Bidiagonal triples and the quantum group $U_q(sl_2)$

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The Basis of \( sl_2(\mathbb{K}) \)

\[
\begin{align*}
    h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
    e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
    f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

\( h, e, f \) satisfy

\[
\begin{align*}
    [h, e] &= 2e, \\
    [h, f] &= -2f, \\
    [e, f] &= h.
\end{align*}
\]
Another Basis of $sl_2(\mathbb{K})$

\[
A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}.
\]

$A, B, C$ satisfy

\[
\]
Let $X$ denote a square matrix. We say $X$ is **upper bidiagonal** whenever both (i) each nonzero entry of $X$ is on the diagonal or superdiagonal; (ii) each entry on the superdiagonal of $X$ is nonzero. We say $X$ is **lower bidiagonal** whenever the transpose of $X$ is upper bidiagonal.
Examples

\[
A = \begin{pmatrix}
q^{-3} & q^3 - q^{-3} & 0 & 0 \\
0 & q^{-1} & q^3 - q^{-1} & 0 \\
0 & 0 & q & q^3 - q \\
0 & 0 & 0 & q^3
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
q^{-3} & 0 & 0 & 0 \\
q^{-3} - q^{-1} & q^{-1} & 0 & 0 \\
0 & q^{-3} - q & q & 0 \\
0 & 0 & q^{-3} - q^3 & q^3
\end{pmatrix},
\]

where \( q^2 \neq 1, q^4 \neq 1, q^6 \neq 1. \)
Let $\mathbb{K}$ be an algebraically closed field with characteristic 0. Let $\mathbb{V}$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a **bidiagonal triple** on $\mathbb{V}$ we mean a sequence of linear transformations $A, B, C : \mathbb{V} \to \mathbb{V}$ that satisfy the following three conditions:
Bidiagonal Triple

(i) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing $A, B, C$ are upper bidiagonal, diagonal, and lower bidiagonal, respectively.

(ii) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing $B, C, A$ are upper bidiagonal, diagonal, and lower bidiagonal, respectively.

(iii) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing $C, A, B$ are upper bidiagonal, diagonal, and lower bidiagonal, respectively.
Example

$A, C$ as before and

$$B = \begin{pmatrix}
q^3 & 0 & 0 & 0 \\
0 & q & 0 & 0 \\
0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & q^{-3}
\end{pmatrix}.$$ 

Then $A, B, C$ is a bidiagonal triple.
Proof

\[ P = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & q^{-2} & -1 - q^{-2} & 1 \\
-q^{-6} & q^{-2} + q^{-4} + q^{-6} & -1 - q^{-2} - q^{-4} & 1 \\
\end{pmatrix}. \]

Then

\[ P^{-1}BP = A, P^{-1}CP = B, P^{-1}AP = C. \]
(i) $A$ is upper bidiagonal with entries $A_{ii} = q^{2i-n}$ for $0 \leq i \leq n$ and $A_{i,i+1} = q^n - q^{2i-n}$ for $0 \leq i \leq n - 1$.

(ii) $B$ is diagonal with $B_{ii} = q^{n-2i}$ for $0 \leq i \leq n$.

(iii) $C$ is lower bidiagonal with entries $C_{ii} = q^{2i-n}$ for $0 \leq i \leq n$ and $C_{i,i-1} = q^{-n} - q^{2i-n}$ for $1 \leq i \leq n$.

Then the sequence $A, B, C$ is a bidiagonal triple on $\mathbb{K}^{n+1}$ (with base $q$).
(i) $A$ is upper bidiagonal with entries $A_{ii} = 2i - n$ for $0 \leq i \leq n$ and $A_{i,i+1} = 2n - 2i$ for $0 \leq i \leq n - 1$.

(ii) $B$ is diagonal with $B_{ii} = n - 2i$ for $0 \leq i \leq n$.

(iii) $C$ is lower bidiagonal with entries $C_{ii} = 2i - n$ for $0 \leq i \leq n$ and $C_{i,i-1} = -2i$ for $1 \leq i \leq n$.

Then the sequence $A, B, C$ is a bidiagonal triple on $\mathbb{K}^{n+1}$ (with base $q = 1$).
Normalized Bidiagonal Triples

We refer all of the above mentioned bidiagonal triples as normalized bidisgonal triples with base $q$. 
Lemma

Let $A, B, C$ denote a bidiagonal triple on $\mathbb{V}$. Let $\alpha^\pm, \beta^\pm, \gamma^\pm$ denote scalars in $\mathbb{K}$ with $\alpha^+, \beta^+, \gamma^+$ nonzero. Then the sequence

$$\alpha^+ A + \alpha^- I, \quad \beta^+ B + \beta^- I, \quad \gamma^+ C + \gamma^- I$$

is a bidiagonal triple on $\mathbb{V}$. 
Affine Equivalence

Let $A, B, C$ and $A', B', C'$ denote two bidiagonal triples on $\mathbb{V}$. We say these two sequences are affine equivalent whenever

$$A' = \alpha^+ A + \alpha^- I, \quad B' = \beta^+ B + \beta^- I, \quad C' = \gamma^+ C + \gamma^- I$$

for some scalars $\alpha^\pm, \beta^\pm, \gamma^\pm \in \mathbb{K}$ with $\alpha^+, \beta^+, \gamma^+$ nonzero.
Main Theorem

Each bidiagonal triple is affine equivalent to a normalized bidiagonal triple with base $q$. 
This algebra has a basis $e, f, h$ satisfying

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,
\]

where $[,]$ denotes the Lie bracket.
There exists a family

$$\mathbb{V}_n \quad n = 0, 1, 2, \ldots$$

(1)

of finite dimensional irreducible $sl_2$-modules with the following properties. The module $\mathbb{V}_n$ has a basis $v_0, v_1, \ldots, v_n$ satisfying $hv_i = (n - 2i)v_i$ for $0 \leq i \leq n$, $fv_i = (i + 1)v_{i+1}$ for $0 \leq i \leq n - 1$, $fv_d = 0$, $ev_i = (n - i + 1)v_{i-1}$ for $1 \leq i \leq n$, $ev_0 = 0$. 
Irreducible $sl_2$-Modules

Every irreducible $sl_2$-module of dimension $n + 1$ is isomorphic to the $\mathcal{V}_n$ in previous slide.
Set $x = -h + 2e, y = h, z = -h - 2f$ in $sl_2$. Then $x, y, z$ is another basis of $sl_2$ satisfying

$$[x, y] = -2x - 2y, [y, z] = -2y - 2z, [z, x] = -2z - 2x.$$
The alternate basis $x, y, z$ of $sl_2$ act on $\nabla_n$ as a bidiagonal triple.
Quantum algebra $U_q(sl_2)$ is the unital associative $\mathbb{K}$-algebra with generators $e, f, k, k^{-1}$ and the following relations:

\[
kk^{-1} = k^{-1}k = 1, \\
kek^{-1} = q^2 e, \ kfk^{-1} = q^{-2} f, \\
ef - fe = \frac{k - k^{-1}}{q - q^{-1}},
\]

where $q \in \mathbb{K}$ is not a root of unity.
The quantum algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x, y, z, z^{-1}$ and the following relations:

$$yy^{-1} = y^{-1}y = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$
Proof

An isomorphism is given by:

\[ y^{\pm 1} \rightarrow k^{\pm 1}, \]
\[ z \rightarrow k^{-1} + f, \]
\[ x \rightarrow k^{-1} - q(q - q^{-1})^2 k^{-1} e. \]

The inverse of this isomorphism is given by:

\[ k^{\pm 1} \rightarrow y^{\pm 1}, \]
\[ f \rightarrow z - y^{-1}, \]
\[ e \rightarrow \frac{1 - yx}{q(q - q^{-1})^2}. \]
Irreducible $U_q(sl_2)$-Modules

There exists a family

$$\mathcal{V}_{\varepsilon,n} \quad \varepsilon \in \{1, -1\}, \quad n = 0, 1, 2 \ldots$$

of finite dimensional irreducible $U_q(sl_2)$-modules with the following properties. The module $\mathcal{V}_{\varepsilon,n}$ has a basis $u_0, u_1, \ldots, u_n$ such that $ku_i = \varepsilon q^{n-2i}u_i$ for $0 \leq i \leq n$, $fu_i = [i+1]_q u_{i+1}$ for $0 \leq i \leq n-1$, $fu_n = 0$, $eu_i = \varepsilon [n-i+1]_q u_{i-1}$ for $1 \leq i \leq n$, $eu_0 = 0$. 
Every irreducible $U_q(sl_2)$-module of dimension $n + 1$ is isomorphic to $\mathbb{V}_{-1,n}$ or $\mathbb{V}_{1,n}$. 
Let $V_{\varepsilon,n}$ denote the finite dimensional irreducible $U_q(sl_2)$-module. Then the alternate generators $\varepsilon x, \varepsilon y, \varepsilon z$ act on $V_{\varepsilon,n}$ as a bidiagonal triple.
Thank You