Normalized matching and nested chain orders

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Abstract

A long-standing conjecture states that posets with the normalized matching property are nested chain orders. We verify this conjecture for posets of rank 2.

Keywords: Posets; bipartite graphs; Matching; Normalized matching; Nested chain orders

1. Introduction

Let $P$ be a partially ordered set, or poset. If every maximal chain of $P$ has the same length $n$, then we say that $P$ is graded of rank $n$. In this case there is a rank function $r : P \to \{0, 1, \ldots, n\}$ such that $r(x) = 0$ if $x$ is a minimal element of $P$ and $r(z) = r(y) + 1$ if $z$ covers $y$ in $P$. Throughout the posets we considered are finite and graded. Our terminology and notation will be standard except as indicated. See any of the texts [2, 3, 10] for undefined terms.

For $0 \leq i \leq n$, let $P_i = \{x \in P : r(x) = i\}$ be the $i$th rank set of $P$. For $X \subseteq P_i$, denote $\Gamma_j(X) = \{y \in P_j : y$ is comparable with some $x \in X\}$. We say that $P$ has the normalized matching property (NM for short), if

$$\frac{|\Gamma_j(X)|}{|P_j|} \geq \frac{|X|}{|P_i|}$$

for all $0 \leq i, j \leq n$ and $X \subseteq P_i$, where $|\ast|$ denotes the cardinality of the set $\ast$. We say that $P$ is a nested chain order if there exists a chain partition $P = C_1 \cup C_2 \cup \cdots$ such that $r(C_i) \subseteq r(C_j)$ whenever $|C_i| \leq |C_j|$, where $r(C) = \{r(x) : x \in C\}$. See [6, 8] for detail description of these properties of posets. Figure 1 shows the Hasse diagrams of an NM poset and a non NM poset. Both of them are nested and have the nested chain partition

$$\{x_1 < y_1 < z_1\} \cup \{x_2 < y_2 < z_2\} \cup \{y_3 < z_3\} \cup \{y_4\}.$$
A long-standing conjecture states that an NM poset is a nested chain order (see [7] for instance). Anderson[1] and Griggs[7] independently settled the conjecture for posets whose rank numbers are symmetric and unimodal, but there has been no answer in general on this problem. The object of this note is to verify the conjecture for posets of rank 2.

**Theorem 1.** An NM poset of rank 2 is a nested chain order.

### 2. Matching of bipartite graphs

Let \( G = (A, B; E) \) be a finite bipartite graph with vertex sets \( A \) and \( B \) and the edge set \( E \subseteq A \times B \). If there exists an injection \( f : A \to B \) such that \((a, f(a)) \in E\) for all \( a \in A \), then we say that \( G \) has a **matching** from \( A \) into \( B \). In particular, if there exists such a bijection then we say that \( G \) has a **perfect matching**. The following is the classical criterion for the existence of a matching.

**Hall Theorem**([9]). A bipartite graph \( G = (A, B; E) \) has a matching from \( A \) into \( B \) if and only if \( |\Gamma(X)| \geq |X| \) for every \( X \subseteq A \), where \( \Gamma(X) \) denotes the set of members of \( B \) which are connected to members of \( X \) by an edge.

We consider the bipartite graph \( G = (A, B; E) \) without isolated vertices. Clearly, \( G \) may be regarded as a graded poset of rank 1. We say that the bipartite graph \( G \) has the **NM property** if \( G \) has this property as a poset (the normalized matching property was originally defined on bipartite graphs, see [5]). By Hall Theorem, if \( G \) has the NM property and \( |A| \leq |B| \), then \( G \) has a matching from \( A \) into \( B \). On the other hand, two rank sets \( P_i \) and \( P_j \) of a graded poset \( P \) may induce a bipartite graph \( P_{ij} = (P_i, P_j; E_{ij}) \),
where $E_{ij} = \{(x, y) : x \in P_i, y \in P_j, x \text{ and } y \text{ are comparable in } P\}$. If the poset $P$ has the NM property, then so does $P_{ij}$ as a bipartite graph, which yields that $P_{ij}$ has the matching property.

3. Proof of Theorem 1

Let $P = P_0 \cup P_1 \cup P_2$ be an NM poset of rank 2. Denote $|P_0| = r, |P_1| = s$ and $|P_2| = t$. Without loss of generality, we may assume that $r \leq t$ (otherwise we consider the dual poset $P^*$ of $P$ which is also graded. Clearly, $P$ has the NM property or the nested chain partition if and only if $P^*$ has the corresponding property). There are three cases to be considered.

Case 1 $r \leq s \leq t$.

In this case, there exist a matching $f$ from $P_0$ into $P_1$ and a matching $g$ from $P_1$ into $P_2$. Suppose that $P_0 = \{x_1, \ldots, x_r\}$. Let $y_i = f(x_i)(1 \leq i \leq r)$ and set $\{y_{r+1}, \ldots, y_s\} = P_1 \setminus \{y_1, \ldots, y_r\}$. Then $P_1 = \{y_1, \ldots, y_s\}$ and $x_i < y_i$ for $1 \leq i \leq r$. Let $z_i = g(y_i)(1 \leq i \leq s)$ and set $\{z_{s+1}, \ldots, z_t\} = P_2 \setminus \{z_1, \ldots, z_s\}$. Then $P_2 = \{z_1, \ldots, z_t\}$ and $y_i < z_i$ for $1 \leq i \leq s$. Thus we may obtain a nested chain partition of $P$:

$$P = \left( \bigcup_{i=1}^{r} \{x_i < y_i < z_i\} \right) \bigcup \left( \bigcup_{i=r+1}^{s} \{y_i < z_i\} \right) \bigcup \left( \bigcup_{i=s+1}^{t} \{z_i\} \right).$$

Case 2 $r \leq t < s$.

Let $P_0 = \{x_1, \ldots, x_r\}$ and $P_1 = \{y_1, \ldots, y_s\}$. Construct a bipartite graph $G = (A, B; E)$ as follows: $A = P_1 \cup P_0 \cup P_0'$, $B = P_1' \cup P_2$ and $E = E_{01} \cup E_{01} \cup E_{11} \cup E_{12}$, where

$$P_0' = \{x_{r+1}, \ldots, x_t\} \text{ is a set of } t - r \text{ elements},$$

$$E_{01} = P_0 \times P_0',$$

$$E_{01} = \{(x, j) \subseteq P_0 \times P_1' : 1 \leq j \leq s, x < y_j\},$$

$$E_{11} = \{(y_j, j) \subseteq P_1 \times P_1' : j = 1, \ldots, s\}, \text{ and}$$

$$E_{12} = \{(y, z) \subseteq P_1 \times P_2 : y < z\}.$$
We claim that $P$ is nested if and only if $G$ has a perfect matching. Indeed, suppose that there exists a matching $f$ of $G$ from $A$ into $B$. Then $f(P_0 \cup \overline{P}_0) \subseteq P'_1$. Thus there is a permutation $\pi$ of $\{1, 2, \ldots, s\}$ such that $f(x_i) = \pi(i)$ for $1 \leq i \leq t$, which implies that $x_i < y_{\pi(i)}$ for $1 \leq i \leq r$. Further, we have $f(\{y_{\pi(1)}, \ldots, y_{\pi(t)}\}) = P_2$. Let $f(y_{\pi(i)}) = z_i$ for $1 \leq i \leq t$. Then $P_2 = \{z_1, \ldots, z_t\}$ and $y_{\pi(i)} < z_i$ for $1 \leq i \leq t$. Thus we may obtain a nested chain partition of $P$:

$$P = \left( \bigcup_{i=1}^{r} \{x_i < y_{\pi(i)} < z_i\} \right) \cup \left( \bigcup_{i=r+1}^{t} \{y_{\pi(i)} < z_i\} \right) \cup \left( \bigcup_{i=t+1}^{s} \{y_{\pi(i)}\} \right).$$

Conversely, it is easy to check that a matching of $G$ may induce a nested chain partition of $P$.

So, to show that $P$ is nested, it suffices to show that for every $X \subseteq A$, $|\Gamma(X)| \geq |X|$. Let $X = \overline{X}_0 \cup X_0 \cup X_1$, where $\overline{X}_0 \subseteq \overline{P}_0$, $X_0 \subseteq P_0$ and $X_1 \subseteq P_1$.

Suppose first that $\overline{X}_0 \neq \emptyset$. Recall that $\Gamma_j(X_i)$ denotes the set of elements of $P_j$ which are comparable with some element of $X_i$ in $P$. We have $\Gamma(X) = P'_1 \cup \Gamma_2(X_1)$. Thus

$$|\Gamma(X)| - |X| = (s + |\Gamma_2(X_1)|) - (|\overline{X}_0| + |X_0| + |X_1|) \geq \left( s + \frac{t}{s} |X_1| \right) - (t + |X_1|) = \frac{s - t}{s} (s - |X_1|) \geq 0.$$

Suppose next that $\overline{X}_0 = \emptyset$. Then $|\Gamma(X)| = |\Gamma'_1(X_1) \cup \Gamma'_1(X_0)| + |\Gamma_2(X_1)|$, where

$$\Gamma'_1(X_1) = \{j : 1 \leq j \leq s, y_j \in X_1\}$$
and
\[ \Gamma'_1(X_0) = \{ j : 1 \leq j \leq s, y_j \text{ is comparable with some } x \in X_0 \text{ in } P \}. \]
Clearly, \(|\Gamma'_1(X_1)| = |X_1|\) and \(|\Gamma'_1(X_0)| = |\Gamma_1(X_0)| \geq \frac{s}{r}|X_0|\).

When \(|\Gamma_2(X_1)| \geq |X_0|\), we have
\[ |\Gamma(X)| \geq |\Gamma'_1(X_1)| + |\Gamma_2(X_1)| \geq |X_1| + |X_0| = |X|. \]

When \(|\Gamma_2(X_1)| < |X_0|\), we have
\[
|\Gamma(X)| - |X| \geq (|\Gamma'_1(X_0)| + |\Gamma_2(X_1)|) - (|X_0| + |X_1|) \\
\geq \frac{s}{r}|X_0| + |\Gamma_2(X_1)| - |X_0| - |X_1| \\
= \left(\frac{s}{r} - 1\right) |X_0| + |\Gamma_2(X_1)| - |X_1| \\
> \left(\frac{s}{r} - 1\right) |\Gamma_2(X_1)| - |X_1| \\
= \frac{s}{r} |\Gamma_2(X_1)| - |X_1| \\
\geq \frac{s}{t} |\Gamma_2(X_1)| - |X_1| \geq 0.
\]
Hence \(|\Gamma(X)| \geq |X|\) for every \(X \subseteq A\), as required.

**Case 3** \(s < r \leq t\).

As did in Case 2, we construct an auxiliary bipartite graph \(G = (A, B; E)\). Let \(\mathcal{P}_0, \mathcal{P}_1, A\) and \(B\) be as above but \(E = E_{02} \cup E_{01} \cup E_{02} \cup E_{12}\), where
\[
\mathcal{E}_{02} = \mathcal{P}_0 \times \mathcal{P}_2, \\
E_{01} = \{ (x, j) \subseteq \mathcal{P}_0 \times \mathcal{P}_1' : 1 \leq j \leq s, x < y_j \}, \\
E_{02} = \{ (x, z) \subseteq \mathcal{P}_0 \times \mathcal{P}_2 : x < z \}, \text{ and} \\
E_{12} = \{ (y, z) \subseteq \mathcal{P}_1 \times \mathcal{P}_2 : y < z \}.
\]

We first show that \(G\) has a perfect matching. Let \(X = \overline{X}_0 \cup X_0 \cup X_1 \subseteq A\), where \(\overline{X}_0 \subseteq \mathcal{P}_0, X_0 \subseteq \mathcal{P}_0\) and \(X_1 \subseteq \mathcal{P}_1\). If \(\overline{X}_0 \neq \emptyset\), then \(\Gamma(X) = \mathcal{P}_2 \cup \Gamma'_1(X_0)\). It follows that
\[
|\Gamma(X)| - |X| = (t + |\Gamma'_1(X_0)|) - (|\overline{X}_0| + |X_0| + |X_1|) \\
\geq \left( t + \frac{s}{r}|X_0| \right) - [(t - r) + |X_0| + s] \\
= \frac{r - s}{r} (r - |X_0|) \geq 0.
\]
Now assume that $\overline{X}_0 = \emptyset$. Then $|\Gamma(X)| = |\Gamma_1(X_0)| + |\Gamma_2(X_0) \cup \Gamma_2(X_1)|$. Note that $|\Gamma_1(X_0)| = |\Gamma_1(X_0)|$. If $|\Gamma_1(X_0)| \geq |X_1|$, then

$$|\Gamma(X)| \geq |\Gamma_1(X_0)| + |\Gamma_2(X_0)| \geq |X_1| + \frac{t}{r}|X_0| \geq |X_1| + |X_0| = |X|.$$

If $|\Gamma_1(X_0)| < |X_1|$, then

$$|\Gamma(X)| - |X| \geq (|\Gamma_1(X_0)| + |\Gamma_2(X_1)|) - (|X_0| + |X_1|)
\geq |\Gamma_1(X_0)| + \left(\frac{t}{s} - 1\right)|X_1| - |X_0|
> |\Gamma_1(X_0)| + \left(\frac{t}{s} - 1\right)|\Gamma_1(X_0)| - |X_0|
= \frac{t}{s}|\Gamma_1(X_0)| - |X_0|
\geq \frac{r}{s}|\Gamma_1(X_0)| - |X_0| \geq 0.$$

Hence $|\Gamma(X)| \geq |X|$ for every $X \subseteq A$. Thus there exists a matching $f$ of $G$ from $A$ into $B$.

We next show that $f$ induces a nested chain partition of $P$. Note that $f^{-1}(P'_1) \subseteq P_0$. Hence there exists a permutation $\pi$ of $\{1, 2, \ldots, r\}$ such that $f(x_{\pi(i)}) = i$ for $1 \leq i \leq s$, which implies that $x_{\pi(i)} < y_i$ for $1 \leq i \leq s$. Let $f(y_i) = z_i$ for $1 \leq i \leq s$ and $f(x_{\pi(i)}) = z_i$ for $s + 1 \leq i \leq r$, and set $\{z_{s+1}, \ldots, z_r\} = P_2 - \{z_1, \ldots, z_s, z_{s+1}, \ldots, z_r\}$. Then we may obtain a nested chain partition of $P$:

$$P = \left(\bigcup_{i=1}^{s}\{x_{\pi(i)} < y_i < z_i\}\right) \cup \left(\bigcup_{i=s+1}^{r}\{x_{\pi(i)} < z_i\}\right) \cup \left(\bigcup_{i=r+1}^{t}\{z_i\}\right).$$

This completes the proof of the theorem. $\Box$

4. Remarks

A graded poset $P = P_0 \cup P_1 \cup \cdots \cup P_n$ is rank-symmetric and rank-unimodal (property RSU), if $|P_0| = |P_n| \leq |P_1| = |P_{n-1}| \leq \cdots \leq |P_{\frac{n}{2}}| = |P_{\frac{n}{2}}|$. A nested chain order $P$ with the RSU property is called a symmetric chain order. From Theorem 1 it follows that if a poset $P$ of rank 2 has the NM and RSU properties then $P$ is a symmetric chain order, which can also be deduced from the result of Ford and Fulkerson[4] about systems
of distinct representatives. From this and by “compressing” the middle ranks of the poset, it is not difficult to deduce the result of Anderson[1] and Griggs[7]: a poset with the NM and RSU properties is a symmetric chain order. This means that the conjecture is true for RSU posets. Similarly, we may show that if the rank numbers of an NM poset $P$ satisfy the condition $|P_0| = |P_n| \geq |P_1| = |P_{n-1}| \geq \cdots \geq |P_{\lfloor \frac{n}{2} \rfloor}| = |P_{\lceil \frac{n}{2} \rceil}|$ then $P$ is nested. Although the conjecture is true for some special posets, the problem remains open. We believe that the general answer depends on the discussion for posets of rank 3.

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References


