Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition*

Paul Terwilliger

Abstract

Let $K$ denote a field and let $d$ denote a nonnegative integer. Let $\mathcal{A}$ denote a $K$-algebra isomorphic to $\text{Mat}_{d+1}(K)$. An element of $\mathcal{A}$ is called multiplicity-free whenever its eigenvalues are mutually distinct and contained in $K$. Let $A$ and $A^*$ denote multiplicity-free elements in $\mathcal{A}$. Let $\{E_i\}_{i=0}^d$ (resp. $\{E_i^*\}_{i=0}^d$) denote an ordering of the primitive idempotents of $A$ (resp. $A^*$). For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) for $E_i$ (resp. $E_i^*$). Let $V$ denote an irreducible left $\mathcal{A}$-module. By a decomposition of $V$ we mean a sequence $\{U_i\}_{i=0}^d$ consisting of 1-dimensional subspaces of $V$ such that $V = \sum_{i=0}^d U_i$. A decomposition $\{U_i\}_{i=0}^d$ of $V$ is said to be split (with respect to the orderings $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$) whenever both (i) $(A - \theta_i I) U_i = U_{i+1}$ ($0 \leq i \leq d-1$), $(A - \theta_d I) U_d = 0$; and (ii) $(A^* - \theta_i^* I) U_i = U_{i-1}$ ($1 \leq i \leq d$), $(A^* - \theta_0^* I) U_0 = 0$. We show there exists at most one decomposition of $V$ which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$. We show the following are equivalent: (i) there exists a decomposition of $V$ which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) both

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases}$$

for $0 \leq i, j \leq d$. We call the sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ a Leonard system whenever both

$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} \quad E_i A^* E_j = \begin{cases} 0, & \text{if } |j - i| > 1; \\ \neq 0, & \text{if } |j - i| = 1 \end{cases}$$

for $0 \leq i, j \leq d$. We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i) there exists a decomposition of $V$ which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) there exists a decomposition of $V$ which is split with respect to $\{E_{d-i}\}_{i=0}^d, \{E_{d-i}^*\}_{i=0}^d$. We also show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if both (i) there exists a decomposition of $V$ which is split with respect to $\{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$; (ii) there exists an antiisomorphism $\dagger$ of $\mathcal{A}$ such that $A^\dagger = A$ and $A^*\dagger = A^*$.

*Keywords. Leonard pair, Tridiagonal pair, Askey-Wilson polynomial, q-Racah polynomial.

2000 Mathematics Subject Classification. 05E30, 17B37, 33C45, 33D45.
1 Leonard pairs and Leonard systems

We begin by recalling the notion of a Leonard pair \([6], [13], [14], [15], [16], [17], [18]\). We will use the following terms. Let \(X\) denote a square matrix. Then \(X\) is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume \(X\) is tridiagonal. Then \(X\) is called irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper \(\mathbb{K}\) will denote a field.

**Definition 1.1** \([13]\) Let \(V\) denote a vector space over \(\mathbb{K}\) with finite positive dimension. By a Leonard pair on \(V\), we mean an ordered pair of linear transformations \(A : V \to V\) and \(A^* : V \to V\) which satisfy (i), (ii) below.

(i) There exists a basis for \(V\) with respect to which the matrix representing \(A\) is irreducible tridiagonal and the matrix representing \(A^*\) is diagonal.

(ii) There exists a basis for \(V\) with respect to which the matrix representing \(A^*\) is irreducible tridiagonal and the matrix representing \(A\) is diagonal.

**Note 1.2** According to a common notational convention \(A^*\) denotes the conjugate-transpose of \(A\). We emphasize we are not using this convention. In a Leonard pair \(A, A^*\) the linear transformations \(A\) and \(A^*\) are arbitrary subject to (i), (ii) above.

Our use of the name “Leonard pair” is motivated by a connection to a theorem of D. Leonard \([2, p. 260], [9]\) which involves the \(q\)-Racah polynomials \([1], [3, p. 162]\) and some related polynomials of the Askey scheme \([7]\). This connection is discussed in \([13, \text{Appendix A}]\) and \([15, \text{Section 16}]\). See \([4], [5], [8], [10], [19]\) for related topics.

Before proceeding we recall a few more terms. Let \(d\) denote a nonnegative integer. Let \(\text{Mat}_{d+1}(\mathbb{K})\) denote the \(\mathbb{K}\)-algebra consisting of all \(d+1\) by \(d+1\) matrices which have entries in \(\mathbb{K}\). We index the rows and columns by \(0, 1, \ldots, d\). Let \(\mathbb{K}^{d+1}\) denote the \(\mathbb{K}\)-vector space consisting of all \(d+1\) by 1 matrices which have entries in \(\mathbb{K}\). We index the rows by \(0, 1, \ldots, d\). We view \(\mathbb{K}^{d+1}\) as a left module for \(\text{Mat}_{d+1}(\mathbb{K})\). We observe this module is irreducible. For the rest of this paper we let \(A\) denote a \(\mathbb{K}\)-algebra isomorphic to \(\text{Mat}_{d+1}(\mathbb{K})\). When we refer to an \(A\)-module we mean a left \(A\)-module. For the rest of this paper we let \(V\) denote an irreducible \(A\)-module. Let \(v_0, v_1, \ldots, v_d\) denote a basis for \(V\). For \(X \in \text{Mat}_{d+1}(\mathbb{K})\) and for \(Y \in A\), we say \(X\) represents \(Y\) with respect to \(v_0, v_1, \ldots, v_d\) whenever \(Y v_j = \sum_{i=0}^{d} X_{ij} v_i\) for \(0 \leq j \leq d\). Let \(A\) denote an element of \(A\). By an eigenvalue of \(A\) we mean a root of the minimal polynomial of \(A\). The eigenvalues of \(A\) are contained in the algebraic closure of \(\mathbb{K}\). The element \(A\) will be called multiplicity-free whenever it has \(d+1\) distinct eigenvalues, all of which are in \(\mathbb{K}\). Let \(A\) denote a multiplicity-free element of \(A\). Let \(\theta_0, \theta_1, \ldots, \theta_d\) denote an ordering of the eigenvalues of \(A\), and for \(0 \leq i \leq d\) put

\[
E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j},
\]
where $I$ denotes the identity of $\mathcal{A}$. We observe (i) $AE_i = \theta_i E_i$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$); (iii) $\sum_{i=0}^{d} E_i = I$. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Using (i)-(iii) we find that as a $K$-vector space, $\mathcal{D}$ has a basis $E_0, E_1, \ldots, E_d$. We refer to $E_i$ as the \textit{primitive idempotent} of $A$ associated with $\theta_i$. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \quad \text{(direct sum)}.$$  

For $0 \leq i \leq d$, $E_i V$ is the (one dimensional) eigenspace of $A$ in $V$ associated with the eigenvalue $\theta_i$, and $E_i$ acts on $V$ as the projection onto this eigenspace. We remark that the sequence $\mathcal{A}^i$ ($0 \leq i \leq d$) is a basis for $\mathcal{D}$ and that $\prod_{i=0}^{d}(A - \theta_i I) = 0$. By a \textit{Leonard pair in $\mathcal{A}$} we mean an ordered pair of elements taken from $\mathcal{A}$ which act on $V$ as a Leonard pair in the sense of Definition 1.1. When working with a Leonard pair, it is often convenient to consider a closely related object which we call a \textit{Leonard system}. A Leonard system is defined as follows.

**Definition 1.3** [13] By a \textit{Leonard system in $\mathcal{A}$}, we mean a sequence $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ which satisfies (i)-(v) below.

(i) Each of $A, A^*$ is a multiplicity-free element of $\mathcal{A}$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of $A$.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\
\neq 0, & \text{if } |i-j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

(v) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\
\neq 0, & \text{if } |i-j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

Leonard pairs and Leonard systems are related as follows. Let $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ denote a Leonard system in $\mathcal{A}$. For $0 \leq i \leq d$ let $v_i$ denote a nonzero vector in $E_i V$. Then $v_0, v_1, \ldots, v_d$ is a basis for $V$ which satisfies Definition 1.1(ii). For $0 \leq i \leq d$ let $v_i^*$ denote a nonzero vector in $E_i^* V$. Then $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for $V$ which satisfies Definition 1.1(i). By these comments the pair $A, A^*$ is a Leonard pair in $\mathcal{A}$. Conversely let $A, A^*$ denote a Leonard pair in $\mathcal{A}$. By [13, Lemma 1.3] each of $A, A^*$ is multiplicity-free. Let $v_0, v_1, \ldots, v_d$ denote a basis for $V$ which satisfies Definition 1.1(ii). For $0 \leq i \leq d$ the vector $v_i$ is an eigenvector for $A$; let $E_i$ denote the corresponding primitive idempotent of $A$. Let $v_0^*, v_1^*, \ldots, v_d^*$ denote a basis for $V$ which satisfies Definition 1.1(i). For $0 \leq i \leq d$ the vector $v_i^*$ is an eigenvector for $A^*$; let $E_i^*$ denote the corresponding primitive idempotent of $A^*$. Then $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ is a Leonard system in $\mathcal{A}$ [15, Lemma 2.3]. In summary we have the following.

**Lemma 1.4** Let $A$ and $A^*$ denote elements in $\mathcal{A}$. Then the pair $A, A^*$ is a Leonard pair in $\mathcal{A}$ if and only if the following (i), (ii) hold.

(i) Each of $A, A^*$ is multiplicity-free.
(ii) There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $A$ and there exists an ordering $E^*_0, E^*_1, \ldots, E^*_d$ of the primitive idempotents of $A^*$ such that $(A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ is a Leonard system in $A$.

Later in this paper we will obtain two characterizations of Leonard systems. These characterizations are based on a concept which we call the split decomposition. This concept is explained in the next section.

## 2 The split decomposition

In this section we introduce the split decomposition. We will refer to the following set-up.

**Definition 2.1** Let $A$ and $A^*$ denote multiplicity-free elements in $A$. Let $E_0, E_1, \ldots, E_d$ denote an ordering of the primitive idempotents of $A$ and for $0 \leq i \leq d$ let $\theta_i$ denote the eigenvalue of $A$ for $E_i$. Let $E^*_0, E^*_1, \ldots, E^*_d$ denote an ordering of the primitive idempotents of $A^*$ and for $0 \leq i \leq d$ let $\theta^*_i$ denote the eigenvalue of $A^*$ for $E^*_i$. We let $D$ (resp. $D^*$) denote the subalgebra of $A$ generated by $A$ (resp. $A^*$). We let $V$ denote an irreducible $A$-module.

With reference to Definition 2.1, by a decomposition of $V$ we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of 1-dimensional subspaces of $V$ such that

$$ V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)}. $$

Let $u_0, u_1, \ldots, u_d$ denote a basis for $V$ and for $0 \leq i \leq d$ let $U_i$ denote the subspace of $V$ spanned by $u_i$. Then the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of $V$. Conversely, let $U_0, U_1, \ldots, U_d$ denote a decomposition of $V$. For $0 \leq i \leq d$ let $u_i$ denote a nonzero vector in $U_i$. Then $u_0, u_1, \ldots, u_d$ is a basis for $V$.

**Definition 2.2** With reference to Definition 2.1, let $U_0, U_1, \ldots, U_d$ denote a decomposition of $V$. We say this decomposition is split (with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$) whenever both

$$ (A - \theta_i I)U_i = U_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_d I)U_d = 0, \quad (1) $$

$$ (A^* - \theta^*_i I)U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta^*_0 I)U_0 = 0. \quad (2) $$

We consider the existence and uniqueness of the split decomposition. We start with uniqueness.

**Lemma 2.3** With reference to Definition 2.1, the following (i), (ii) hold.

(i) Assume there exists a decomposition $U_0, U_1, \ldots, U_d$ of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. Then $U_i = \prod_{h=0}^{i-1}(A - \theta_h I)E_0 V$ and $U_i = \prod_{h=i+1}^d(A^* - \theta^*_h I)E_d V$ for $0 \leq i \leq d$.

(ii) There exists at most one decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. 

4
Proof: (i) From the equation on the right in (2) we find \( U_0 = E_0^* V \). Using this and (1) we obtain \( U_i = \prod_{h=0}^{i-1}(A - \theta_h I)E_0^* V \) for \( 0 \leq i \leq d \). From the equation on the right in (1) we find \( U_d = E_d V \). Using this and (2) we obtain \( U_i = \prod_{h=i+1}^{d}(A^* - \theta_h^* I)E_d V \) for \( 0 \leq i \leq d \). (ii) Immediate from (i) above.

We turn our attention to the existence of the split decomposition. In Section 4 we will give necessary and sufficient conditions for this existence. We will use the following result.

**Lemma 2.4** With reference to Definition 2.1, assume there exists a decomposition \( U_0, U_1, \ldots, U_d \) of \( V \) which is split with respect to \( E_0, E_1, \ldots, E_d \) and \( E_0^*, E_1^*, \ldots, E_d^* \). Then the following (i)-(v) hold for \( 0 \leq i \leq d \).

(i) \( \sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E_0^* V \).

(ii) \( \sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E_h^* V \).

(iii) \( \sum_{h=i}^{d} U_h = \sum_{h=0}^{d-i} A^h E_d V \).

(iv) \( \sum_{h=i}^{d} U_h = \sum_{h=0}^{d-i} E_h V \).

(v) \( U_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_i V + E_{i+1} V + \cdots + E_d V) \).

Proof: (i) For \( 0 \leq j \leq d \) we have \( U_j = \prod_{h=0}^{j-1}(A - \theta_h I)E_0^* V \) by Lemma 2.3(i) so \( U_j \subseteq \sum_{h=0}^{j} A^h E_0^* V \). Apparently \( \sum_{h=0}^{i} U_h \subseteq \sum_{h=0}^{i} A^h E_0^* V \). In this inclusion the sum on the left has dimension \( i + 1 \) since \( U_0, U_1, \ldots, U_d \) is a decomposition. The sum on the right has dimension at most \( i + 1 \). Therefore \( \sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E_0^* V \).

(ii) For \( 0 \leq j \leq d \) we have \( \prod_{h=0}^{j}(A^* - \theta_h^* I)U_j = 0 \) by (2) so \( U_j \subseteq \sum_{h=0}^{j} E_h^* V \). Apparently \( \sum_{h=0}^{i} U_i \subseteq \sum_{h=0}^{i} E_h^* V \). In this inclusion each side has dimension \( i + 1 \) so equality holds.

(iii) Similar to the proof of (i) above.

(iv) Similar to the proof of (ii) above.

(v) Combine (ii), (iv) above.

\[ \square \]

### 3 Some products

With reference to Definition 2.1, our next goal is to display necessary and sufficient conditions for the existence of the split decomposition. These conditions involve the following products.

\[ E_i^* A E_j^*, \quad E_i A^* E_j \quad (0 \leq i, j \leq d). \]

We are interested in which of these products are 0. We begin by considering just one of these products.
Lemma 3.1 With reference to Definition 2.1. for $0 \leq i \leq d$ let $v_i^*$ denote a nonzero vector in $E_i^* V$ and observe $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for $V$. Let $B$ denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents $A$ with respect to this basis, so that

$$A v_j^* = \sum_{i=0}^{d} B_{ij} v_i^* \quad (0 \leq j \leq d). \quad (3)$$

Then for $0 \leq i, j \leq d$ the following are equivalent: (i) $E_i^* A E_j^* = 0$; (ii) $B_{ij} = 0$.

Proof: Let the integers $i, j$ be given. Observe $E_i^* v_s^* = \delta_{r,s} v_r^*$ for $0 \leq r, s \leq d$. By this and (3) we find $E_i^* A E_j^* V$ is spanned by $B_{ij} v_i^*$. The result follows. \qed

In the next lemma we consider a certain pattern of vanishing products among the $E_i^* A E_j^*$. We will use the following notation. Let $\lambda$ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in $\lambda$ which have coefficients in $\mathbb{K}$. Let $f_0, f_1, \ldots, f_d$ denote a sequence of polynomials taken from $\mathbb{K}[\lambda]$. We say this sequence is graded whenever $f_0 = 1$ and $f_i$ has degree exactly $i$ for $0 \leq i \leq d$.

Lemma 3.2 With reference to Definition 2.1, the following (i)-(iii) are equivalent.

(i) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d).$

(ii) There exists a graded sequence of polynomials $f_0, f_1, \ldots, f_d$ taken from $\mathbb{K}[\lambda]$ such that $E_i^* V = f_i(A) E_0^* V$ for $0 \leq i \leq d$.

(iii) For $0 \leq i \leq d$,

$$\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V. \quad (4)$$

Proof: (i) $\Rightarrow$ (ii) For $0 \leq i \leq d$ let $v_i^*$ denote a nonzero vector in $E_i^* V$ and observe $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for $V$. Let $B$ denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents $A$ with respect to this basis. By Lemma 3.1,

$$B_{ij} = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d). \quad (5)$$

Let $f_0, f_1, \ldots, f_d$ denote the polynomials in $\mathbb{K}[\lambda]$ which satisfy $f_0 = 1$ and

$$\lambda f_j = \sum_{i=0}^{j+1} B_{ij} f_i \quad (0 \leq j \leq d - 1). \quad (6)$$

We observe $f_i$ has degree exactly $i$ for $0 \leq i \leq d$ so the sequence $f_0, f_1, \ldots, f_d$ is graded. Comparing (3) and (6) in light of (5) we find $f_i(A)v_0^* = v_i^*$ for $0 \leq i \leq d$. The result follows.

(ii) $\Rightarrow$ (iii) For $0 \leq j \leq d$ we have $E_j^* V = f_j(A) E_0^* V$. The degree of $f_j$ is $j$ so $E_j^* V \subseteq$
\[ \sum_{h=0}^{d} A^h E_0^* V. \] Apparently \( \sum_{h=0}^{i} E_h^* V \subseteq \sum_{h=0}^{i} A^h E_0^* V. \) In this inclusion the sum on the left has dimension \( i + 1 \) and the sum on the right has dimension at most \( i + 1 \). Therefore \( \sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V. \)

\((iii) \Rightarrow (i)\) For \( 0 \leq i \leq d \) let \( V_i \) denote the subspace on the left or right in (4). From the right-hand side of (4) we find \( V_i + AV_i = V_{i+1} \) for \( 0 \leq i \leq d - 1 \). From the left-hand side of (4) we find \( E_s^* V_s = 0 \) for \( 0 \leq s < r \leq d \). Let \( i, j \) denote integers \( 0 \leq i, j \leq d \) and first assume \( i - j > 1 \). We show \( E_i^* A E_j^* = 0 \). Observe \( E_i^* V \subseteq V_j \) and \( AV_j \subseteq V_{j+1} \) so \( AE_j^* V \subseteq V_{j+1} \). However \( E_i^* V_{j+1} = 0 \) since \( i - j > 1 \) so \( E_i^* A E_j^* V = 0 \). It follows \( E_i^* A E_j^* = 0 \). Next we assume \( i - j = 1 \) and show \( E_i^* A E_j^* \neq 0 \). Suppose \( E_i^* A E_j^* = 0 \). By this and our previous remarks we find \( E_i^* A E_h^* = 0 \) for \( 0 \leq h \leq j \). By this and since \( V_i = \sum_{h=0}^{i} E_h^* V \) we find \( E_i^* A V_j = 0 \). However \( V_i = V_j + AV_j \) and \( E_i^* V_j = 0 \) so \( E_i^* V_i = 0 \). This contradicts the construction so \( E_i^* A E_j^* \neq 0 \).

**Corollary 3.3** Referring to Lemma 3.2, assume the equivalent conditions \((i)-(iii)\) hold. Let \( v_0^* \) denote a nonzero vector in \( E_0^* V \). Then the homomorphism of \( \mathbb{K} \)-vector spaces from \( D \) to \( V \) which sends each \( X \) to \( X v_0^* \) is an isomorphism.

**Proof:** Since \( D \) and \( V \) have the same dimension it suffices to show the map is surjective. Setting \( i = d \) in (4) we find \( V = D v_0^* \) so the map is surjective. \( \square \)

Replacing \( (A; A^*; \{E_i\}_{i=0}^{d}; \{E^*_{d-i}\}_{i=0}^{d}) \) by \( (A^*; A; \{E^*_{d-i}\}_{i=0}^{d}; \{E_{d-i}\}_{i=0}^{d}) \) in Lemma 3.2 and Corollary 3.3 we routinely obtain the following results.

**Lemma 3.4** With reference to Definition 2.1, the following \((i)-(iii)\) are equivalent.

\( (i) \) \( E_i^* A E_j^* = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} \) \( (0 \leq i, j \leq d) \).

\( (ii) \) There exists a graded sequence of polynomials \( f_0^*, f_1^*, \ldots, f_d^* \) taken from \( \mathbb{K}[\lambda] \) such that \( E_i V = f_{d-i}(A^*) E_d V \) for \( 0 \leq i \leq d \).

\( (iii) \) For \( 0 \leq i \leq d \),

\[ \sum_{h=i}^{d} E_h V = \sum_{h=0}^{d-i} A^* E_d V. \]

**Corollary 3.5** Referring to Lemma 3.4, assume the equivalent conditions \((i)-(iii)\) hold. Let \( v_d \) denote a nonzero vector in \( E_d V \). Then the homomorphism of \( \mathbb{K} \)-vector spaces from \( D^* \) to \( V \) which sends each \( X \) to \( X v_d \) is an isomorphism.

**4 The existence of the split decomposition**

We now display necessary and sufficient conditions for the existence of the split decomposition.
Theorem 4.1  With reference to Definition 2.1, the following (i), (ii) are equivalent.

(i) There exists a decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$.

(ii) Both

\[
E_i^*A_iE_j = \begin{cases} 
0, & \text{if } i - j > 1; \\
\neq 0, & \text{if } i - j = 1
\end{cases} 
(0 \leq i, j \leq d), \tag{7}
\]

\[
E_iA_i^*E_j = \begin{cases} 
0, & \text{if } j - i > 1; \\
\neq 0, & \text{if } j - i = 1
\end{cases} 
(0 \leq i, j \leq d). \tag{8}
\]

Proof: (i) $\Rightarrow$ (ii) By assumption there exists a decomposition $U_0, U_1, \ldots, U_d$ of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. For $0 \leq i \leq d$ we have $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E_0^* V$ by Lemma 2.4(i) and $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E_h^* V$ by Lemma 2.4(ii) so $\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V$. This gives Lemma 3.2(iii). Applying that lemma we obtain (7). For $0 \leq i \leq d$ we have $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E_0^* V$ by Lemma 2.4(iii) and $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E_h^* V$ by Lemma 2.4(iv) so $\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V$. This gives Lemma 3.4(iii). Applying that lemma we obtain (8).

(ii) $\Rightarrow$ (i). For $0 \leq i \leq d$ we define $\tau_i = \prod_{h=0}^{i-1} (A - \theta_h I)$. We observe $\tau_i$ ($0 \leq i \leq d$) is a basis for $D$. Let $v_0^*$ denote a nonzero vector in $E_0^* V$. Observe Lemma 3.2(i) holds by (7) so Corollary 3.3 applies; by that corollary $\tau_i v_0^*$ ($0 \leq i \leq d$) is a basis for $V$. We define $U_i = \text{Span}(\tau_i v_0^*)$ for $0 \leq i \leq d$ and observe $U_0, U_1, \ldots, U_d$ is a decomposition of $V$. We show this decomposition is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. To do this we show $U_0, U_1, \ldots, U_d$ satisfies (1) and (2). Concerning (1), from the construction $(A - \theta_i I) \tau_i = \tau_{i+1}$ for $0 \leq i \leq d - 1$ and $(A - \theta_d I) \tau_d = 0$. Applying both sides of these equations to $v_0^*$ we find $(A - \theta_i I) U_i = U_{i+1}$ for $0 \leq i \leq d - 1$ and $(A - \theta_d I) U_d = 0$. We have now shown (1). Concerning (2), this will follow if we can show (a) $(A^* - \theta_i^* I) U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leq i \leq d$; (b) $(A^* - \theta_i^* I) U_i \subseteq \sum_{h=0}^{d} U_h$ for $1 \leq i \leq d$; (c) $(A^* - \theta_i^* I) U_i \neq 0$ for $1 \leq i \leq d$. We begin with (a). For $0 \leq j \leq d$ the elements $\tau_h (0 \leq h \leq j)$ and the elements $A^h (0 \leq h \leq j)$ span the same subspace of $D$. Therefore $\sum_{h=0}^{j} U_h = \sum_{h=0}^{j} A^h E_0^* V$. We mentioned Lemma 3.2(i) holds so Lemma 3.2(iii) holds; therefore $\sum_{h=0}^{j} E_h^* V = \sum_{h=0}^{j} A^h E_0^* V$ so $\sum_{h=0}^{j} U_h = \sum_{h=0}^{j} E_h^* V$. Observe $(A^* - \theta_i^* I) \sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i-1} E_h^* V$ for $0 \leq i \leq d$. Combining these comments we find $(A^* - \theta_i^* I) U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leq i \leq d$. We now have (a). Next we prove (b). From the construction for $0 \leq j \leq d$ we have $\prod_{h=j}^{i} (A - \theta_h I) U_j = 0$ so $\prod_{h=j}^{d} (A - \theta_h I) U_j = 0$. From this we find $U_j \subseteq \sum_{h=j}^{d} E_h V$. Apparently $\sum_{h=i}^{d} U_h \subseteq \sum_{h=i}^{d} E_h V$ for $0 \leq i \leq d$. By this and since $U_0, U_1, \ldots, U_d$ is a decomposition we find $\sum_{h=i}^{d} U_h = \sum_{h=i}^{d} E_h V$ for $0 \leq i \leq d$. From (8) we find $A^* E_j V \subseteq \sum_{h=j}^{d} E_h V$ for $1 \leq j \leq d$. Therefore $(A^* - \theta_j^* I) \sum_{h=j}^{d} E_h V \subseteq \sum_{h=j}^{d} E_h V$ for $1 \leq j \leq d$. From these comments we find $(A^* - \theta_j^* I) U_j \subseteq \sum_{h=j}^{d} U_h$ for $1 \leq j \leq d$. We now have (b). Next we show (c). Suppose there exists an integer $i$ ($1 \leq i \leq d$) such that $(A^* - \theta_i^* I) U_i = 0$. We assume $i$ is maximal subject to this. We obtain a contradiction as follows. For $0 < i \leq d$ we find $(A^* - \theta_i^* I) U_{i+1} \subseteq U_{i+1}$ by (a), (b). In this inclusion the left-hand side is nonzero and the right-hand side has dimension 1 so we have equality. We mentioned earlier $(A - \theta_d I) U_d = 0$ so $U_d = E_d V$. Apparently $U_d = \prod_{h=d+1}^{d} (A^* - \theta_h^* I) E_d V$ for $1 \leq j \leq d$. Therefore $(A^* - \theta_i^* I) U_i \subseteq \sum_{h=i}^{d} E_h V$ for $0 \leq i \leq d$.
for $i < j \leq d$ and $0 = \prod_{h=i}^{d}(A^* - \theta_h I)E_dV$. Let $v_d$ denote a nonzero vector in $E_dV$ and observe $0 = \prod_{h=i}^{d}(A^* - \theta_h I)v_d$. This is inconsistent with Corollary 3.5 and the fact that $0 \neq \prod_{h=i}^{d}(A^* - \theta_h I)$. We now have a contradiction and (c) is proved. Combining (a)-(c) we obtain (2). We have shown the decomposition $U_0, U_1, \ldots, U_d$ satisfies (1), (2). Applying Definition 2.2 we find $U_0, U_1, \ldots, U_d$ is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. \qed

5 Two characterizations of Leonard systems

In this section we obtain two characterizations of Leonard systems. Both characterizations involve the split decomposition. We will first state the characterizations, then prove a few lemmas, and then prove the characterizations. Our first characterization is stated as follows.

**Theorem 5.1** With reference to Definition 2.1, the sequence $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ is a Leonard system if and only if the following (i), (ii) hold.

(i) There exists a decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$.

(ii) There exists a decomposition of $V$ which is split with respect to $E_d, E_{d-1}, \ldots, E_0$ and $E_0^*, E_1^*, \ldots, E_d^*$.

In order to state our second characterization we recall a definition. Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ denote any map. We call $\sigma$ an antiautomorphism of $\mathcal{A}$ whenever $\sigma$ is an isomorphism of $\mathbb{K}$-vector spaces and $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$. For example assume $\mathcal{A} = \text{Mat}_{d+1}(\mathbb{K})$. Then $\sigma$ is an antiautomorphism of $\mathcal{A}$ if and only if there exists an invertible $R \in \mathcal{A}$ such that $X^\sigma = R^{-1}X^tR$ for all $X \in \mathcal{A}$, where $t$ denotes transpose. This follows from the Skolem-Noether Theorem [11, Cor. 9.122].

We now state our second characterization of Leonard systems.

**Theorem 5.2** With reference to Definition 2.1, the sequence $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ is a Leonard system if and only if the following (i), (ii) hold.

(i) There exists a decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$.

(ii) There exists an antiautomorphism $\dagger$ of $\mathcal{A}$ such that $A^\dagger = A$ and $A^*\dagger = A^*$.

We now prove some lemmas which we will use to obtain Theorem 5.1 and Theorem 5.2. We have a preliminary remark. With reference to Definition 2.1, let us consider the following four conditions:

\[
E_i^*AE_j^* = \begin{cases} 
0, & \text{if } i-j > 1; \\
0, & \text{if } i-j = 1 \\
0, & \text{if } j-i > 1; \\
0, & \text{if } j-i = 1 
\end{cases} \quad (0 \leq i, j \leq d), \quad (9)
\]

\[
E_i^*AE_j^* = \begin{cases} 
0, & \text{if } i-j > 1; \\
0, & \text{if } i-j = 1 \\
0, & \text{if } j-i > 1; \\
0, & \text{if } j-i = 1 
\end{cases} \quad (0 \leq i, j \leq d), \quad (10)
\]
\[ E_i A^* E_j = \begin{cases} 0, & \text{if } i - j > 1; \\ \neq 0, & \text{if } i - j = 1 \end{cases} (0 \leq i, j \leq d), \] (11)

\[ E_i A^* E_j = \begin{cases} 0, & \text{if } j - i > 1; \\ \neq 0, & \text{if } j - i = 1 \end{cases} (0 \leq i, j \leq d). \] (12)

We observe \((A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)\) is a Leonard system if and only if each of (9)–(12) holds.

**Lemma 5.3** With reference to Definition 2.1, assume conditions (9) and (10) hold. Then \(A, E_0^*\) together generate \(\mathcal{A}\). Moreover \(A, A^*\) together generate \(\mathcal{A}\).

**Proof:** The elements \(A^r E_0^s (0 \leq r, s \leq d)\) form a basis for the \(K\)-vector space \(\mathcal{A}\). This assertion is immediate from the proof of [15, Lemma 3.1]. It follows that \(A, E_0^*\) together generate \(\mathcal{A}\). The elements \(A, A^*\) together generate \(\mathcal{A}\) since \(E_0^*\) is a polynomial in \(A^*\). \(\square\)

**Lemma 5.4** With reference to Definition 2.1, assume conditions (9) and (10) hold. Then there exists a unique antiautomorphism \(\dagger\) of \(\mathcal{A}\) such that \(A^\dagger = A\) and \(A^{*\dagger} = A^*\). Moreover \(X^{\dagger\dagger} = X\) for all \(X \in \mathcal{A}\).

**Proof:** Concerning the existence of \(\dagger\), for \(0 \leq i \leq d\) let \(v_i^*\) denote a nonzero element of \(E_i^* V\) and recall \(v_0^*, v_1^*, \ldots, v_d^*\) is a basis for \(V\). For \(X \in \mathcal{A}\) let \(X^\dagger\) denote the matrix in \(\text{Mat}_{d+1}(K)\) which represents \(X\) with respect to the basis \(v_0^*, v_1^*, \ldots, v_d^*\). We observe \(\dagger : \mathcal{A} \to \text{Mat}_{d+1}(K)\) is an isomorphism of \(K\)-algebras. We abbreviate \(B = A^\dagger\) and \(B^* = A^{*\dagger}\). We observe \(B\) is irreducible tridiagonal and \(B^* = \text{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)\). Let \(D\) denote the diagonal matrix in \(\text{Mat}_{d+1}(K)\) which has \(ii\) entry

\[ D_{ii} = \frac{B_{0i} B_{12} \cdots B_{i-1,i} B_{i,i-1}}{B_{10} B_{21} \cdots B_{i-1,i-1}} \] \((0 \leq i \leq d)\).

It is routine to verify \(D^{-1} B^\dagger D = B\). Each of \(D, B^*\) is diagonal so \(DB^* = B^* D\); also \(B^{*\dagger} = B^*\) so \(D^{-1} B^* D = B^*\). Let \(\sigma : \text{Mat}_{d+1}(K) \to \text{Mat}_{d+1}(K)\) denote the map which satisfies \(X^\sigma = D^{-1} X^\dagger D\) for all \(X \in \text{Mat}_{d+1}(K)\). We observe \(\sigma\) is an antiautomorphism of \(\text{Mat}_{d+1}(K)\) such that \(B^{\sigma} = B\) and \(B^{*\sigma} = B^*\). We define the map \(\dagger : \mathcal{A} \to \mathcal{A}\) to be the composition \(\dagger := \sigma^{-1} \circ \sigma \circ \dagger\). We observe \(\dagger\) is an antiautomorphism of \(\mathcal{A}\) such that \(A^\dagger = A\) and \(A^{*\dagger} = A^*\). We have now shown there exists an antiautomorphism \(\dagger\) of \(\mathcal{A}\) such that \(A^\dagger = A\) and \(A^{*\dagger} = A^*\). This antiautomorphism is unique since \(A, A^*\) together generate \(\mathcal{A}\). The map \(X \to X^{\dagger\dagger}\) is an isomorphism of \(K\)-algebras from \(\mathcal{A}\) to itself. This map is the identity since \(A^{\dagger\dagger} = A\), \(A^{*\dagger\dagger} = A^*\), and since \(A, A^*\) together generate \(\mathcal{A}\). \(\square\)

**Lemma 5.5** With reference to Definition 2.1, assume there exists an antiautomorphism \(\dagger\) of \(\mathcal{A}\) such that \(A^\dagger = A\) and \(A^{*\dagger} = A^*\). Then \(E_i^\dagger = E_i\) and \(E_i^{*\dagger} = E_i^*\) for \(0 \leq i \leq d\).

**Proof:** Recall \(E_i\) (resp. \(E_i^*\)) is a polynomial in \(A\) (resp. \(A^*\)) for \(0 \leq i \leq d\). \(\square\)
Lemma 5.6 With reference to Definition 2.1, assume there exists an antiautomorphism ↑ of \( \mathcal{A} \) such that \( A↑ = A \) and \( A^*↑ = A^* \). Then for \( 0 ≤ i, j ≤ d \), (i) \( E_i^* A E_j^* = 0 \) if and only if \( E_j^* A E_i^* = 0 \); and (ii) \( E_i A^* E_j = 0 \) if and only if \( E_j A^* E_i = 0 \).

Proof: By Lemma 5.5 and since ↑ is an antiautomorphism,

\[
(E_i^* A E_j^*)^↑ = E_j^* A E_i^* \quad (0 ≤ i, j ≤ d).
\]

Assertion (i) follows since ↑ : \( \mathcal{A} \rightarrow \mathcal{A} \) is a bijection. To obtain (ii) interchange the roles of \( A \) and \( A^* \) in the proof of (i).

Lemma 5.7 With reference to Definition 2.1, assume at least three of (9)–(12) hold. Then each of (9)–(12) hold; in other words \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system.

Proof: Interchanging \( A \) and \( A^* \) if necessary, we may assume without loss of generality that (9) and (10) hold. By Lemma 5.4 there exists an antiautomorphism ↑ of \( \mathcal{A} \) such that \( A↑ = A \) and \( A^*↑ = A^* \). By assumption at least one of (11), (12) holds. Combining this with Lemma 5.6 we find (11), (12) both hold. The result follows.

We are now ready to prove Theorem 5.1.

Proof: By Theorem 4.1 we find (i) holds if and only if each of (9), (12) holds. Applying Theorem 4.1 again, this time with \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) replaced by \( (A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \), we find (ii) holds if and only if each of (9), (11) holds. Suppose \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system. Then each of (9)–(12) holds. In particular each of (9), (11), (12) holds so (i), (ii) hold by our above remarks. Conversely suppose (i), (ii) hold. Then each of (9), (11), (12) holds. At least three of (9)–(12) hold so \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system by Lemma 5.7.

We are now ready to prove Theorem 5.2.

Proof: First assume \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system. Then (i) holds by Theorem 5.1 and (ii) holds by Lemma 5.4. Conversely assume (i), (ii) hold. Combining (i) and Theorem 4.1 we obtain (9), (12). Combining this with (ii) and using Lemma 5.6 we obtain (10), (11). Now each of (9)–(12) holds so \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system.

We would like to emphasize the following fact.

Theorem 5.8 Let \( A; A^* \) denote a Leonard pair in \( \mathcal{A} \). Then there exists a unique antiautomorphism ↑ of \( \mathcal{A} \) such that \( A↑ = A \) and \( A^*↑ = A^* \). Moreover \( X↑↑ = X \) for all \( X \in \mathcal{A} \).

Proof: Since \( A, A^* \) is a Leonard pair there exists an ordering \( E_0, E_1, \ldots, E_d \) of the primitive idempotents of \( A \) and an ordering \( E_0^*, E_1^*, \ldots, E_d^* \) of the primitive idempotents of \( A^* \) such that \( (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \) is a Leonard system. These orderings satisfy (9)–(12). In particular (9), (10) are satisfied so the result follows by Lemma 5.4.

We finish this section with a comment.
Lemma 5.9 With reference to Definition 2.1, assume there exists a decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Then the following (i), (ii) are equivalent.

(i) The pair $A, A^*$ is a Leonard pair.

(ii) The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.

Proof: (i) $\Rightarrow$ (ii) We assume there exists a decomposition of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Therefore each of (9), (12) holds by Theorem 4.1. Since $A, A^*$ is a Leonard pair there exists an anti-automorphism $A^\dagger$ of $A$ such that $A^\dagger = A$ and $A^\dagger = A^*$. Applying Lemma 5.6 we find each of (10), (11) holds. Now each of (9)-(12) holds so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.
(ii) $\Rightarrow$ (i) Clear. 

\[\square\]

6 The two characterizations in terms of matrices

In this section we restate Theorem 5.1 and Theorem 5.2 in terms of matrices. We first set some notation. With reference to Definition 2.1, suppose there exists a decomposition $U_0, U_1, \ldots, U_d$ of $V$ which is split with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Pick an integer $i$ ($1 \leq i \leq d$). By (2) we find $(A^\dagger - \theta_i^* I)U_i = U_{i-1}$ and by (1) we find $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently $U_i$ is an eigenspace for $(A - \theta_{i-1} I)(A^\dagger - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero element of $K$. Let us denote this eigenvalue by $\varphi_i$. We call $\varphi_1, \varphi_2, \ldots, \varphi_d$ the split sequence for $A, A^\dagger$ with respect to $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. The split sequence has the following interpretation. For $0 \leq i \leq d$ let $u_i$ denote a nonzero vector in $U_i$ and recall $u_0, u_1, \ldots, u_d$ is a basis for $V$. We normalize the $u_i$ so that $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leq i \leq d - 1$. With respect to the basis $u_0, u_1, \ldots, u_d$ the matrices which represent $A$ and $A^\dagger$ are as follows.

$$
A = \begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
& & 1 & \theta_2 \\
& & & & & \ddots \\
& & & & & & 1 & \theta_d \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \theta_0 \\
\end{pmatrix}, \quad A^\dagger = \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
1 & \theta_1^* & \varphi_2 \\
& & 1 & \theta_2^* \\
& & & & & \ddots \\
& & & & & & 1 & \theta_d^* \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \theta_0 \\
\end{pmatrix}.
$$

Motivated by this we consider the following set-up.

Definition 6.1 Let $d$ denote a nonnegative integer. Let $A$ and $A^\dagger$ denote matrices in $\text{Mat}_{d+1}(K)$ of the form

$$
A = \begin{pmatrix}
\theta_0 & 0 \\
1 & \theta_1 \\
& & 1 & \theta_2 \\
& & & & \ddots \\
& & & & & 1 & \theta_d \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \theta_0 \\
\end{pmatrix}, \quad A^\dagger = \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
1 & \theta_1^* & \varphi_2 \\
& & 1 & \theta_2^* \\
& & & & & \ddots \\
& & & & & & 1 & \theta_d^* \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \theta_0 \\
\end{pmatrix},
$$

12
where
\[ \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if} \quad i \neq j \quad (0 \leq i, j \leq d), \]
\[ \varphi_i \neq 0 \quad (1 \leq i \leq d). \]

We observe \( A \) (resp. \( A^* \)) is multiplicity-free, with eigenvalues \( \theta_0, \theta_1, \ldots, \theta_d \) (resp. \( \theta_0, \theta_1, \ldots, \theta_d \)). For \( 0 \leq i \leq d \) we let \( E_i \) (resp. \( E_i^* \)) denote the primitive idempotent for \( A \) (resp. \( A^* \)) associated with \( \theta_i \) (resp. \( \theta_i^* \)).

We make some comments. With reference to Definition 6.1, for \( 0 \leq i \leq d \) let \( u_i \) denote the vector in \( \mathbb{K}^{d+1} \) which has \( i \)th entry 1 and all other entries 0. We observe \( u_0, u_1, \ldots, u_d \) is a basis for \( \mathbb{K}^{d+1} \). From the form of \( A \) we have \( (A - \theta_i I)u_i = u_{i+1} \) for \( 0 \leq i \leq d - 1 \) and \( (A - \theta_d I)u_d = 0 \). From the form of \( A^* \) we have \( (A^* - \theta_i^* I)u_i = \varphi_i u_{i-1} \) for \( 1 \leq i \leq d \) and \( (A^* - \theta_0^* I)u_0 = 0 \). For \( 0 \leq i \leq d \) let \( U_i \) denote the subspace of \( \mathbb{K}^{d+1} \) spanned by \( u_i \). Then \( U_0, U_1, \ldots, U_d \) is a decomposition of \( \mathbb{K}^{d+1} \). This decomposition satisfies \( (A - \theta_i I)U_i = U_{i+1} \) for \( 0 \leq i \leq d - 1 \) and \( (A - \theta_d I)U_d = 0 \). Similarly \( (A^* - \theta_i^* I)U_i = U_{i-1} \) for \( 1 \leq i \leq d \) and \( (A^* - \theta_0^* I)U_0 = 0 \). In other words the decomposition \( U_0, U_1, \ldots, U_d \) is split with respect to \( E_0, E_1, \ldots, E_d \) and \( E_0^*, E_1^*, \ldots, E_d^* \). We observe \( \varphi_1, \varphi_2, \ldots, \varphi_d \) is the corresponding split sequence for \( A, A^* \). We now consider when is the pair \( A, A^* \) a Leonard pair. We begin with a remark.

**Lemma 6.2** With reference to Definition 6.1, the following (i), (ii) are equivalent.

(i) The pair \( A, A^* \) is a Leonard pair.

(ii) The sequence \( (A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d}) \) is a Leonard system.

**Proof:** We mentioned there exists a decomposition of \( \mathbb{K}^{d+1} \) which is split with respect to \( E_0, E_1, \ldots, E_d \) and \( E_0^*, E_1^*, \ldots, E_d^* \). Therefore Lemma 5.9 applies and the result follows. \( \square \)

We now give a matrix version of Theorem 5.1.

**Theorem 6.3** Referring to Definition 6.1, the following (i), (ii) are equivalent.

(i) The pair \( A, A^* \) is a Leonard pair.

(ii) There exists an invertible \( G \in \text{Mat}_{d+1}(\mathbb{K}) \) and there exists nonzero \( \phi_i \in \mathbb{K} \) (1 \( \leq i \leq d \)) such that
\[
G^{-1}AG = \begin{pmatrix}
\theta_d \\
1 & \theta_{d-1} \\
1 & \theta_{d-2} \\
& \ddots \\
0 & \cdots & 1 & \theta_0
\end{pmatrix}, \quad G^{-1}A^*G = \begin{pmatrix}
\theta_0^* & \phi_1 \\
\theta_1^* & \phi_2 \\
& \ddots \\
0 & \cdots & \phi_d & \theta_d
\end{pmatrix}.
\]

Suppose (i), (ii) hold. Then the sequence \( \phi_1, \phi_2, \ldots, \phi_d \) is the split sequence for \( A, A^* \) associated with \( E_d, E_{d-1}, \ldots, E_0 \) and \( E_0^*, E_1^*, \ldots, E_d^* \).
Proof: (i) \implies (ii) The sequence \((A; A^*; \{E_i\}_i^d; \{E^*_i\}_i^d)\) is a Leonard system by Lemma 6.2. By Theorem 5.1 there exists a decomposition of \(K^{d+1}\) which is split with respect to \(E_d, E_{d-1}, \ldots, E_0\) and \(E^*_0, E^*_1, \ldots, E^*_d\). Let \(V_0, V_1, \ldots, V_d\) denote this decomposition. By the definition of a split decomposition we have \((A - \theta_{d-i}I)V_i = V_{i+1}\) for \(0 \leq i \leq d-1\) and \((A - \theta_{d-0}I)V_d = 0\). Moreover \((A^* - \theta_{d-i}^*I)V_i = V_{i-1}\) for \(1 \leq i \leq d\) and \((A^* - \theta_{d-0}^*I)V_0 = 0\). For \(0 \leq i \leq d\) let \(v_i\) denote a nonzero vector in \(V_i\) and observe \(v_0, v_1, \ldots, v_d\) is a basis for \(K^{d+1}\). We normalize the \(v_i\) so that \((A - \theta_{d-i}I)v_i = v_{i+1}\) for \(0 \leq i \leq d-1\). Let \(\phi_1, \phi_2, \ldots, \phi_d\) denote the split sequence for \(A, A^*\) with respect to \(E_d, E_{d-1}, \ldots, E_0\) and \(E^*_0, E^*_1, \ldots, E^*_d\). Then \(\phi_i \neq 0\) \((1 \leq i \leq d)\) and moreover \((A^* - \theta_{d-i}^*I)v_i = \phi_{i}v_{i-1}\) \((1 \leq i \leq d)\), \((A^* - \theta_{d-0}^*I)v_0 = 0\). Let \(G\) denote the matrix in \(\text{Mat}_{d+1}(K)\) which has column \(i\) equal to \(v_i\) for \(0 \leq i \leq d\). We observe \(G\) is invertible. Moreover the matrices \(G^{-1}AG\) and \(G^{-1}A^*G\) have the form shown above.

(ii) \implies (i) We show \((A; A^*; \{E_i\}_i^d; \{E^*_i\}_i^d)\) is a Leonard system. In order to do this we apply Theorem 5.1. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of \(K^{d+1}\) which is split with respect to \(E_0, E_1, \ldots, E_d\) and \(E^*_0, E^*_1, \ldots, E^*_d\). Therefore Theorem 5.1(ii) holds. We show Theorem 5.1(iii) holds. For \(0 \leq i \leq d\) let \(v_i\) denote column \(i\) of \(G\) and observe \(v_0, v_1, \ldots, v_d\) is a basis for \(K^{d+1}\). From the form of \(G^{-1}AG\) we find \((A - \theta_{d-i}I)v_i = v_{i+1}\) for \(0 \leq i \leq d-1\) and \((A - \theta_{d-0}I)v_d = 0\). From the form of \(G^{-1}A^*G\) we find \((A^* - \theta_{d-i}^*I)v_i = \phi_{i}v_{i-1}\) for \(1 \leq i \leq d\) and \((A^* - \theta_{d-0}^*I)v_0 = 0\). For \(0 \leq i \leq d\) let \(V_i\) denote the subspace of \(K^{d+1}\) spanned by \(v_i\). Then \(V_0, V_1, \ldots, V_d\) is a decomposition of \(K^{d+1}\). Also \((A - \theta_{d-i}I)V_i = V_{i+1}\) for \(0 \leq i \leq d-1\) and \((A - \theta_{d-0}I)V_d = 0\). Moreover \((A^* - \theta_{d-i}^*I)V_i = V_{i-1}\) for \(1 \leq i \leq d\) and \((A^* - \theta_{d-0}^*I)V_0 = 0\). Apparently \(V_0, V_1, \ldots, V_d\) is split with respect to \(E_d, E_{d-1}, \ldots, E_0\) and \(E^*_0, E^*_1, \ldots, E^*_d\). Now Theorem 5.1(ii) holds; applying that theorem we find \((A; A^*; \{E_i\}_i^d; \{E^*_i\}_i^d)\) is a Leonard system. In particular \(A, A^*\) is a Leonard pair.

Assume (i), (ii) both hold. From the proof of (ii) \implies (i) we find that for \(1 \leq i \leq d\), \(\phi_i\) is the eigenvalue of \((A - \theta_{d-i+1}I)(A^* - \theta_i^*I)\) associated with \(V_i\). Therefore \(\phi_1, \phi_2, \ldots, \phi_d\) is the split sequence for \(A, A^*\) associated with \(E_d, E_{d-1}, \ldots, E_0\) and \(E^*_0, E^*_1, \ldots, E^*_d\). \(\square\)

We now give a matrix version of Theorem 5.2.

**Theorem 6.4** Referring to Definition 6.1, the following (i), (ii) are equivalent.

(i) The pair \(A, A^*\) is a Leonard pair.

(ii) There exists an invertible \(H \in \text{Mat}_{d+1}(K)\) such that \(H^{-1}A^tH = A^*\) and \(H^{-1}A^{st}H = A^*\).

Proof: (i) \implies (ii) By Theorem 5.8 there exists an anti-automorphism \(\dagger\) of \(\text{Mat}_{d+1}(K)\) such that \(A^\dagger = A\) and \(A^{s\dagger} = A^*\). Since \(\dagger\) is an anti-automorphism there exists an invertible \(H \in \text{Mat}_{d+1}(K)\) such that \(X^\dagger = H^{-1}X^tH\) for all \(X \in \text{Mat}_{d+1}(K)\). Setting \(X = A\) we have \(H^{-1}A^tH = A\). Setting \(X = A^*\) we have \(H^{-1}A^{st}H = A^*\).

(ii) \implies (i) We show \((A; A^*; \{E_i\}_i^d; \{E^*_i\}_i^d)\) is a Leonard system. In order to do this we apply Theorem 5.2. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of \(K^{d+1}\) which is split with respect to \(E_0, E_1, \ldots, E_d\) and \(E^*_0, E^*_1, \ldots, E^*_d\). Therefore Theorem 5.2(i) holds. Let \(\dagger : \text{Mat}_{d+1}(K) \rightarrow \text{Mat}_{d+1}(K)\) denote the map which satisfies \(X^\dagger = H^{-1}X^tH\) for all \(X \in \text{Mat}_{d+1}(K)\). Then \(\dagger\) is an anti-automorphism of \(\text{Mat}_{d+1}(K)\) such
that $A^t = A$ and $A^{*t} = A^*$. Now Theorem 5.2(ii) holds; applying that theorem we find $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In particular $A, A^*$ is a Leonard pair.

7 Remarks

Referring to Definition 6.1, presumably condition (ii) of Theorem 6.3 or Theorem 6.4 can be translated into a condition on the entries of $A$ and $A^*$. It turns out such a condition is already in the literature; we cite it here for the sake of completeness.

**Theorem 7.1** [13, Corollary 14.2] With reference to Definition 6.1, the pair $A, A^*$ is a Leonard pair if and only if there exists nonzero $\phi_i \in \mathbb{K}$ ($1 \leq i \leq d$) such that (i)-(iii) hold below.

(i) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_0^* - \theta_0)(\theta_{i-1} - \theta_d)$

(ii) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_0^* - \theta_0)(\theta_{d-i+1} - \theta_0)$

(iii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$.

Suppose (i)-(iii) hold. Then $\phi_1, \phi_2, \ldots, \phi_d$ is the split sequence for $A, A^*$ with respect to $E_d, E_{d-1}, \ldots, E_0$ and $E_0^*, E_1^*, \ldots, E_d^*$.

8 Acknowledgement

The author would like to thank ...for giving the manuscript a close reading and offering many valuable suggestions.

References


Paul Terwilliger
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, Wisconsin, 53706 USA
email: terwilli@math.wisc.edu