. We observe $s_i \in E_i^*V$. Let $v$ denote a nonzero vector 
in $E_i^*V$ which is orthogonal to $s_1$. We define 

$$(M; v) := \{ P \in M \mid Pv \in E_D^*V \}.$$ 

We observe $(M; v)$ is a subspace of $M$. We consider the dimension of $(M; v)$. 
We first observe $(M; v) \neq 0$. To see this, let $J$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ 
which has all entries 1. It is known $J$ is contained in $M$ [2, p. 64]. In fact 
$J \in (M; v)$; the reason is $Jv = 0$ since $v$ is orthogonal to $s_1$. Apparently 
$(M; v)$ is nonzero so it has dimension at least 1. We now consider when 
does $(M; v)$ have dimension at least 2? To answer this question we recall the 
Terwilliger algebra. Let $T$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by 
$A, E_0^*, E_1^*, \ldots, E_D^*$, where $A$ denotes the adjacency matrix of $\Gamma$. The algebra 
$T$ is known as the Terwilliger algebra (or subconstituent algebra) of $\Gamma$ with 
respect to $x$ [19, 20, 21]. By a $T$-module we mean a subspace $W \subseteq V$ such 
that $TW \subseteq W$. Let $W$ denote a $T$-module. We say $W$ is irreducible whenever $W \neq 0$ and $W$ does not contain a $T$-module other than 0 and $W$. Let $W$ 
denote an irreducible $T$-module. By the endpoint of $W$ we mean the minimal integer $i$ ($0 \leq i \leq D$) such that $E_i^*W \neq 0$. We say $W$ is thin whenever $E_i^*W$ 
has dimension at most 1 for $0 \leq i \leq D$. We now state our main theorem.

**Theorem 1.1.** Let $v$ denote a nonzero vector in $E_1^*V$ which is orthogonal to 
s_1. Then the following (i), (ii) are equivalent.
(i) $(M; v)$ has dimension at least 2.

(ii) $v$ is contained in a thin irreducible $T$-module with endpoint 1.

Suppose (i),(ii) hold above. Then $(M; v)$ has dimension exactly 2.

With reference to Theorem 1.1, suppose for the moment that (i), (ii) hold. We find a basis for $(M; v)$. To describe our basis we need some notation. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of $A$, and for $0 \leq i \leq D$ let $E_i$ denote the primitive idempotent of $M$ associated with $\theta_i$. We recall $E_i$ satisfies $(A - \theta_i I)E_i = 0$. We introduce a type of element in $M$ which generalizes the $E_0, E_1, \ldots, E_D$. We call this type of element a pseudo primitive idempotent for $\Gamma$. In order to define the pseudo primitive idempotents, we first define for each $\theta \in \mathbb{C} \cup \infty$ a subspace of $M$ which we call $M(\theta)$. For $\theta \in \mathbb{C}$, $M(\theta)$ consists of those elements $Y$ of $M$ such that $(A - \theta I)Y \in CA_D$, where $A_D$ is the $D$th distance matrix of $\Gamma$. We define $M(\infty) = CA_D$. We show $M(\theta)$ has dimension 1 for all $\theta \in \mathbb{C} \cup \infty$. Given distinct $\theta, \theta'$ in $\mathbb{C} \cup \infty$, we show $M(\theta) \cap M(\theta') = 0$. For $0 \leq i \leq D$ we show $M(\theta_i) = CE_i$. Let $\theta \in \mathbb{C} \cup \infty$. By a pseudo primitive idempotent for $\theta$, we mean a nonzero element of $M(\theta)$. Before proceeding we define an involution on $\mathbb{C} \cup \infty$. For $\eta \in \mathbb{C} \cup \infty$ we define

$$\tilde{\eta} = \begin{cases} 
\infty & \text{if } \eta = -1, \\
-1 & \text{if } \eta = \infty, \\
-1 - \frac{\eta}{\eta + 1} & \text{if } \eta \neq -1, \eta \neq \infty.
\end{cases}$$

We observe $\tilde{\eta} = \eta$ for $\eta \in \mathbb{C} \cup \infty$. Let $W$ denote a thin irreducible $T$-module with endpoint 1. Observe $E_1^\ast W$ is a one dimensional eigenspace for $E_1^\ast AE_1^\ast$; let $\eta$ denote the corresponding eigenvalue. We call $\eta$ the local eigenvalue of $W$.

**Theorem 1.2.** Let $v$ denote a nonzero vector in $E_1^\ast V$ which is orthogonal to $s_1$. Suppose $v$ satisfies the equivalent conditions (i), (ii) in Theorem 1.1. Let $W$ denote the $T$-module from part (ii) of that theorem and let $\eta$ denote the local eigenvalue for $W$. Let $E$ denote a pseudo primitive idempotent for $\tilde{\eta}$. Then $J, E$ form a basis for $(M; v)$.

We comment on when the scalar $\tilde{\eta}$ from Theorem 1.2 is an eigenvalue of $\Gamma$. Let $W$ denote a thin irreducible $T$-module with endpoint 1 and local
eigenvalue $\eta$. It is known $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$ [18, Theorem 1]. If $\eta = \tilde{\theta}_1$ then $\tilde{\eta} = \theta_1$. If $\eta = \tilde{\theta}_D$ then $\tilde{\eta} = \theta_D$. We show that if $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$ then $\tilde{\eta}$ is not an eigenvalue of $\Gamma$.

The paper is organized as follows. In section 2 we give some preliminaries on distance-regular graphs. In section 3 and section 4 we review some basic results on the Terwilliger algebra and its modules. We prove Theorem 1.1 in section 5. In section 6 we discuss pseudo primitive idempotents. In section 7 we discuss local eigenvalues. We prove Theorem 1.2 in section 8.

## 2 Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [2] or Brouwer, Cohen, and Neumaier [4] for more background information.

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We endow $V$ with the Hermitean inner product $\langle \cdot, \cdot \rangle$ which satisfies $\langle u, v \rangle = u^t \overline{v}$ for all $u, v \in V$, where $t$ denotes transpose and $-$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. We observe $\{ \hat{y} \mid y \in X \}$ is an orthonormal basis for $V$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set $X$, edge set $R$, path-length distance function $\vartheta$ and diameter $D := \max\{\vartheta(x, y) \mid x, y \in X\}$. We say $\Gamma$ is distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and for all $x, y \in X$ with $\vartheta(x, y) = h$, the number

$$p_{ij}^h = |\{z \in X \mid \vartheta(x, z) = i, \vartheta(z, y) = j \}|$$

is independent of $x$ and $y$. The integers $p_{ij}^h$ are called the intersection numbers for $\Gamma$. Observe $p_{ij}^h = p_{ji}^h$ ($0 \leq h, i, j \leq D$). We abbreviate $c_i := p_{i1}^0$ ($1 \leq i \leq D$), $a_i := p_{ii}^1$ ($0 \leq i \leq D$), $b_i := p_{i+1}^i$ ($0 \leq i \leq D - 1$), $k_i := p_{ii}^D$. 

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\((0 \leq i \leq D)\), and for convenience we set \(c_0 := 0\) and \(b_D := 0\). Note that \(b_{i-1}c_i \neq 0\) \((1 \leq i \leq D)\).

For the rest of this paper we assume \(\Gamma = (X, R)\) is distance-regular with diameter \(D \geq 3\). By (2.1) and the triangle inequality,

\[
\begin{align*}
    p_{h}^{i} &= 0 \quad \text{if } |h - i| > 1 \quad (0 \leq h, i \leq D), \quad (2.2) \\
    p_{i}^{j} &= 0 \quad \text{if } |i - j| > 1 \quad (0 \leq i, j \leq D). \quad (2.3)
\end{align*}
\]

Observe \(\Gamma\) is regular with valency \(k = k_1 = b_0\), and that \(k = c_i + a_i + b_i\) for \(0 \leq i \leq D\). By [4, p. 127] we have

\[
k_{i-1}b_{i-1} = k_{i}c_{i} \quad (1 \leq i \leq D). \quad (2.4)
\]

We recall the Bose-Mesner algebra of \(\Gamma\). For \(0 \leq i \leq D\) let \(A_i\) denote the matrix in \(\text{Mat}_X(\mathbb{C})\) which has \(yz\) entry

\[
(A_i)_{yz} = \begin{cases} 
1 & \text{if } \partial(y, z) = i \\
0 & \text{if } \partial(y, z) \neq i 
\end{cases} \quad (y, z \in X).
\]

We call \(A_i\) the \(i\)th \textit{distance matrix} of \(\Gamma\). For notational convenience we define \(A_i = 0\) for \(i < 0\) and \(i > D\). Observe (ai) \(A_0 = I\); (aii) \(\sum_{i=0}^{D} A_i = J\); (aiii) \(\overline{A_i} = A_i\) \((0 \leq i \leq D)\); (aiv) \(A_i^{D} = A_i\) \((0 \leq i \leq D)\); (av) \(A_i A_j = \sum_{h=0}^{D} p_{i}^{h} A_{h}\) \((0 \leq i, j \leq D)\), where \(I\) denotes the identity matrix and \(J\) denotes the all ones matrix. We abbreviate \(A := A_1\) and call this the \textit{adjacency matrix} of \(\Gamma\). Let \(\mathbf{M}\) denote the subalgebra of \(\text{Mat}_X(\mathbb{C})\) generated by \(A\). Using (ai)–(av) we find \(A_0, A_1, \ldots, A_D\) form a basis of \(\mathbf{M}\). We call \(\mathbf{M}\) the \textit{Bose-Mesner algebra} of \(\Gamma\). By [2, p. 59, p. 64], \(\mathbf{M}\) has a second basis \(E_0, E_1, \ldots, E_D\) such that (ei) \(E_0 = |X|^{-1} J\); (eii) \(\sum_{i=0}^{D} E_i = I\); (eiii) \(E_i^{D} = E_i\) \((0 \leq i \leq D)\); (eiv) \(E_i E_j = \delta_{ij} E_i\) \((0 \leq i, j \leq D)\). We call \(E_0, E_1, \ldots, E_D\) the \textit{primitive idempotents} for \(\Gamma\). Since \(E_0, E_1, \ldots, E_D\) form a basis for \(\mathbf{M}\) there exists complex scalars \(\theta_0, \theta_1, \ldots, \theta_D\) such that \(A = \sum_{i=0}^{D} \theta_i E_i\). By this and (ev) we find \(AE_i = \theta_i E_i\) for \(0 \leq i \leq D\). Using (aiii) and (eii) we find each of \(\theta_0, \theta_1, \ldots, \theta_D\) is a real number. Observe \(\theta_0, \theta_1, \ldots, \theta_D\) are mutually distinct since \(A\) generates \(\mathbf{M}\). By [2, p.197] we have \(\theta_0 = k\) and \(-k \leq \theta_i \leq k\) for \(0 \leq i \leq D\). Throughout this paper, we assume \(E_0, E_1, \ldots, E_D\) are indexed so that \(\theta_0 > \theta_1 > \cdots > \theta_D\). We call \(\theta_i\) the \(i\)th \textit{eigenvalue} of \(\Gamma\).
We recall some polynomials. To motivate these we make a comment. Setting $i = 1$ in (4av) and using (2.2),
\begin{equation}
AA_j = b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1} \quad (0 \leq j \leq D - 1),
\end{equation}
where $b_{-1} = 0$. Let $\lambda$ denote an indeterminate and let $\mathbb{C}[\lambda]$ denote the $\mathbb{C}$-algebra consisting of all polynomials in $\lambda$ which have coefficients in $\mathbb{C}$. Let $f_0, f_1, \cdots, f_D$ denote the polynomials in $\mathbb{C}[\lambda]$ which satisfy $f_0 = 1$ and
\begin{equation}
\lambda f_j = b_{j-1}f_{j-1} + a_jf_j + c_{j+1}f_{j+1} \quad (0 \leq j \leq D - 1),
\end{equation}
where $f_{-1} = 0$. For $0 \leq j \leq D$ the degree of $f_j$ is exactly $j$. Comparing (2.5) and (2.6) we find $A_j = f_j(A)$.

3 The Terwilliger algebra

For the remainder of this paper we fix $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ which has $yy$ entry
\begin{equation}
(E_i^*)_{yy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i \\
0 & \text{if } \partial(x, y) \neq i 
\end{cases} \quad (y \in X).
\end{equation}
We call $E_i^*$ the $i$th dual idempotent of $\Gamma$ with respect to $x$. For convenience we define $E_i^* = 0$ for $i < 0$ and $i > D$. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $E_i^* = E_i^*$ (0 $\leq i \leq D$), (iii) $E_i^* E_j^* = E_j^*$ (0 $\leq i \leq D$), (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ (0 $\leq i, j \leq D$). The $E_i^*$ have the following interpretation. Using (3.1) we find
\begin{equation*}
E_i^* V = \text{span}\{y|y \in X, \ \partial(x, y) = i\} \quad (0 \leq i \leq D).
\end{equation*}
By this and since $\{y|y \in X\}$ is an orthonormal basis for $V$,
\begin{equation*}
V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}.
\end{equation*}
For $0 \leq i \leq D$, $E_i^*$ acts on $V$ as the projection onto $E_i^* V$. We call $E_i^* V$ the $i$th subconstituent of $\Gamma$ with respect to $x$. For $0 \leq i \leq D$ we define $s_i = \sum \hat{y}$, where the sum is over all vertices $y \in X$ such that $\partial(x, y) = i$. We observe $s_i \in E_i^* V$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$, $E_0^*$, $E_1^*$, $\cdots$, $E_D^*$. The algebra $T$ is semisimple but not commutative in general [19, Lemma 3.4]. We call $T$ the Terwilliger algebra.
(or subconstituent algebra) of \( \Gamma \) with respect to \( x \). We refer the reader to [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24] for more information on the Terwilliger algebra. We will use the following facts. Pick any integers \( h, i, j \) (\( 0 \leq h, i, j \leq D \)). By [19, Lemma 3.2] we have 
\[ E_j^* A_h E_i^* = 0 \] if and only if \( p_{ij}^h = 0 \). By this and (2.2), (2.3) we find
\[
E_i^* A_h E_i^* = \begin{cases} 0 & \text{if } |h - i| > 1 \quad (0 \leq h, i \leq D), \\ E_i^* A E_j^* = 0 & \text{if } |i - j| > 1 \quad (0 \leq i, j \leq D). \end{cases}
\]

Lemma 3.1. The following (i), (ii) hold for \( 0 \leq i \leq D \).

(i) \( E_i^* J E_1^* = E_i^* A_{i-1} E_i^* + E_i^* A_i E_i^* + E_i^* A_{i+1} E_i^* \).

(ii) \( A_i E_i^* = E_{i-1}^* A_i E_i^* + E_{i}^* A_i E_i^* + E_{i+1}^* A_i E_i^* \).

Proof. (i) Recall \( J = \sum_{h=0}^D A_h \) so \( E_i^* J E_1^* = \sum_{h=0}^D E_i^* A_h E_1^* \). Evaluating this using (3.2) we obtain the result.

(ii) Recall \( J = \sum_{h=0}^D E_h^* \) so \( A_i E_i^* = \sum_{h=0}^D E_h^* A_i E_1^* \). Evaluating this using (3.2) we obtain the result. \( \square \)

Lemma 3.2. For \( 0 \leq i \leq D - 1 \) we have
\[
E_{i+1}^* A_i E_1^* - E_i^* A_{i+1} E_1^* = \sum_{h=0}^i A_h E_1^* - \sum_{h=0}^i E_h^* J E_1^*.
\]

Proof. Evaluate each term in the right-hand side of (3.4) using Lemma 3.1 and simplify the result. \( \square \)

Corollary 3.3. Let \( v \) denote a vector in \( E_1^* V \) which is orthogonal to \( s_1 \). Then for \( 0 \leq i \leq D - 1 \) we have
\[
E_{i+1}^* A_i v - E_i^* A_{i+1} v = \sum_{h=0}^i A_h v.
\]

Moreover \( E_0^* A v = 0 \).

Proof. To obtain (3.5) apply all terms of (3.4) to \( v \) and evaluate the result using \( E_1^* v = v \) and \( J v = 0 \). Setting \( i = 0 \) in (3.5) we find \( v - E_0^* A v = v \) so \( E_0^* A v = 0 \). \( \square \)

Lemma 3.4. The following (i), (ii) hold for \( 1 \leq i \leq D - 1 \).
(i) \( E_{i+1}^* AE_i^* A_{i-1} E_i^* = c_i E_{i+1}^* A_i E_i^* \)

(ii) \( E_{i-1}^* AE_i^* A_{i+1} E_i^* = b_i E_{i-1}^* A_i E_i^* \).

Proof. (i) For all \( y, z \in X \), on either side the \( yz \) entry is equal to \( c_i \) if \( \partial(x, y) = i + 1, \partial(x, z) = 1, \partial(y, z) = i \), and zero otherwise.

(ii) For all \( y, z \in X \), on either side the \( yz \) entry is equal to \( b_i \) if \( \partial(x, y) = i - 1, \partial(x, z) = 1, \partial(y, z) = i \), and zero otherwise.  \( \square \)

Corollary 3.5. Let \( v \) denote a vector in \( E_i^* V \). Then the following (i), (ii) hold for \( 1 \leq i \leq D - 1 \).

(i) Suppose \( E_i^* A_{i-1} v = 0. \) Then \( E_{i+1}^* A_i v = 0. \)

(ii) Suppose \( E_i^* A_{i+1} v = 0. \) Then \( E_{i-1}^* A_i v = 0. \)

Proof. In Lemma 3.4(i),(ii) apply both sides to \( v \) and use \( E_i^* v = v. \)  \( \square \)

4 The modules of the Terwilliger algebra

Let \( T \) denote the Terwilliger algebra of \( \Gamma \) with respect to \( x \). By a \( T \)-module we mean a subspace \( W \subseteq V \) such that \( BW \subseteq W \) for all \( B \in T \). Let \( W \) denote a \( T \)-module. Then \( W \) is said to be irreducible whenever \( W \) is nonzero and \( W \) contains no \( T \)-modules other than 0 and \( W \). Let \( W \) denote an irreducible \( T \)-module. Then \( W \) is the orthogonal direct sum of the nonzero spaces among \( E_0^* W, E_1^* W, \ldots, E_D^* W \) [19, Lemma 3.4]. By the endpoint of \( W \) we mean \( \min \{0 \leq i \leq D, E_i^* W \neq 0 \} \). By the diameter of \( W \) we mean \( \lfloor \{0 \leq i \leq D, E_i^* W \neq 0 \} \rfloor - 1 \). We say \( W \) is thin whenever \( E_i^* W \) has dimension at most 1 for \( 0 \leq i \leq D \). There exists a unique irreducible \( T \)-module which has endpoint 0 [10, Prop. 8.4]. This module is called \( V_0 \). For \( 0 \leq i \leq D \) the vector \( s_i \) is a basis for \( E_i^* V_0 \) [19, Lemma 3.6]. Therefore \( V_0 \) is thin with diameter \( D \). The module \( V_0 \) is orthogonal to each irreducible \( T \)-module other than \( V_0 \) [6, Lem. 3.3]. For more information on \( V_0 \) see [6, 10]. We will use the following facts.

Lemma 4.1. [19, Lemma 3.9] Let \( W \) denote an irreducible \( T \)-module with endpoint \( r \) and diameter \( d \). Then

\[ E_i^* W \neq 0 \quad (r \leq i \leq r + d). \]  (4.1)
Moreover
\[ E_i^* A E_j^* W \neq 0 \quad \text{if} \quad |i - j| = 1, \quad (r \leq i, j \leq r + d). \quad (4.2) \]

Lemma 4.2. [6, Lemma 3.4] Let \( W \) denote a \( T \)-module. Suppose there exists an integer \( i \) (0 \leq i \leq D) such that \( \dim(E_i^* W) = 1 \) and \( W = TE_i^* W \). Then \( W \) is irreducible.

Theorem 4.3. [12, Lemma 10.1], [22, Theorem 11.1] Let \( W \) denote a thin irreducible \( T \)-module with endpoint one, and let \( v \) denote a nonzero vector in \( E_1^* W \). Then \( W = M v \). Moreover the diameter of \( W \) is \( D - 2 \) or \( D - 1 \).

Theorem 4.4. [12, Corollary 8.6, Theorem 9.8] Let \( v \) denote a nonzero vector in \( E_1^* V \) which is orthogonal to \( s_1 \). Then the dimension of \( M v \) is \( D - 1 \) or \( D \). Suppose the dimension of \( M v \) is \( D - 1 \). Then \( M v \) is a thin irreducible \( T \)-module with endpoint 1 and diameter \( D - 2 \).

5 The proof of Theorem 1.1

We now give a proof of Theorem 1.1.

Proof. (i) \( \implies \) (ii) We show \( M v \) is a thin irreducible \( T \)-module with endpoint 1. By Theorem 4.4 the dimension of \( M v \) is either \( D - 1 \) or \( D \). First assume the dimension of \( M v \) is equal to \( D - 1 \). Then by Theorem 4.4, \( M v \) is a thin irreducible \( T \)-module with endpoint 1. Next assume the dimension of \( M v \) is equal to \( D \). The space \( (M; v) \) contains \( J \) and has dimension at least 2, so there exists \( P \in (M; v) \) such that \( J, P \) are linearly independent. From the construction \( P v \in E_D^* V \). Observe \( P v \neq 0 \); otherwise the dimension of \( M v \) is not \( D \). The elements \( A_0, A_1, \ldots, A_D \) form a basis for \( M \). Therefore the elements \( A_0 + A_1 + \cdots + A_i \) (0 \leq i \leq D) form a basis for \( M \). Apparently there exist complex scalars \( \rho_i \) (0 \leq i \leq D) such that \( P = \sum_{i=0}^D \rho_i (A_0 + A_1 + \cdots + A_i) \). Recall \( J = \sum_{h=0}^D A_h \). Subtracting a scalar multiple of \( J \) from \( P \) if necessary, we may assume \( \rho_D = 0 \). We consider \( P v \) from two points of view. On one hand we have \( P v \in E_D^* V \). Therefore \( E_D^* P v = P v \) and \( E_i^* P v = 0 \) for 0 \leq i \leq D - 1. On the other hand using (3.5),

\[
P v = \sum_{i=0}^{D-1} \rho_i (E_{i+1}^* A_i v - E_i^* A_{i+1} v).
\]
Combining these two points of view we find $Pv = \rho_{D-1}E_D^*A_{D-1}v$, $\rho_0E_0^*Av = 0$, and

$$\rho_{i-1}E_i^*A_{i-1}v = \rho_iE_i^*A_{i+1}v \quad (1 \leq i \leq D-1). \quad (5.1)$$

We mentioned $Pv \neq 0$; therefore $\rho_{D-1} \neq 0$ and $E_D^*A_{D-1}v \neq 0$. Applying Corollary 3.5(i) we find $E_i^*A_{i-1}v \neq 0$ for $1 \leq i \leq D$. We claim $E_i^*A_{i+1}v$ and $E_i^*A_{i-1}v$ are linearly dependent for $1 \leq i \leq D - 1$. Suppose there exists an integer $i$ ($1 \leq i \leq D - 1$) such that $E_i^*A_{i+1}v$ and $E_i^*A_{i-1}v$ are linearly independent. Then $E_i^*A_{i+1}v \neq 0$. Applying Corollary 3.5(ii) we find $E_j^*A_{j+1}v \neq 0$ for $i \leq j \leq D - 1$. Using these facts and (5.1) we routinely find $\rho_j = 0$ for $i \leq j \leq D - 1$. In particular $\rho_{D-1} = 0$ for a contradiction. We have now shown $E_i^*A_{i+1}v$ and $E_i^*A_{i-1}v$ are linearly dependent for $1 \leq i \leq D - 1$. Observe $Mv$ is spanned by the vectors

$$(A_0 + A_1 + \cdots + A_i)v \quad (0 \leq i \leq D-1).$$

By Corollary 3.3 and our above comments we find $Mv$ is contained in the span of

$$(0 \leq i \leq D - 1).$$

Since $Mv$ has dimension $D$ we find $Mv$ is equal to the span of (5.2). Apparently $Mv$ is a $T$-module. Moreover $Mv$ is irreducible by Lemma 4.2. Apparently $Mv$ is thin with endpoint 1.

$((iii) \implies (i))$ We show $(M;v)$ has dimension at least 2. Since $J \in (M;v)$ it suffices to exhibit an element $P \in (M;v)$ such that $J, P$ are linearly independent. Let $W$ denote a thin irreducible $T$-module which has endpoint 1 and contains $v$. By Theorem 4.3 we have $W = Mv$; also by Theorem 4.3 the diameter of $W$ is $D - 2$ or $D - 1$. First suppose $W$ has diameter $D - 2$. Then $W$ has dimension $D - 1$. Consider the map $\sigma : M \to V$ which sends each element $P$ to $Pv$. The image of $M$ under $\sigma$ is $Mv$ and the kernel of $\sigma$ is contained in $(M;v)$. The image has dimension $D - 1$ and $M$ has dimension $D + 1$ so the kernel has dimension 2. It follows $(M;v)$ has dimension at least 2. Next assume $W$ has diameter $D - 1$. In this case $E_D^*W \neq 0$ by (4.1). Since $W = Mv$ there exists $P \in M$ such that $Pv$ is a nonzero element in $E_D^*W$. Now $P \in (M;v)$. Observe $P, J$ are linearly independent since $Pv \neq 0$ and $Jv = 0$. Apparently the dimension of $(M;v)$ is at least 2.
Now assume (i), (ii) hold. We show the dimension of \((M; v)\) is 2. To do this, we show the dimension of \((M; v)\) is at most 2. Let \(H\) denote the subspace of \(M\) spanned by \(A_0, A_1, \ldots, A_{D-2}\). We show \(H\) has 0 intersection with \((M; v)\). By Theorem 4.4 the dimension of \(Mv\) is at least \(D-1\). Recall \(M\) is generated by \(A\) so the vectors \(A_i v\) \((0 \leq i \leq D-2)\) are linearly independent. Apparently the vectors \(A_i v\) \((0 \leq i \leq D-2)\) are linearly independent. For \(0 \leq i \leq D-2\) the vector \(A_i v\) is contained in \(\sum_{h=0}^{D-1} E_h^* V\) by Lemma 3.1(ii); therefore \(A_i v\) is orthogonal to \(E_h^* V\). We now see the vectors \(A_i v\) \((0 \leq i \leq D-2)\) are linearly independent and orthogonal to \(E_h^* V\). It follows \(H\) has 0 intersection with \((M; v)\). Observe \(H\) is codimension 2 in \(M\) so the dimension of \((M; v)\) is at most 2. We conclude the dimension of \((M; v)\) is 2. \(\square\)

6 Pseudo primitive idempotents

In this section we introduce the notion of a pseudo primitive idempotent.

Definition 6.1. For each \(\theta \in \mathbb{C} \cup \infty\) we define a subspace of \(M\) which we call \(M(\theta)\). For \(\theta \in \mathbb{C}\), \(M(\theta)\) consists of those elements \(Y\) of \(M\) such that \((A - \theta I)Y \in \mathbb{C}A_D\). We define \(M(\infty) = \mathbb{C}A_D\).

With reference to Definition 6.1, we will show each \(M(\theta)\) has dimension 1. To establish this we display a basis for \(M(\theta)\). We will use the following result.

Lemma 6.2. Let \(Y\) denote an element of \(M\) and write \(Y = \sum_{i=0}^{D} \rho_i A_i\). Let \(\theta\) denote a complex number. Then the following (i), (ii) are equivalent.

(i) \((A - \theta I)Y \in \mathbb{C}A_D\).

(ii) \(\rho_i = \rho_0 f_i(\theta) k_i^{-1}\) for \(0 \leq i \leq D\).

Proof. Evaluating \((A - \theta I)Y\) using \(Y = \sum_{i=0}^{D} \rho_i A_i\) and simplifying the result using (2.5) we obtain

\[(A - \theta I)Y = \sum_{i=0}^{D} A_i(c_i \rho_{i-1} + a_i \rho_i + b_i \rho_{i+1} - \theta \rho_i),\]

where \(\rho_{-1} = 0\) and \(\rho_{D+1} = 0\). Observe by (2.4), (2.6) that \(\rho_i = \rho_0 f_i(\theta) k_i^{-1}\) for \(0 \leq i \leq D\) if and only if \(c_i \rho_{i-1} + a_i \rho_i + b_i \rho_{i+1} = \theta \rho_i\) for \(0 \leq i \leq D - 1\). The result follows. \(\square\)
Corollary 6.3. For $\theta \in \mathbb{C}$ the following is a basis for $M(\theta)$.

$$\sum_{i=0}^{D} f_i(\theta)k_i^{-1}A_i.$$  \hspace{1cm} (6.1)

Proof. Immediate from Lemma 6.2. \hfill $\Box$

Corollary 6.4. The space $M(\theta)$ has dimension 1 for all $\theta \in \mathbb{C} \cup \infty$.

Proof. Suppose $\theta = \infty$. Then $M(\theta)$ has basis $A_D$ and therefore has dimension 1. Suppose $\theta \in \mathbb{C}$. Then $M(\theta)$ has dimension 1 by Corollary 6.3. \hfill $\Box$

Lemma 6.5. Let $\theta$ and $\theta'$ denote distinct elements of $\mathbb{C} \cup \infty$. Then $M(\theta) \cap M(\theta') = 0$.

Proof. This is a routine consequence of Corollary 6.3 and the fact that $M(\infty) = CA_D$. \hfill $\Box$

Corollary 6.6. For $0 \leq i \leq D$ we have $M(\theta_i) = CE_i$.

Proof. Observe $(A - \theta_i I)E_i = 0$ so $E_i \in M(\theta_i)$. The space $M(\theta_i)$ has dimension 1 by Corollary 6.4 and $E_i$ is nonzero so $E_i$ is a basis for $M(\theta_i)$. \hfill $\Box$

Remark 6.7. [2, p. 63] For $0 \leq j \leq D$ we have

$$E_j = m_j |X|^{-1} \sum_{i=0}^{D} f_i(\theta_j)k_i^{-1}A_i,$$

where $m_j$ denotes the rank of $E_j$.

Definition 6.8. Let $\theta \in \mathbb{C} \cup \infty$. By a \textit{pseudo primitive idempotent} for $\theta$ we mean a nonzero element of $M(\theta)$, where $M(\theta)$ is from Definition 6.1.

7 The local eigenvalues

Definition 7.1. Define a function $\tilde{\eta} : \mathbb{C} \cup \infty \rightarrow \mathbb{C} \cup \infty$ by

$$\tilde{\eta} = \begin{cases} \infty & \text{if } \eta = -1, \\
-1 & \text{if } \eta = \infty, \\
-1 - \frac{b_1}{1+\eta} & \text{if } \eta \neq -1, \eta \neq \infty. \end{cases}$$

Observe $\tilde{\eta} = \eta$ for all $\eta \in \mathbb{C} \cup \infty$. 

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Let \( v \) denote a nonzero vector in \( E^*_1 V \) which is orthogonal to \( s_1 \). Assume \( v \) is an eigenvector for \( E^*_1 AE^*_1 \) and let \( \eta \) denote the corresponding eigenvalue. We recall a few facts concerning \( \eta \) and \( \tilde{\eta} \). We have \( \tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D \) [18, Theorem 1].

If \( \eta = \tilde{\theta}_1 \) then \( \tilde{\eta} = \theta_1 \). If \( \eta = \tilde{\theta}_D \) then \( \tilde{\eta} = \theta_D \). We have \( \theta_D < -1 < \theta_1 \) by [18, Lemma 3] so \( \tilde{\theta}_1 < -1 < \tilde{\theta}_D \). If \( \tilde{\theta}_1 < \eta < -1 \) then \( \theta_1 < \tilde{\eta} \). If \( -1 < \eta < \tilde{\theta}_D \) then \( \tilde{\eta} < \theta_D \). We will show that if \( \tilde{\theta}_1 < \eta < \tilde{\theta}_D \) then \( \tilde{\eta} \) is not an eigenvalue of \( \Gamma \).

Given the above inequalities, to prove this it suffices to prove the following result.

**Proposition 7.2.** Let \( v \) denote a nonzero vector in \( E^*_1 V \). Assume \( v \) is an eigenvector for \( E^*_1 AE^*_1 \) and let \( \eta \) denote the corresponding eigenvalue. Then \( \tilde{\eta} \neq k \).

**Proof.** Suppose \( \tilde{\eta} = k \). Then \( \eta = \tilde{k} \) so by Definition 7.1,

\[
\eta = -1 - \frac{b_1}{k + 1}.
\]

By this and since \( b_1 < k \) we see \( \eta \) is a rational number such that \(-2 < \eta < -1\). In particular \( \eta \) is not an integer. Observe \( \eta \) is an eigenvalue of the subgraph of \( \Gamma \) induced on the set of vertices adjacent \( x \); therefore \( \eta \) is an algebraic integer. A rational algebraic integer is an integer so we have a contradiction. We conclude \( \tilde{\eta} \neq k \). \( \square \)

**Corollary 7.3.** Let \( v \) denote a nonzero vector in \( E^*_1 V \) which is orthogonal to \( s_1 \). Assume \( v \) is an eigenvector for \( E^*_1 AE^*_1 \) and let \( \eta \) denote the corresponding eigenvalue. Suppose \( \tilde{\theta}_1 < \eta < \tilde{\theta}_D \). Then \( \tilde{\eta} \) is not an eigenvalue of \( \Gamma \).

## 8 The proof of Theorem 1.2

We now give a proof of Theorem 1.2.

**Proof.** We first show \( E \) is contained in \((M; v)\). To do this we show \( Ev \in E^*_D V \). First suppose \( \eta \neq -1 \). Then \( \tilde{\eta} \in \mathbb{C} \) by Definition 7.1. By Definition 6.1 there exists \( \epsilon \in \mathbb{C} \) such that \((A - \tilde{\eta}I)E = \epsilon A_D \). By this and Lemma 3.1(ii),

\[
AEv = \tilde{\eta}Ev + \epsilon A_D v \\
\in \mathbb{C}Ev + E^*_D W + E^*_D W.
\]

(8.1)

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In order to show $Ev \in E_1^*V$ we show $E_i^*Ev = 0$ for $0 \leq i \leq D - 1$. Observe $E_0^*Ev = 0$ since $E_0^*Ev \in E_0^*W$ and $W$ has endpoint 1. We show $E_1^*Ev = 0$. By Corollary 6.3 there exists a nonzero $m \in \mathbb{C}$ such that

$$E = m \sum_{h=0}^{D} f_h(\bar{\eta})k_h^{-1}A_h.$$  

Let us abbreviate

$$\rho_h = m f_h(\bar{\eta})k_h^{-1} \quad (0 \leq h \leq D),$$  

so that $E = \sum_{h=0}^{D} \rho_h A_h$. By this and (3.2) we find $E_1^*E_1^* = \sum_{h=0}^{2} \rho_h E_1^*A_h E_1^*$. Applying this to $v$ we find

$$E_1^*Ev = \sum_{h=0}^{2} \rho_h E_1^*A_h v.$$  

Setting $i = 1$ in Lemma 3.1(i), applying each term to $v$, and using $Jv = 0$ we find

$$0 = \sum_{h=0}^{2} E_1^*A_h v.$$  

By (8.3), (8.4), and since $E_1^*Av = \eta v$ we find $E_1^*Ev = \gamma v$ where $\gamma = \rho_0 - \rho_2 + \eta(\rho_1 - \rho_2)$. Evaluating $\gamma$ using (2.6), (8.2), and Definition 7.1 we routinely find $\gamma = 0$. Apparently $E_1^*Ev = 0$. We now show $E_i^*Ev = 0$ for $2 \leq i \leq D - 1$. Suppose there exists an integer $j$ ($2 \leq j \leq D - 1$) such that $E_j^*Ev \neq 0$. We choose $j$ minimal so that

$$E_j^*Ev = 0 \quad (0 \leq i \leq j - 1).$$  

Combining this with (8.1) we find

$$E_i^*AEv = 0 \quad (0 \leq i \leq j - 1).$$  

Since $W$ is thin and since $E_j^*Ev \neq 0$ we find $E_j^*Ev$ is a basis for $E_j^*W$. Apparently $E_{j-1}^*AE_j^*Ev$ spans $E_{j-1}^*AE_j^*W$. The space $E_{j-1}^*AE_j^*W$ is nonzero by (4.2) and since the diameter of $W$ is at least $D - 2$. Therefore $E_{j-1}^*AE_j^*Ev \neq 0$. We may now argue

$$E_{j-1}^*AEv = \sum_{i=0}^{D} E_{j-1}^*AE_i^*Ev$$

$$= E_{j-1}^*AE_j^*Ev \quad \text{by (3.3), (8.5)}$$

$$\neq 0$$  

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which contradicts (8.6). We conclude $E^*_i Ev = 0$ for $2 \leq i \leq D - 1$. We have now shown $E^*_i Ev = 0$ for $0 \leq i \leq D - 1$ so $Ev \in E_D^* V$ in the case $\eta \neq -1$. Next suppose $\eta = -1$, so that $\tilde{\eta} = \infty$. By Definition 6.1 there exists a nonzero $t \in \mathbb{C}$ such that $E = tA_D$. In order to show $Ev \in E_D^* V$ we show $A_D v \in E_D^* V$. Since $A_D v$ is contained in $E_{D-1}^* V + E_D^* V$ by Lemma 3.1(ii), it suffices to show $E_{D-1}^* A_D v = 0$. To do this it is convenient to prove a bit more, that $E_i^* A_{i+1} v = 0$ for $1 \leq i \leq D - 1$. We prove this by induction on $i$. First assume $i = 1$. Setting $i = 1$ in Lemma 3.1(i), applying each term to $v$ and using $Jv = 0$, $E_1^* A v = -v$, we obtain $E_1^* A_2 v = 0$. Next suppose $2 \leq i \leq D - 1$ and assume by induction that $E_{i-1}^* A_i v = 0$. We show $E_i^* A_{i+1} v = 0$. To do this we assume $E_i^* A_{i+1} v \neq 0$ and get a contradiction. Note that $E_i^* A_{i+1} v$ spans $E_i^* W$ since $W$ is thin. Then $E_{i-1}^* AE_i^* A_{i+1} v \neq 0$ by (4.2). But $E_{i-1}^* AE_i^* A_{i+1} v = b_i E_{i-1}^* A_i v$ by Lemma 3.4(ii). Of course $b_i \neq 0$ so $E_{i-1}^* A_i v \neq 0$, a contradiction. Therefore $E_i^* A_{i+1} v = 0$. We have now shown $E_i^* A_{i+1} v = 0$ for $1 \leq i \leq D - 1$ and in particular $E_{D-1}^* A_D v = 0$. It follows $Ev \in E_D^* V$ for the case $\eta = -1$. We have now shown $Ev \in E_D^* V$ for all cases so $E \in (M; v)$. We now prove $E, J$ form a basis for $(M; v)$. By Theorem 1.1 $(M; v)$ has dimension 2. We mentioned earlier $J \in (M; v)$. We show $E, J$ are linearly independent. Recall $E, J$ are pseudo primitive idempotents for $\tilde{\eta}, k$ respectively. We have $\tilde{\eta} \neq k$ by Proposition 7.2 so $E, J$ are linearly independent in view of Lemma 6.5. \hfill $\Box$

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