Proof of a Conjecture of Ehrenborg and Steingrímsson on Excedance Statistic

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We give an affirmative answer to a conjecture of Ehrenborg and Steingrímsson on the general log-concavity of the excedance statistic.

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1. INTRODUCTION

Very recently, Ehrenborg and Steingrímsson [7] studied enumerative properties of the excedance statistic. Let $S_n$ denote the permutation group on the set $\{1, 2, \ldots, n\}$ and $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$. An excedance in $\pi$ is an index $i$ such that $\pi_i > i$. Following [7], we encode the excedance set of a permutation as a word in the letters $a$ and $b$. The excedance word $w(\pi)$ of $\pi$ is the $ab$-word $w_1w_2\cdots w_{n-1}$ of length $n - 1$, where $w_i = b$ if $i$ is an excedance in $\pi$ and $w_i = a$ otherwise. Denote the number of permutations in $S_n$ with excedance word $w$ by the bracket $[w]$. Ehrenborg and Steingrímsson have shown, among other things, that the sequence $\{[b^ka^n-k]\}_{k=0}^n$ is unimodal and that for any $ab$-word $u$ the sequence $\{[ua^n]\}_{n \geq 0}$ is log-concave. Furthermore, they conjectured the following.

CONJECTURE 1.1 ([7]). For any three $ab$-words $u$, $v$ and $w$ the following four inequalities hold:

$$[uvw][uavbw] \leq [uavw][uvbw]$$

(1)

$$[uvw][uavbw] \geq [uavw][uvbw]$$

(2)

$$[uvw][uvbw] \leq [uvbw][uvbw]$$

(3)

$$[uvw][uvbw] \leq [uvbw][uvbw].$$

(4)

It is easy to see that inequality (1) implies the log-concavity of the sequence $\{[ua^n]\}_{n \geq 0}$. Moreover, Ehrenborg and Steingrímsson have observed that Conjecture 1.1 implies the log-concavity of the sequence $\{[ub^kv^n-k]\}_{k=0}^n$. So Conjecture 1.1 can be viewed as a general log-concavity property of the excedance statistic.

The main object of this paper is to verify Conjecture 1.1.

2. LOG-CONCAVITY RESULTS ON SEQUENCES

In this section we present some necessary log-concavity results on sequences for proving Conjecture 1.1.

PROPOSITION 2.1. Let $\{a_i\}$, $\{b_j\}$, $\{x_i\}$ and $\{y_j\}$ be four sequences of positive real numbers. If the sequence $\{b_j/a_i\}$ is increasing and the sequence $\{y_j/x_i\}$ is decreasing, then for any $0 \leq n \leq m$,

$$\sum_{i=0}^{n} a_i x_i \sum_{j=0}^{m} b_j y_j \leq \sum_{i=0}^{n} a_i y_i \sum_{j=0}^{m} b_j x_j.$$

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PROOF. It is not difficult to verify that
\[
\sum_{i=0}^{n} a_i x_i \sum_{i=0}^{n} b_i y_i - \sum_{i=0}^{n} a_i y_i \sum_{i=0}^{n} b_i x_i = \sum_{0 \leq i < j \leq n} (a_i b_j - a_j b_i)(x_i y_j - x_j y_i)
\]
and
\[
\sum_{i=0}^{n} a_i x_i \sum_{j=0}^{m} b_j y_j - \sum_{i=0}^{n} a_i y_i \sum_{j=0}^{m} b_j x_j = \sum_{0 \leq i < j \leq n} (a_i b_j - a_j b_i)(x_i y_j - x_j y_i) + \sum_{i=0}^{n} \sum_{j=n+1}^{m} a_i b_j (x_i y_j - y_i x_j)
\]
for \(n < m\). By the assumption for the four sequences, we have that \(a_i b_j - a_j b_i \geq 0\) and \(x_i y_j - x_j y_i \leq 0\) for \(i < j\), so the statement follows.

A sequence \(a_0, a_1, a_2, \ldots\) of non-negative real numbers is called log-concave if \(a_i a_{i+1} \leq a_i^2\) for all \(i \geq 1\). It is said to have no internal zeros if there are not three indices \(i < j < k\) such that \(a_i, a_k \neq 0\) and \(a_j = 0\).

**Corollary 2.2.** Let \(\{a_i\}, \{x_i\}\) and \(\{y_i\}\) be three sequences of positive numbers. If the sequence \(\{a_i\}\) is log-concave and the sequence \(\{y_i / x_i\}\) is decreasing, then
\[
\sum_{i=0}^{n} a_i x_i \sum_{i=0}^{n} a_i y_{i+1} \leq \sum_{i=0}^{n} a_i x_{i+1} \sum_{i=0}^{n} a_i y_i.
\]

**Proof.** Since the log-concavity of the sequence \(\{a_i\}\) implies that the sequence \(\{a_{i-1} / a_i\}\) is increasing, it follows from Proposition 2.1 that
\[
\sum_{i=0}^{n} a_i x_i \sum_{i=0}^{n} a_i y_{i+1} = a_0 x_0 \sum_{i=0}^{n} a_i y_{i+1} + \sum_{i=1}^{n} a_i x_i \sum_{i=1}^{n} a_{i-1} y_i + a_n y_{n+1} \sum_{i=1}^{n} a_i x_i \\
\leq a_0 y_0 \sum_{i=0}^{n} a_i x_{i+1} + \sum_{i=1}^{n} a_i y_i \sum_{i=1}^{n} a_{i-1} x_i + a_n x_{n+1} \sum_{i=1}^{n} a_i y_i \\
= \sum_{i=0}^{n} a_i x_{i+1} \sum_{i=0}^{n} a_i y_i.
\]

**Proposition 2.3.** Let \(x_0, x_1, \ldots, x_n\) be a log-concave sequence with no internal zeros. Then for any \(\ell \geq 0\), the following two sequences:
\[
y_i = \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x_j, \quad i = 0, 1, 2, \ldots, \ell + n \tag{5}
\]
and
\[
y_i = \sum_{j=-\ell}^{n} \binom{\ell + j + 1}{i} x_j, \quad i = \ell, \ell + 1, \ldots, \ell + n \tag{6}
\]
are log-concave respectively.
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PROOF. We have to prove that \( D_i = y_i^2 - y_{i-1}y_{i+1} \geq 0 \) for each \( i \). For \( u \leq v \) and \( k \leq 2n \), let \( C_{u,v} \) be the coefficient of \( x_u x_v \) in \( D_i \) and denote \( S_k = \sum_{j=1}^{\lfloor k/2 \rfloor} C_{j,k-j} x_j x_{k-j} \). Then \( D_i = \sum_{k \geq 2n} S_k \). Thus it suffices to show that \( S_k \geq 0 \) for all \( k \). We prove this by the same technique as used in [7, Lemma 5.2]. More precisely, we show that \( C_k = \sum_{j=1}^{\lfloor k/2 \rfloor} C_{j,k-j} \geq 0 \) for all \( k \) and that there exists an index \( r \) such that \( C_{j,k-j} < 0 \) if \( j < r \) and \( C_{j,k-j} \geq 0 \) if \( j \geq r \). Since the log-concavity of the sequence \( \{x_j\} \) implies that \( x_u x_{k-u} \leq x_r x_{k-r} \) for \( 0 \leq u \leq v \leq \lfloor k/2 \rfloor \) (see, e.g., [1, Proposition 2.5.1]). Thus we have

\[
S_k = \sum_{j=1}^{\lfloor k/2 \rfloor} C_{j,k-j} x_j x_{k-j} \geq \sum_{j=1}^{\lfloor k/2 \rfloor} C_{j,k-j} x_r x_{k-r} = C_k x_r x_{k-r} \geq 0.
\]

We first show that \( C_k \geq 0 \) for all \( k \).

For the first sequence (5), the generating function of \( C_k \) is

\[
\sum_{k=0}^{2n} C_k x^k = \left\{ \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j \right\}^2 - \left\{ \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j \right\} \left\{ \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j \right\}.
\]

Denote

\[
f_i(x) = \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j.
\]

Then

\[
f_{i-1}(x) = \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j = \sum_{j=1}^{n+1} \binom{\ell + j + 1}{i} x^{j-1} - \sum_{j=0}^{n} \binom{\ell + j + 1}{i} x^j = (x^{-1} - 1) f_i(x) + \left( \binom{\ell + n + 2}{i+1} x^n - \binom{\ell + 1}{i} x^{-1} \right).
\]

Thus

\[
\sum_{k=0}^{2n} C_k x^k = f_i^2(x) - f_{i-1}(x) f_{i+1}(x) = f_i(x) \left( f_i(x) - (x^{-1} - 1) f_{i+1}(x) \right) - \left\{ \binom{\ell + n + 2}{i} x^n - \binom{\ell + 1}{i} x^{-1} \right\} f_{i+1}(x) = \left( \binom{\ell + n + 2}{i+1} x^n - \binom{\ell + 1}{i+1} x^{-1} \right) f_i(x) - \left\{ \binom{\ell + n + 2}{i} x^n - \binom{\ell + 1}{i} x^{-1} \right\} f_{i+1}(x).
\]

When \( 0 \leq k < n \), we have

\[
C_k = -\frac{\binom{\ell + 1}{i+1} \binom{\ell + k + 2}{i} + \binom{\ell + 1}{i} \binom{\ell + k + 2}{i+1}}{\ell+1} = \frac{(\ell+1)!(\ell+k+2)!}{i!(\ell-i+1)!(\ell+k-i+2)!}(k+1).
\]

When \( n \leq k \leq 2n \), we have

\[
C_k = \frac{\binom{\ell + n + 2}{i+1} \binom{\ell + k - n - 1}{i} - \binom{\ell + n + 2}{i} \binom{\ell + k - n + 1}{i+1}}{\ell+1} = \frac{(\ell+n+2)!(\ell+k-n+1)!}{i!(\ell+n-i+2)!(\ell+k-n-i+1)!}(2n-k+1).
\]
For the second sequence (6), the generating function of $C_k$ may be obtained similarly:
\[
\sum_{k=2(i-\ell)}^{2n} C_k x^k = \left\{ \binom{\ell + n + 2}{i + 1} x^n - x^{i-\ell} \right\} f_i(x) - \left\{ \binom{\ell + n + 2}{i} x^n - x^{i-\ell} \right\} f_{i+1}(x),
\]
where
\[
f_i(x) = \sum_{j=i-\ell}^{n} \binom{\ell + j + 1}{i} x^j.
\]

When $2(i-\ell) \leq k < n + i - \ell$, we have
\[
C_k = -\binom{\ell + k - (i - \ell) + 1}{i} + \binom{\ell + k - (i - \ell - 1) + 1}{i + 1} = \binom{2\ell + k - i + 1}{i + 1}.
\]

When $k = n + i - \ell$, we have
\[
C_k = \binom{\ell + n + 2}{i + 1} \binom{\ell + (i - \ell) + 1}{i} - \binom{\ell + n + 1}{i} = \frac{(\ell + n + 1)!}{i!(\ell + n - i + 1)!} \binom{\ell + n + 1}{i}.
\]

When $n + i - \ell < k \leq 2n$, we have
\[
C_k = \frac{(\ell + n + 2)\binom{\ell + (k - n) + 1}{i} - (\ell + n + 2)\binom{\ell + (k - n) + 1}{i + 1}}{i!(i + 1)!(\ell + n - i + 2)!(\ell + k - n - i + 1)!} = \frac{(\ell + n + 1)!}{i!(\ell + n - i + 1)!} \binom{\ell + n + 1}{i}.
\]

So $C_k \geq 0$ for all $k$, as claimed.

It remains to be shown that for some $r$ the sequence $C_{j,k-j}$ is negative for $j < r$ and non-negative for $j \geq r$. In fact, we have
\[
C_{j,k-j} = 2 \binom{\ell + j + 1}{i} \binom{\ell + k - j + 1}{i} - \binom{\ell + j + 1}{i - 1} \binom{\ell + k - j + 1}{i + 1}
- \binom{\ell + k - j + 1}{i - 1} \binom{\ell + j + 1}{i + 1}
= \frac{(\ell + j + 1)!}{i!(i + 1)!(\ell + j - i + 2)!(\ell + k - j - i + 2)!} A_j
\]
for $j < k/2$ and
\[
C_{j,k-j} = \binom{\ell + j + 1}{i}^2 - \binom{\ell + j + 1}{i - 1} \binom{\ell + j + 1}{i + 1}
= \frac{[\binom{\ell + j + 1}{i}]^2}{2i!(i + 1)!(\ell + j - i + 2)!} A_j
\]
for $j = k/2$ (when $k$ even), where
\[
A_j = 2(i + 1)(\ell + j - i + 2)(\ell + k - j - i + 2)
- i(\ell + k - j - i + 2)(\ell + k - j - i + 1) - i(\ell + k - j - i + 2)(\ell + j - i + 1).
\]

Thus $C_{j,k-j}$ has the same sign as that of $A_j$ for all $j$. However, the derivative of $A_j$ with respect to $j$ is $2(2i + 1)(k - 2j) \geq 0$. Hence the sequence $\{A_j\}$ is increasing and changes sign at most once. Clearly, $A_j$ must eventually be non-negative (otherwise all $C_{j,k-j} < 0$, contradicting $C_k \geq 0$). Let $r$ be the smallest value of $j$ such that $A_j \geq 0$. Then $C_{j,k-j} < 0$ if $j < r$ and $C_{j,k-j} \geq 0$ if $j \geq r$. Thus the proof is complete. □
COROLLARY 2.4. Let \( x_0, x_1, \ldots, x_n \) be a log-concave sequence with no internal zeros. Then the sequence
\[
y_i = \sum_{j=i}^{n} \binom{j+1}{i} x_j, \quad i = 0, 1, 2, \ldots, n
\]
is log-concave.

COROLLARY 2.5. Let \( x_0, x_1, \ldots, x_n \) be a log-concave sequence with no internal zeros. Then for any \( \ell \geq 0 \), the sequence
\[
y_i = \begin{cases} 
\sum_{j=0}^{n} \binom{\ell+j+1}{j} x_j & \text{if } 0 \leq i \leq \ell \\
\sum_{j=-\ell}^{n} \binom{\ell+j+1}{j} x_j & \text{if } \ell \leq i \leq \ell + n
\end{cases}
\]
is log-concave.

PROOF. By Proposition 2.3, it suffices to show that \( y_{\ell-1} y_{\ell+1} \leq y_{\ell}^2 \). In fact, we have
\[
y_{\ell-1} y_{\ell+1} = \sum_{j=0}^{n} \binom{\ell+j+1}{\ell} x_j \sum_{j=0}^{n} \binom{\ell+j+1}{\ell+1} x_j \\
\leq \sum_{j=0}^{n} \binom{\ell+j+1}{\ell} x_j \sum_{j=0}^{n} \binom{\ell+j+1}{\ell+1} x_j \\
\leq \left( \sum_{j=0}^{n} \binom{\ell+j+1}{\ell} x_j \right)^2 = y_{\ell}^2,
\]
where the second inequality follows from the log-concavity of the sequence (5).

Let \( r, m_1, \ldots, m_r, n_1, \ldots, n_r \) be positive integers. We define a sequence
\( C_i^{(r)}(m_1, \ldots, m_1; n_1, \ldots, n_1) \), \( i = 0, 1, 2, \ldots, (n_r + \cdots + n_1) \)
by recursion as follows.
If \( r = 1 \), then let \( C_i^{(1)}(1; n) = \binom{n+1}{i} \), and for \( m > 1 \), let
\[
C_i^{(1)}(m; n) = \sum_{j=i}^{n} C_j^{(1)}(m-1; n) \binom{j+1}{i}.
\]
Next assume that \( r > 1 \). Then let
\[
C_i^{(r)}(1, m_r-1, \ldots, m_1; n_r, n_{r-1}, \ldots, n_1) = \begin{cases} 
\sum_{j=\min(m_r, n_r)+\cdots+n_1}^{n_r} C_j^{(r-1)}(m_r-1, \ldots, m_1; n_r, n_{r-1}, \ldots, n_1) \binom{n_r+j+1}{i} & \text{if } i < n_r \\
\sum_{j=\min(m_r, n_r)+\cdots+n_1}^{n_r} C_j^{(r-1)}(m_r-1, \ldots, m_1; n_r, n_{r-1}, \ldots, n_1) \binom{n_r+j+1}{i} & \text{if } i \geq n_r,
\end{cases}
\]
and for \( m_r > 1 \), let
\[
C_i^{(r)}(m_r, \ldots, m_1; n_r, \ldots, n_1) = \sum_{j=i}^{n_r+\cdots+n_1} C_j^{(r)}(m_r-1, \ldots, m_1; n_r, n_{r-1}, \ldots, n_1) \binom{j+1}{i}.
\]

By induction and Corollaries 2.4 and 2.5 we have the following result, which will play a key role in the proof of inequality (1) in the next section.
PROPOSITION 2.6. For any positive integers \( r, m_1, \ldots, m_r, n_1, \ldots, n_r \), the sequence
\[
C_i^{(\nu)}(m_r, \ldots, m_1; n_r, \ldots, n_1), \quad i = 0, 1, 2, \ldots, (n_r + \cdots + n_1)
\]
is log-concave.

3. PROOF OF CONJECTURE 1.1

To prove Conjecture 1.1 we need to review some basic properties of the excedance statistic. The following lemma is a collection of various propositions in [7].

**Lemma 3.1.** Let \( u \) and \( v \) be ab-words. Then

(i) \([au] = [u]a\), \([ub] = [u]b\).

(ii) \([ubav] = [uabv] + [ubv] + [uav]\).

(iii) \([uba^n] = \sum_{i=0}^{n} \binom{n+1}{i} [ua^i]\), and

(iv) the sequence \([ua^n]\) is log-concave.

For an ab-word \( u = u_1 u_2 \cdots u_n \), define the dual word \( u' \) by \( u' = u'_n \cdots u'_2 u'_1 \), where \( u'_i = b \) if \( u_i = a \) and \( u'_i = a \) if \( u_i = b \). Then \([u'] = [u]\) (see [7, Lemma 2.2]). Thus we see that inequality (4) is equivalent to inequality (1). In fact one can show that all four inequalities in Conjecture 1.1 are equivalent. But we shall not give the details for their equivalence here. We first verify inequality (1). Inequalities (2) and (3) then follow.

We start by proving a special case of inequality (1).

**Lemma 3.2.** Let \( u \) and \( v \) be ab-words. Then
\[
[uav][uava] \leq [uav][uva],
\]
i.e., inequality (1) holds for \( w = \emptyset \).

**Proof.** We proceed by induction on the length of \( v \). The cases \( v = \emptyset \) (the empty word) and \( v = a^n \) follow from the log-concavity of the sequence \([ua^n]\). For the case \( v = xb \), inequality (7) becomes \([ux][uxba] \leq [ux][uxba] \). By Lemma 3.1(ii) this inequality reduces to \([ux][uxa] \leq [ux][uxa] \), which holds by the induction hypothesis. What remains is the case \( v = xba^n \). In this case, inequality (7) becomes
\[
[uxba^n][uxba^{n+1}] \leq [uxaxba^n][uxba^{n+1}].
\]
By Lemma 3.1(iii), the left-hand side of (8) is
\[
\sum_{i=0}^{n} \binom{n+1}{i} [uxa^i] \sum_{j=0}^{n+1} \binom{n+2}{j} [uxa^j]
\]
and the right-hand side of (8) is
\[
\sum_{i=0}^{n} \binom{n+1}{i} [uxa^i] \sum_{j=0}^{n+1} \binom{n+2}{j} [uxa^j].
\]
Note that the sequence \( \{ \binom{n+2}{j} / \binom{n+1}{j} \} \) is increasing for \( i \) since \( \binom{n+2}{j} / \binom{n+1}{j} = (n+2) / (n+2 - i) \), and that the sequence \( \{ [uaxa^i] / [uxa^i] \} \) is decreasing for \( i = 0, 1, \ldots, n \) since \([[uxa^i][uxa^{i+1}] \leq [uxa^i][uxa^{i+1}] \) by the induction hypothesis. Thus inequality (8) follows from Proposition 2.1, and the proof is then complete. \( \square \)
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**Corollary 3.3.** Inequality (1) holds for \( w = a^n \), i.e.,

\[ [uva^n][uva]^{n+1} \leq [uva^n][uva]^{n+1}. \]

In other words, the sequence \([uva^n]/[uva^n]\) is decreasing.

**Proof.** Substitute \( va^n \) for \( v \) in inequality (7).

**Lemma 3.4.** Let \( r, m_1, \ldots, m_r, n_1, \ldots, n_r \) be positive integers and \( w = b^{m_1} a^{r_1} \ldots b^{m_r} a^{n_r} \).

Then for any ab-word \( x \),

\[ [xw] = \sum_{i=0}^{n_r + \cdots + n_1} C_i^{(r)}(m_r, \ldots, m_1; n_r, \ldots, n_1) [xa'] \]

**Proof.** We apply double induction on \( r \) and \( m_r \).

By Lemma 3.1(iii) it follows that

\[ [xba^n] = \sum_{i=0}^{n} \binom{n+1}{i} [xa'] = \sum_{i=0}^{n} C_i^{(1)}(1; n) [xa'] \]

and that for \( m > 1 \),

\[ [xb^m a^n] = [(xb)b^{m-1}a^n] = \sum_{j=0}^{n} C_j^{(1)}(m-1; n) [xa'] \]

Next suppose that \( r > 1 \). If \( m_r = 1 \), then

\[ [xw] = [(xba^{n_r})b^{m_{r-1}}a^{n_{r-1}} \ldots b^{m_1}a^{n_1}] \]

\[ = \sum_{j=0}^{n_r + \cdots + n_1} C_j^{(r-1)}(m_{r-1}, \ldots, m_1; n_{r-1}, \ldots, n_1) [xb^{n_r+j}] \]

\[ = \sum_{j=0}^{n_r + \cdots + n_1} C_j^{(r-1)}(m_{r-1}, \ldots, m_1; n_{r-1}, \ldots, n_1) \sum_{i=0}^{n_r+j} \binom{n_r+j+1}{i} [xa'] \]

\[ = \sum_{j=0}^{n_r + \cdots + n_1} \left( \sum_{i=0}^{n_r-1+j} + \sum_{i=n_r}^{n_r+j} \right) C_j^{(r-1)}(m_{r-1}, \ldots, m_1; n_{r-1}, \ldots, n_1) \binom{n_r+j+1}{i} [xa'] \]

\[ = \sum_{i=0}^{n_r-1+j} \sum_{j=0}^{n_r-1+j} C_j^{(r-1)}(m_{r-1}, \ldots, m_1; n_{r-1}, \ldots, n_1) \binom{n_r+j+1}{i} [xa'] \]

\[ + \sum_{i=n_r}^{n_r+j} \sum_{j=0}^{n_r+j} C_j^{(r-1)}(m_{r-1}, \ldots, m_1; n_{r-1}, \ldots, n_1) \binom{n_r+j+1}{i} [xa'] \]
\[ n_r + \cdots + n_1 = \sum_{i=0}^{n_r + \cdots + n_1} C_i^{(r)}(m_r, \ldots, m_1; n_r, \ldots, n_1)[xa^i], \]

and if \( m_r > 1 \), then

\[
[xw] = [(xb)b^{m_r-1}a^{n_r} \cdots b^{m_1}a^{n_1}]
\]

\[
= \sum_{j=0}^{n_r + \cdots + n_1} C_j^{(r)}(m_r - 1, m_{r-1}, \ldots, m_1; n_r, n_{r-1}, \ldots, n_1)[xba^j]
\]

\[
= \sum_{j=0}^{n_r + \cdots + n_1} C_j^{(r)}(m_r - 1, \ldots, m_1; n_r, \ldots, n_1) \sum_{i=0}^{j} \binom{j+1}{i}[xa^i]
\]

\[
= \sum_{i=0}^{n_r + \cdots + n_1} \sum_{j=i}^{n_r + \cdots + n_1} C_j^{(r)}(m_r - 1, \ldots, m_1; n_r, \ldots, n_1) \binom{j+1}{i}[xa^i]
\]

\[
= \sum_{i=0}^{n_r + \cdots + n_1} C_i^{(r)}(m_r, \ldots, m_1; n_r, \ldots, n_1)[xa^i].
\]

Thus the statement follows by induction. \( \square \)

We are now in a position to verify inequality (1).

**Theorem 3.5.** Inequality (1) holds for any \( u, v \) and \( w \).

**Proof.** We apply induction on the length of \( w \). The base case is \( w = \emptyset \), which is just Lemma 3.2. Now suppose that \( w \neq \emptyset \). For \( w = ax \) inequality (1) can be written in the form

\[
[u(va)x][ua(va)x] \leq [ua(va)x][uva(va)x],
\]

which holds by the induction hypothesis. Thus by Lemma 3.1(i) we can assume, without loss of generality, that \( w = b^{m_r}a^{n_r} \cdots b^{m_1}a^{n_1} \), where \( m_1, \ldots, m_r, n_1, \ldots, n_1 \) are positive integers. Denote \( n = n_r + \cdots + n_1 \) and \( C_i = C_i^{(r)}(m_r, \ldots, m_1; n_r, \ldots, n_1) \) for \( i = 0, 1, 2, \ldots, n \). Then it follows from Lemma 3.4, Corollaries 2.2 and 3.3 that

\[
[uvw][uavaw] = \sum_{i=0}^{n} C_i[uva^i] \sum_{i=0}^{n} C_i[uva^{i+1}]
\]

\[
\leq \sum_{i=0}^{n} C_i[uva^{i+1}] \sum_{i=0}^{n} C_i[uva^i] = [uva^i][uva^i],
\]

as required. \( \square \)

An immediate consequence of Theorem 3.5 is the following corollary, which generalizes the result of Ehrenborg and Steingr"{i}msson on the log-concavity of the sequence \([ua^n]\)_{n \geq 0}.

**Corollary 3.6.** For any ab-words \( u \) and \( w \), the sequence \([ua^n w]\)_{n \geq 0} is log-concave.

**Theorem 3.7.** Inequality (2) holds for any \( u, v \) and \( w \).

**Proof.** Apply induction on the length of \( w \). The case \( w = \emptyset \) is trivial. For the induction step, assume the statement to hold for \( w = x \) and we show that it then also holds for \( w = bx \)
and \( w = ax \), which covers all possibilities. In the case \( w = bx \), inequality (2) can be written in the form
\[
[u_a(vb)x][u(vb)bx] \leq [u(vb)x][u_a(vb)bx],
\]
which holds by the induction hypothesis. What remains is the case \( w = ax \). By Lemma 3.1(ii), we have
\[
[u_{avw}][uvbw] = [u_{avax}][uvbx] = [uavax][uavbx] + [uvbx] + [uvax]
\]
and
\[
[u_{uvw}][u_{avbw}] = [uvax][uavbx] = [uvax][uavbx] + [uvbx] + [uvax].
\]
However, by the induction hypothesis and Theorem 3.5, we have
\[
[u_{av}(va)x][u(va)bx] \leq [u(va)x][u_{av}(va)bx] \quad (9)
\]
and
\[
[u_{uv}(bx)][u_{av}(bx)] \leq [uv_{a}(bx)][u_{av}(bx)]. \quad (10)
\]
Multiplying inequalities (9) and (10) together and cancelling terms, we obtain
\[
[u_{avax}][uvbx] \leq [uvax][uavbx]. \quad (11)
\]
Thus it follows from inequalities (9) and (11) that
\[
[u_{avw}][uvbw] \leq [uvw][u_{avbw}],
\]
and the proof is then complete. \( \square \)

**Theorem 3.8.** Inequality (3) holds for any \( u, v \) and \( w \).

*Proof.* Apply induction on the length of \( v \). It follows from inequality (2) and Lemma 3.1(ii) that
\[
[uaw][ubw] \leq [uw][uaw] \leq [uw][u_{aw}].
\]
Thus inequality (3) holds for \( v = \emptyset \). Now suppose that \( v \neq \emptyset \). There are two cases to check. First suppose that \( v = bx \). Then inequality (3) can be written in the form
\[
[(ub)bxw][(ub)xbw] \leq [(ub)xw][(ub)bxaw],
\]
which holds by the induction hypothesis. Next suppose that \( v = ax \). By Lemma 3.1(ii), we have
\[
[u_{avw}][ubvw] = [uaxw][ubaxw] = [uaxw][uabxw] + [ubxw] + [uaxw]
\]
and
\[
[u_{uvw}][ubvaw] = [uaxw][ubaxw] = [uaxw][uabxw] + [ubxw] + [uaxw].
\]
However, by the induction hypothesis and Theorem 3.5, we have
\[
[(ua)xaw][(ua)bxw] \leq [(ua)xw][(ua)bxaw] \quad (12)
\]
and

\[ [u(bx)w][ua(bx)aw] \leq [ua(bx)w][u(bx)aw]. \quad (13) \]

Multiplying inequalities (12) and (13) together and cancelling terms, we obtain

\[ [uaxaw][ubxw] \leq [uaxw][ubxaw]. \quad (14) \]

Thus it follows from inequalities (12) and (14) that

\[ [uvaw][ubvw] \leq [uvw][ubvw], \]

and the proof is then complete. \( \square \)

Thus we can conclude the following theorem.

**Theorem 3.9.** Conjecture 1.1 is true.

In [7], Ehrenborg and Steingrímsson proved the unimodality of the sequence \([b^k a^{n-k}], k = 0, 1, \ldots, n\). They also observed that Conjecture 1.1 implies the log-concavity of the sequence \([ub^k va^{n-k}w], k = 0, 1, \ldots, n\). For completeness we record this result as a corollary and give its proof.

**Corollary 3.10.** For any ab-words \(u, v \) and \( w\), the sequence \([ub^k va^{n-k}w]\) is log-concave for \(k = 0, 1, \ldots, n\).

**Proof.** For any ab-word \(x\), we have by Theorem 3.9

\[ [uxw][uxa^2w] \leq [uxaw]^2, \quad [uxw][ub^2xw] \leq [ubxw]^2 \]

and

\[ [uxaw][ubxw] \leq [uxw][ubxaw]. \]

From these three inequalities it follows that

\[ [uxa^2w][ub^2xw] \leq [ubxaw]^2. \]

Now taking \(x = [b^{k-1} va^{n-k-1}]\), we have

\[ [ub^{k-1} va^{n-k-1} w][ub^{k+1} va^{n-k-1} w] \leq [ub^k va^{n-k} w]^2. \]

This completes the proof. \( \square \)

**Comment**

Relevant results to permutation statistics and log-concavity of sequences see [2–6, 8–14].

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