On Strongly Closed Subgraphs with Diameter Two and $Q$-Polynomial Property*

(Preliminary Version 2.0.0)

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August 30, 2004

Abstract

In this paper, we study a distance-regular graph $\Gamma = (X, R)$ with an intersection number $a_2 \neq 0$ having a strongly closed subgraph $Y$ of diameter 2. Let $E_0, E_1, \ldots, E_D$ be the primitive idempotents corresponding to the eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of $\Gamma$. Let $V = C^X$ be the vector space consisting of column vectors whose rows are labeled by the vertex set $X$. Let $W$ be the subspace of $V$ consisting of vectors whose supports lie in $Y$. A nonzero vector $v \in W$ is said to be tight if $E_0v = E_i v = 0$ for some $i = 1, 2, \ldots, D$. We show that the existence of a tight vector in $W$ is equivalent to a balanced condition defined by P. Terwilliger. As an application, we study the structure of parallelogram-free distance-regular graphs and conditions for these graphs to be $Q$-polynomial.

1 Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D$ with the vertex set $X$ and the edge set $R$. For vertices $x$ and $y$, $\partial(x, y)$ denotes the distance between $x$ and $y$, i.e., the length of a shortest path connecting $x$ and $y$. For a vertex $u \in X$ and $j \in \{0, 1, \ldots, D\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\} \text{ and } \Gamma(u) = \Gamma_1(u).$$

*Key words: distance-regular graph, association scheme, $Q$-polynomial, strongly closed subgraph, regular near polygon

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For two vertices $u$ and $v \in X$ with $\partial(u, v) = j$ let

$$
C(u, v) = \Gamma_{j-1}(u) \cap \Gamma(v),
A(u, v) = \Gamma_j(u) \cap \Gamma(v), \text{ and}
B(u, v) = \Gamma_{j+1}(u) \cap \Gamma(v).
$$

The cardinalities $c_j = |C(u, v)|$, $a_j = |A(u, v)|$ and $b_j = |B(u, v)|$ depend only on $j = \partial(u, v)$, and they are called the intersection numbers of $\Gamma$. Let $k_i = |\Gamma_i(u)|$ and $k = k_1$. Then $k_0 = 1$,

$$
b_i k_i = c_{i+1} k_{i+1} \text{ for } i = 0, 1, \ldots, D - 1,
$$

and $k_i$ does not depend on the choice of $u$. The number $k_i$ is called the $i$-th valency, and $k$ the valency of $\Gamma$.

A subset $Y$ of the vertex set $X$ is said to be strongly closed if the following condition is satisfied:

$$
C(u, v) \cup A(u, v) \subset Y \text{ for all } u, v \in Y.
$$

We often identify a subset of $X$ with the induced subgraph on it. In particular, when $Y$ is strongly closed, $Y$ is referred to as a strongly closed subgraph of $\Gamma$.

Strongly closed subgraphs were defined in [12]. Many distance-regular graphs have strongly closed subgraphs and conditions to guarantee the existence of such subgraphs have been studied, for example, in [4, 5, 6, 7, 13, 22, 23]. In this paper, we study the connection between the eigenvalues of a strongly closed subgraph and those of $\Gamma$. In order to state our results, we make a few more definitions.

Let $V = C^X$ denote the vector space over the complex number field consisting of column vectors whose coordinates are indexed by $X$ with complex entries. For all $x \in X$, let $\hat{x}$ denote the element of $V$ with a 1 in the $x$-coordinate and 0 in all other coordinates. For a vector $\mathbf{v} = \sum_{x \in X} \alpha(x) \hat{x} \in V$ expressed as a linear combination of $\hat{x}$'s, supp($\mathbf{v}$) denotes the support of $\mathbf{v}$, i.e., supp($\mathbf{v}$) $= \{x \in X \mid \alpha(x) \neq 0\}$.

Let $Y$ be a subset of $X$. Let $E^* = E^*(Y)$ denote the projection onto the subspace spanned by vectors $\hat{y}$ with $y \in Y$.

We now state our main results.

**Theorem 1.1** Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$, and an intersection number $a_2 > 0$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be eigenvalues of $\Gamma$, and $E_i$ the primitive idempotent corresponding to $\theta_i$ for $i = 0, 1, \ldots, D$. Let $Y$ be a strongly closed regular subgraph of $\Gamma$. Then $Y$
is strongly regular, i.e., distance-regular of diameter two, with eigenvalues \( \eta_0 = c_2 + a_2 > \eta_1 > 0 > -1 > \eta_2 \), and

\[
\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}, \quad \theta_D \geq -1 - \frac{b_1}{1 + \eta_1}.
\]

Moreover the following are equivalent.

(i) There is a nonzero vector \( v \in E^*V \) such that \( E_0 v = E_i v = 0 \) for some \( i \in \{1, 2, \ldots, D\} \).

(ii) Either one of the following holds.

(a) For every \( x, y \in Y \) with \( \partial(x, y) = 2 \), \( E_1 u = 0 \) and \( \theta_1 = -1 - \frac{b_1}{1 + \eta_2} \), where

\[
u = \sum_{z \in A(y, x)} \hat{z} - \sum_{w \in A(x, y)} \hat{w} - \eta_2 (\hat{x} - \hat{y}), \quad \text{or}
\]

(b) For every \( x, y \in Y \) with \( \partial(x, y) = 2 \), \( E_D u = 0 \) and \( \theta_1 = -1 - \frac{b_1}{1 + \eta_1} \), where

\[
u = \sum_{z \in A(y, x)} \hat{z} - \sum_{w \in A(x, y)} \hat{w} - \eta_1 (\hat{x} - \hat{y}).
\]

The condition in (ii) above is a balanced condition defined by P. Terwilliger in [17, 18]. Balanced conditions are closely related to \( Q \)-polynomial property of distance-regular graphs.

Recently, in [20], P. Terwilliger and C. Weng showed that if \( \theta_1 \) is the second largest eigenvalue of a regular near polygon with diameter \( D \geq 3 \), valency \( k \) and intersection numbers \( a_1 > 0, c_2 > 1 \), then

\[
\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}.
\]

Equality is attained above if and only if \( \Gamma \) is \( Q \)-polynomial with classical parameters with respect to \( \theta_1 \).

In this case \( \Gamma \) contains a strongly closed strongly regular subgraph with valency \( c_2(1 + a_1) \) and the least eigenvalue is \( -c_2 \). Hence the first inequality in (2) is nothing but the inequality (3).

The second result in this paper is a characterization of \( Q \)-polynomial property of parallelogram-free distance-regular graphs. Recall that a parallelogram of length \( j + 1 \geq 2 \) is a four-vertex configuration \((w, x, y, z)\) such that \( \partial(w, x) = \partial(y, z) = j = \partial(x, z), \partial(x, y) = \partial(z, w) = 1 \) and \( \partial(w, y) = j + 1 \). Regular near polygons do not have parallelograms of any lengths. There is a series of excellent articles on parallelogram-free distance-regular graphs by C. Weng and others. See [10, 19, 21, 22, 23, 24].
Theorem 1.2 Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_2 > 0$ and $b_1 > b_2$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the eigenvalues of $\Gamma$. Then the roots $\eta_1 \geq \eta_2$ of the quadratic equation $x^2 + (c_2 - a_1)x - a_2$ satisfies $\eta_1 > 0 > -1 > \eta_2$, and the following hold.

(i) $\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}$, and $\theta_D \geq -1 - \frac{b_1}{1 + \eta_1}$.

(ii) Suppose $\theta \in \{\theta_1, \theta_D\}$ attains one of the bounds above. Let $q = b_1/\theta + 1$. Then

(a) The intersection numbers of $\Gamma$ are such that $qc_i - b_i - q(qc_{i-1} - b_{i-1})$ is independent of $i$ $(1 \leq i \leq D)$.

(b) $c_3 \geq (c_2 - q)(q^2 + q + 1)$.

(c) If $\theta = \theta_1$, then $q + 1 \geq c_2$ and $q^2 + q + 1 \geq c_3$, and if $\theta = \theta_D$, then $q + 1 \leq -a_1$.

(d) The equality holds in (b) if and only if $\Gamma$ is $Q$-polynomial with classical parameters $(D, q, \alpha, \beta)$ with suitable choices of real numbers $\alpha$ and $\beta$.

If $\Gamma$ is a regular near polygon of diameter $D \geq 3$, then it is parallelogram-free and $c_2a_1 = a_2$. So $\eta_2 = -c_2$. Hence if $\theta_1$ attains the bound in (i), then $q = c_2 - 1$. Now (ii)(b), (c) and (d) imply that $\Gamma$ is $Q$-polynomial with classical parameters.

2 Preliminaries

In this section we recall some facts about distance-regular graphs. For the general theory of distance-regular graphs, we refer the reader to [1, 3].

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(C)$ denote the complex algebra consisting of all matrices whose rows and columns are indexed by $X$ with complex entries. Let $V = C^X$ denote the vector space over the complex number field consisting of column vectors whose coordinates are indexed by $X$ with complex entries. We observe $\text{Mat}_X(C)$ acts on $V$ by left multiplication. We endow $V$ with the Hermitean inner product $\langle \cdot, \cdot \rangle$ defined by

$\langle u, v \rangle = \overline{u} \bar{v}$ \quad $(u, v \in V)$,
where \( u^t \) denotes the transpose of \( u \), and \( \bar{v} \) denotes the complex conjugate of \( v \). We abbreviate \( \|u\|^2 = \langle u, u \rangle \) for all \( u \in V \).

Let \( \Gamma = (X, R) \) denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set \( X \) and edge set \( R \). For \( x, y \in X \), let \( \partial(x, y) \) denote the distance between \( x \) and \( y \), that is the length of a shortest path connecting \( x \) and \( y \). The diameter \( D \) is the maximal distance between vertices. The graph \( \Gamma \) is said to be distance-regular whenever for all integers \( h, i, j \in \{0, 1, \ldots, D\} \) and for all vertices \( x, y \in X \) with \( \partial(x, y) = h \), the number
\[
p^h_{i,j} = |\{ z \in X \mid \partial(x, z) = i, \partial(z, y) = j \}|
\]
is independent of \( x \) and \( y \). We abbreviate \( c_i = p^1_{i-1,1} \) (\( 1 \leq i \leq D \)), \( a_i = p^i_{i,1} \) (\( 0 \leq i \leq D \)), \( b_i = p^i_{i+1,1} \) (\( 0 \leq i \leq D - 1 \)) and \( k = b_0 \). For notational convenience, we define \( c_0 = 0 \) and \( b_D = 0 \). \( \Gamma \) is regular of valency \( k \) and we have
\[
k = c_i + a_i + b_i \quad \text{for all } i = 0, 1, \ldots, D.
\]
A strongly regular graph, in this paper, is a distance-regular graph of diameter 2.

For the rest of this paper we assume \( \Gamma \) is distance-regular with diameter \( D \).

For \( i \in \{0, 1, \ldots, D\} \) let \( A_i \) denote the matrix in \( \text{Mat}_X(C) \) whose \((x, y)\)-entry is defined by
\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise}.
\end{cases}
\]
The matrix \( A_i \) is called the \( i \)-th adjacency matrix of \( \Gamma \). For \( i, j \in \{0, 1, \ldots, D\} \) we have
\[
A_iA_j = \sum_{h=0}^{D} p^h_{i,j}A_h.
\]
In particular, using triangular inequalities we have for \( i \in \{1, \ldots, D\} \)
\[
A_iA_1 = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}
\]
by setting \( A_{D+1} = 0 \) and \( c_{D+1} = 1 \). Observe that by (6) and (7), the linear span \( \mathcal{M} = \text{Span}(A_0, A_1, \ldots, A_D) \) is closed under multiplication and it is algebraically generated by \( A = A_1 \). \( \mathcal{M} \) is called the Bose-Mesner algebra of \( \Gamma \). Since \( \mathcal{M} \) is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents. Let \( E_0, E_1, E_2, \ldots, E_D \) be the primitive idempotents. We write
\[
A_i = \sum_{j=0}^{D} p_i(j)E_j \quad \text{for all } i \in \{0, 1, \ldots, D\},
\]
and
\[ E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j)A_j \quad \text{for all } i \in \{0, 1, \ldots, D\}. \]  
(9)

Set \( m_i = q_i(0) \) and \( \theta_i = p_i(i) \). Then \( \theta_0, \theta_1, \ldots, \theta_D \) are the distinct eigenvalues of \( A = A_1 \), and \( m_i \) is the multiplicity of \( \theta_i \) in \( A \). We order \( E_0, E_1, \ldots, E_D \) so that
\[
\theta_0 > \theta_1 > \cdots > \theta_D.
\]

Since \( \Gamma \) is a connected \( k \)-regular graph, its adjacency matrix has largest eigenvalue \( k \) with multiplicity 1. Hence 
\( E_0 = \frac{1}{|X|} \) in this ordering, where 
\[ J \in \text{Mat}_X(C) \] is the all 1’s matrix. We use the following well-known formulas.

For all \( i, j \in \{0, 1, \ldots, D\} \),
\[ p_i(j) = \frac{q_j(i)}{m_j}. \]  
(10)

Let \( v_0(t), v_1(t), \ldots, v_D(t), v_{D+1}(t) \) denote polynomials in \( R[t] \) satisfying
\[ v_0(t) = 1 \] and for \( i \in \{0, 1, \ldots, D\} \),
\[ tv_i(t) = b_{i-1}v_{i-1}(t) + a_iv_i(t) + c_{i+1}v_{i+1}(t), \]  
(11)

where \( b_{-1} = 0, c_{D+1} = 1 \) and \( v_{-1}(t) = 0 \). Then for each integer \( i \in \{0, 1, \ldots, D+1\} \), the polynomial \( v_i(t) \) has degree \( i \), the leading coefficient \( (c_1c_2\cdots c_i)^{-1} \). Moreover, by (7), we have \( v_i(A_1) = A_i \) with \( A_{D+1} = O \). Hence \( p_i(j) = v_i(\theta_j) \) by (8). For each eigenvalue \( \theta_j \), let
\[ \sigma_i = \sigma_i(\theta_j) = \frac{q_j(i)}{m_j} = \frac{p_i(j)}{k_i}. \]

The numbers \( \sigma_0, \sigma_1, \ldots, \sigma_D \) are called the cosine sequence associated with \( \theta_j \). Let \( \sigma_0, \sigma_1, \ldots, \sigma_D \) denote the cosine sequence associated with \( k = \theta_0 \). Then \( \sigma_i = 1 \) (0 \leq i \leq D). By the trivial cosine sequence of \( \Gamma \) we mean the cosine sequence associated with \( k \).

**Lemma 2.1** Let \( \Gamma \) be a distance-regular graph of diameter \( D \). Let \( E \) be a primitive idempotent associated with the eigenvalue \( \theta \), and let \( \sigma_0, \sigma_1, \ldots, \sigma_D \) be its cosine sequence. Let \( \sigma_{-1} \) and \( \sigma_{D+1} \) be indeterminate. Then the following hold.

(i) \( c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \quad \text{for } i = 0, 1, \ldots, D. \)

(ii) \( c_i(\sigma_{i-1} - \sigma_i) + b_i(\sigma_{i+1} - \sigma_i) = (\theta - k)\sigma_i \quad \text{for } i = 0, 1, \ldots, D. \)
\( \sigma_0 = 1, \sigma_1 = \theta/k, \)

\[
\begin{align*}
\sigma_2 &= \frac{\theta^2 - a_1 \theta - k}{kb_1}, \\
\sigma_3 &= \frac{\theta^3 - (a_1 + a_2) \theta^2 + (a_1a_2 - k - c_2b_1) \theta + a_2k}{kb_1b_2}.
\end{align*}
\]

(iv) \( \sigma_0 - \sigma_1 = (k-\theta)/k, \)

\[
\begin{align*}
\sigma_0 - \sigma_2 &= \frac{1}{kb_1}(k - \theta)(\theta + 1 + b_1), \\
\sigma_0 - \sigma_3 &= \frac{1}{kb_1b_2}(k - \theta)(\theta^2 + (k - a_1 - a_2) \theta + b_1b_2 - a_2), \\
\sigma_1 - \sigma_2 &= \frac{1}{kb_1}(k - \theta)(\theta + 1), \\
\sigma_1 - \sigma_3 &= \frac{1}{kb_1b_2}(k - \theta)(\theta^2 + (k - a_1 - a_2) \theta - a_2), \text{ and} \\
\sigma_2 - \sigma_3 &= \frac{1}{kb_1b_2}(k - \theta)(\theta^2 + (c_2 - a_1) \theta + c_2 - k).
\end{align*}
\]

**Proof.** (i) follows from (10), (11) and the definition of \( \sigma_i \) using (1). Since \( k = c_i + a_i + b_i \), we have (ii). The remaining follows from (i) and (ii). Most of the formulas above can be found in [1, 3, 9].

**Lemma 2.2** ([9, Lemma 2.6]) Suppose \( \theta \neq k \). Then the following hold.

(i) \( k > \theta_1 > 0. \)

(ii) \( -1 > \theta_D \geq -1 - b_1. \)

(iii) Suppose \( \Gamma \) is not bipartite. Then \( \theta_D > -1 - b_1. \)

### 3 Existence of Strongly Closed Subgraphs

In this section we review the results on the existence of strongly closed subgraphs. In this paper, we need only the case when \( b_1 > b_2 \) and \( a_2 \neq 0 \). For more general cases, see [4, 5, 6, 7]. We first recall two conditions defined in [5].

**Definition 3.1** Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( D \). Let \( j \) be an integer \( 1 \leq j \leq D - 1. \)
For every pair of vertices \( x \) and \( y \) with \( \partial(x, y) = j \), there is a strongly closed subgraph containing \( x \) and \( y \) of diameter \( j \).

**SS\( j \):** For all vertices \( x, y \) and \( z \in X \) such that \( \partial(x, z) = \partial(y, z) = j \) and \( \partial(x, y) = 1 \), \( B(x, z) = B(y, z) \).

Note that the condition \( SS_j \) is nothing but the nonexistence of a parallelogram of length \( j + 1 \).

**Proposition 3.1** ([23, Theorem 1], [13, Theorem 1.1])  
Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( D \geq 3 \). Suppose \( b_1 > b_2 \).

(i) If \( a_1 \neq 0 \), then for a positive integer \( i < D \), the following conditions are equivalent.

(a) \( SC_j \) holds for every \( j \in \{1, 2, \ldots, i\} \).

(b) \( SS_j \) holds for every \( j \in \{1, 2, \ldots, i\} \).

(ii) If \( a_2 \neq 0 \), then the condition \( SC_2 \) holds if and only if the conditions \( SS_1 \) and \( SS_2 \) hold.

Moreover, if the conditions are satisfied, then all strongly closed subgraphs of diameter \( j \leq q \) are distance-regular.

**Proposition 3.2**  
Let \( \Gamma = (X, R) \) be a distance-regular graph with diameter \( D \). Let \( \ell \) be an integer \( 2 \leq \ell \leq D \). Suppose \( \Gamma \) satisfies the conditions \( SS_m \) for \( m = 1, 2, \ldots, \ell - 1 \). Let \( h, i \) and \( j \) be integers such that \( h \geq 0 \), \( 1 \leq j \leq i \leq \ell - h \). If \( u, v, w, x \) are vertices of \( \Gamma \) such that

\[
\partial(x, v) = h + i, \ \partial(x, u) = h, \ \partial(u, v) = i, \ \partial(u, w) = j, \ \partial(w, v) = i - j + 1.
\]

Then \( \partial(x, w) = h + j \).

**Proof.** We first prove a special case \( i = j \) by induction on \( h \). There is nothing to prove if \( h = 0 \). Let \( h \geq 1 \) and \( y \in C(u, x) \). Then by induction hypothesis \( \partial(y, w) = h + i - 1 \). Since \( \partial(y, v) = h + i - 1 \), the condition \( SS_{h+i-1} \) forces \( \partial(x, w) = h + i \) as \( x \in B(v, y) = B(w, y) \). This proves the special case.

We now prove the general case by induction on \( h \). There is nothing to prove if \( h = 0 \). Let \( h \geq 1 \) and \( y \in C(u, x) \). Suppose \( \partial(x, w) \neq h + j \). Then by a triangular inequality, \( \partial(x, w) = h + j - 1 \). By induction hypothesis, \( \partial(y, w) = h + j - 1 \). Let \( h' = i - j + 1, i' = j' = h + j - 1, x' = v, u' = w, \)
\( v' = x \) and \( w' = y \). Then by the special case we treated in the previous paragraph, \( \partial(y, v) = h + i \). This is a contradiction as
\[
\partial(y, v) \leq \partial(y, u) + \partial(u, v) = h - 1 + i.
\]
This proves the assertion.

Recall that a kite of length \( i \geq 2 \) is a four-vertex configuration \((w, x, y, z)\) such that \( \partial(w, x) = \partial(w, z) = i - 1, \partial(x, y) = \partial(y, z) = \partial(z, x) = 1 \) and \( \partial(w, y) = i \).

**Corollary 3.3** Let \( \Gamma = (X, R) \) be a distance-regular graph with diameter \( D \). Let \( \ell \) be an integer \( 2 \leq \ell \leq D \). Suppose \( \Gamma \) satisfies the conditions \( SS_j \) for \( j = 1, 2, \ldots, \ell - 1 \). Let \( i \) be a positive integer such that \( i \leq \ell \), and let \( u, v, x \) be vertices of \( \Gamma \) such that
\[
\partial(x, v) = \ell, \quad \partial(x, u) = \ell - i \quad \text{and} \quad \partial(u, v) = i.
\]

Then \( A(u, v) \subset \Gamma_\ell(x) \) and \( A(v, u) \subset \Gamma_{\ell-i+1}(x) \). In particular, \( \Gamma \) does not have a kite of length \( \ell \).

**Proof.** The assertions follow directly from Proposition 3.2 by setting \( j = i \) or \( j = 1 \).

To show that there is no kite, set \( i = 1 \).

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**4 Strongly Closed Subgraph of Diameter 2**

In this section, we prove Theorem 1.1. Throughout this section, we assume the following.

**Hypothesis 4.1** Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( D \geq 3 \) with \( a_2 \neq 0 \). Let \( Y \) be a strongly closed subset of \( X \). Suppose the induced subgraph on \( Y \) is regular with diameter 2, and \( Y \) does not contain a parallelogram of length 2.

Let \( \Delta = (Y, R_{Y \times Y}) \) be the subgraph of \( \Gamma \) induced on \( Y \). Since \( \Delta \) is strongly closed and regular, it is connected and strongly regular of valency \( \kappa = c_2 + a_2 \). Set \( \lambda = a_1 \) and \( \mu = c_2 \). Since \( \Delta \) does not contain a parallelogram of length 2, \( 0 \neq a_2 \geq c_2a_1 \). For a strongly regular graph we adopt the notation \((\nu, \kappa, \lambda, \mu)\) to describe its parameters, here \( \nu \) is the number of vertices. Recall that a conference graph is a strongly regular graph with parameters \((4\mu + 1, 2\mu, \mu - 1, \mu)\).
Lemma 4.1 Let $\Delta = (Y, R_{|Y \times Y})$ be the subgraph of $\Gamma$ induced on $Y$. Then $\Delta$ is strongly regular with parameters $(\nu, \kappa, \lambda, \mu)$ with $\kappa = c_2 + a_2$, $\lambda = a_1$ and $\mu = c_2$ and the following hold.

(i) $0 \neq \kappa - \mu \geq \lambda \mu$.

(ii) The eigenvalues of $\Delta$ are $\eta_0 > \eta_1 > \eta_2$ with $\eta_0 = \kappa$, $\eta_1 > 0$, $\eta_2 < -1$, where $\eta_1$ and $\eta_2$ are the roots of the equation

$$t^2 + (\mu - \lambda)t + (\mu - \kappa) = 0.$$ 

(iii) All eigenvalues of $\Delta$ are integers if $\kappa > 2$.

(iv) Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the distinct eigenvalues of $\Gamma$. Then

$$\theta_1 \geq \eta_1 > 0,\quad -1 > \eta_2 \geq \theta_D.$$ 

Proof. See [3, Theorem 1.3.1]. For integrality of eigenvalues in (iii), it suffices to consider the case when $\Delta$ is a conference graph. Since $\lambda \mu \leq \kappa - \mu = \mu$, either $\lambda = 1$ or 0. If $\lambda = 1$, the eigenvalues are integral. If $\lambda = 0$, then $\kappa = 2$, which is excluded. (v) follows from Corollary 3.3.2 in [3] using (ii).

Let $E^*_i = E^*_i(Y)$ ($i = 0, 1, \ldots, D$) denote the diagonal matrices in $\text{Mat}_X(C)$ defined by the following.

$$(E^*_i)_{x,y} = \begin{cases} 1 & \text{if } x = y \text{ and } \partial(x,Y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

In [15] we defined the subalgebra $T = T(Y)$ of $\text{Mat}_X(C)$ generated by $A$ and $E^*_0, E^*_1, \ldots, E^*_D$. We review a couple of results in [15]. Since we need only $E^*_0$ in this paper, write $E^* = E^*_0$ and $W = E^*V$, where $V = C^X$. Set $\hat{A} = E^*A E^*$. Then $W$ is the vector subspace of $V$ consisting of the vectors whose supports are in $Y$. Let $W_0, W_1$ and $W_2$ be the eigenspaces of $\hat{A}$ in $W$ corresponding to eigenvalues $\eta_0, \eta_1$ and $\eta_2$, respectively. Note that the width of $Y$ denoted $w(Y)$, i.e., the maximal distance of the vertices of $Y$ in $\Gamma$, in our case equals two. Let $1_Y$ denote the characteristic vector of $Y$ defined by

$$1_Y = \sum_{y \in Y} \hat{y} \in W.$$ 

Definition 4.1 A nonzero vector $v \in W$ is said to be tight, if

$$|\{i \mid i \in \{0, 1, \ldots, D\}, \ E_i v = 0\}| = w(Y) = 2.$$
Proposition 4.2 Let $\mathbf{v} \in W$ be a nonzero vector such that $E_0 \mathbf{v} = 0$.

(i) For $i \in \{0, 1, \ldots, D\}$,
\[
\frac{\|E_i \mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{m_i(k - \theta_i)((1 + \eta(\mathbf{v}))(1 + \theta_i) + b_1)}{kb_1|X|} \geq 0, \text{ with } \eta(\mathbf{v}) = \frac{\mathbf{v} \mathbf{A} \mathbf{v}}{\|\mathbf{v}\|^2}.
\]

(ii) The following hold.
\[
-1 - \frac{b_1}{1 + \theta_D} \geq \eta_1 \geq \eta(\mathbf{v}) \geq \eta_2 \geq -1 - \frac{b_1}{1 + \theta_1}.
\]

(iii) The following are equivalent.

(a) $\mathbf{v}$ is tight.

(b) One of the following holds.

1. $\eta(\mathbf{v}) = \eta_1 = -1 - \frac{b_1}{1 + \theta_D}$ and $\tilde{\mathbf{A}} \mathbf{v} = \eta_1 \mathbf{v}$, or

2. $\eta(\mathbf{v}) = \eta_2 = -1 - \frac{b_1}{1 + \theta_1}$ and $\tilde{\mathbf{A}} \mathbf{v} = \eta_2 \mathbf{v}$.

Proof. First observe that $\theta_1 \geq \eta_1 > -1$ and $\theta_D \leq \eta_2 < -1$ by Lemma 4.1 (iv).

Since $E_0 \mathbf{v} = 0$, $\mathbf{v}$ can be expressed as a sum of two mutually orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $W$ such that $\tilde{\mathbf{A}} \mathbf{v}_1 = \eta_1 \mathbf{v}_1$ and $\tilde{\mathbf{A}} \mathbf{v}_2 = \eta_2 \mathbf{v}_2$. Hence
\[
\mathbf{v} \mathbf{A} \mathbf{v} = \tilde{\mathbf{v}} \tilde{\mathbf{A}} \mathbf{v} = \eta_1 \|\mathbf{v}_1\|^2 + \eta_2 \|\mathbf{v}_2\|^2
\]
and
\[
\eta_1 \|\mathbf{v}\|^2 \geq \eta_1 \|\mathbf{v}_1\|^2 + \eta_2 \|\mathbf{v}_2\|^2 \geq \eta_2 \|\mathbf{v}\|^2.
\]
Thus $\eta_1 \geq \eta(\mathbf{v}) \geq \eta_2$ and that one of the equalities holds if and only if $\mathbf{v}_2 = 0$ or $\mathbf{v}_1 = 0$.

The formula in (i) can be computed directly using the fact that $w(Y) = 2$ or see [15, Lemma 11.4]. (ii) is from (i) and the observation above. See also [15, Lemma 11.5].

Since $E_0 \mathbf{v} = 0$, $\mathbf{v}$ is tight if and only if $E_i \mathbf{v} = 0$ for some $i \in \{1, 2, \ldots, D\}$. Hence by (i), if $E_i \mathbf{v} = 0$, then $i = 1$ or $D$, and one of the two conditions in (b) holds. The converse is clear. ■

Lemma 4.3 The following hold.

\[
W_1 = \text{Span}(\alpha(u, v) - \eta_2(\hat{u} - \hat{v}) \mid u, v \in Y, \partial(u, v) = 2), \text{ and}
\]
\[
W_2 = \text{Span}(\alpha(u, v) - \eta_1(\hat{u} - \hat{v}) \mid u, v \in Y, \partial(u, v) = 2),
\]
where
\[
\alpha(u, v) = \sum_{z \in \partial(v, u)} \hat{z} - \sum_{w \in \partial(u, v)} \hat{w}.
\]
Proof. It is clear that \( W_0 = \text{Span}(1_Y) \). Hence

\[
W_1 + W_2 = W_0^\perp \cap W = \text{Span}(\hat{u} - \hat{v} \mid u, v \in Y).
\]

We claim that

\[
W_1 + W_2 = \text{Span}(\hat{u} - \hat{v} \mid u, v \in Y, \partial(u, v) = 2).
\]

Suppose \( \partial(u, v) = 1 \). Since \( \kappa - \mu = a_2 \neq 0 \) by Hypothesis 4.1, there exists a vertex \( w \in \Gamma_2(u) \cap \Gamma_2(v) \cap Y \). Hence

\[
\hat{u} - \hat{v} = (\hat{u} - \hat{w}) + (\hat{w} - \hat{v}).
\]

This proves the claim.

It follows from Lemma 4.1 that

\[
(\tilde{A} - \eta_1 I)(\tilde{A} - \eta_2 I)
\]

vanishes on \( W_1 + W_2 \), and hence

\[
W_1 = \text{Span}(\tilde{A} - \eta_2 I)(\hat{u} - \hat{v} \mid u, v \in Y \partial(u, v) = 2), \quad \text{and}
\]

\[
W_2 = \text{Span}(\tilde{A} - \eta_1 I)(\hat{u} - \hat{v} \mid u, v \in Y \partial(u, v) = 2).
\]

We now compute \( \tilde{A}(\hat{u} - \hat{v}) \) using the fact that \( Y \) is strongly closed and that \( C(v, u) = C(u, v) \).

\[
\tilde{A}(\hat{u} - \hat{v}) = \tilde{A}\hat{u} - \tilde{A}\hat{v}
\]

\[
= \sum_{z \in A(v, u) \cup C(v, u)} \hat{z} - \sum_{w \in A(u, v) \cup C(u, v)} \hat{w}
\]

\[
= \sum_{z \in A(v, u)} \hat{z} - \sum_{w \in A(u, v)} \hat{w}
\]

\[
= \alpha(u, v).
\]

This proves the assertions. \( \blacksquare \)

Proof of Theorem 1.1. The induced subgraph on \( Y \) is strongly regular, and

\[
\eta_0 = \kappa = c_2 + a_2 > \eta_1 > 0 > -1 > \eta_2
\]

by Lemma 4.1. Next we apply Proposition 4.2 to have (2) by (i). Moreover, a nonzero vector \( v \in W \) satisfies \( E_0v = E_i v = 0 \) for some \( i \in \{1, 2, \ldots, D\} \) if and only if \( v \) is an eigenvector of \( \tilde{A} \) for \( \eta_1 \) or \( \eta_2 \).

Now we have the equivalence of (i) and (ii) in Theorem 1.1 by Lemma 4.3. \( \blacksquare \)
5 Balanced Conditions

In this section, we discuss the conditions in Theorem 1.1 (ii) from a different point of view by reviewing the balanced conditions defined by P. Terwilliger in [17, 18]. Throughout this section we assume the following hypothesis.

**Hypothesis 5.1** Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( D \geq 3 \). Let \( E \) be a primitive idempotent associated with the eigenvalue \( \theta \neq k \) with cosine sequence \( 1 = \sigma_0, \sigma_1, \ldots, \sigma_D \). Let \( V = C^X \), and set \( m = \text{rank}E \).

By (9), we have

\[
E = \frac{m}{|X|} \sum_{i=0}^{D} \sigma_i A_i, \quad AE = \theta E, \quad (12)
\]

and for every pair of vertices \( x, y \) with \( \partial(x, y) = i \),

\[
\langle E \hat{x}, E \hat{y} \rangle = \frac{m}{|X|} \sigma_i. \quad (13)
\]

**Definition 5.1** Let \( x, y \) be vertices at distance \( i \). We define vectors \( \gamma(x, y) \), \( \alpha(x, y) \) and \( \beta(x, y) \) by the following.

\[
\gamma(x, y) = \gamma_i(x, y) = \sum_{z \in C(y,x)} \hat{z} - \sum_{w \in C(x,y)} \hat{w},
\]

\[
\alpha(x, y) = \alpha_i(x, y) = \sum_{z \in A(y,x)} \hat{z} - \sum_{w \in A(x,y)} \hat{w}, \quad \text{and}
\]

\[
\beta(x, y) = \beta_i(x, y) = \sum_{z \in B(y,x)} \hat{z} - \sum_{w \in B(x,y)} \hat{w}.
\]

**Definition 5.2** Suppose \( \sigma_i \neq 1 \).

1. \( \Gamma \) is said to satisfy \( C(i) \)-balanced condition if

\[
E \gamma(x, y) \in \text{Span}(E \hat{x} - \hat{y}) \quad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.
\]

2. \( \Gamma \) is said to satisfy \( A(i) \)-balanced condition if

\[
E \alpha(x, y) \in \text{Span}(E \hat{x} - \hat{y}) \quad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.
\]

3. \( \Gamma \) is said to satisfy \( B(i) \)-balanced condition if

\[
E \beta(x, y) \in \text{Span}(E \hat{x} - \hat{y}) \quad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.
\]
Lemma 5.1 The following hold.

(i) $\alpha_1(x, y) = \gamma_2(x, y) = 0$.

(ii) $E(\gamma(x, y) + \alpha(x, y) + \beta(x, y)) = \theta E(\hat{x} - \hat{y})$.

(iii) $\Gamma$ satisfies $B(2)$-balanced condition if and only if it satisfies $A(2)$-balanced condition.

Proof. (i) This is clear from Definition 5.1.

(ii) Since $E$ is the primitive idempotent associated with an eigenvalue $\theta$, 

$$E(\gamma(x, y) + \alpha(x, y) + \beta(x, y)) = E(A \hat{x} - A \hat{y}) = EA(\hat{x} - \hat{y}) = \theta E(\hat{x} - \hat{y})$$

(iii) This follows from (i) and (ii).

By applications of (9) we easily obtain the following.

Lemma 5.2 Let $u$ and $v$ be vertices in $\Gamma$ with $\partial(u, v) = i$. Then the following hold.

(i) $\|E(\hat{u} - \hat{v})\|^2 = 2 \cdot \frac{m}{|X|} (\sigma_0 - \sigma_i)$.

(ii) $\langle E\gamma(u, v), E(\hat{u} - \hat{v}) \rangle = 2 \cdot c_i \cdot \frac{m}{|X|} (\sigma_1 - \sigma_{i-1})$.

(iii) $\langle E\alpha(u, v), E(\hat{u} - \hat{v}) \rangle = 2 \cdot a_i \cdot \frac{m}{|X|} (\sigma_1 - \sigma_i)$.

(iv) $\langle E\beta(u, v), E(\hat{u} - \hat{v}) \rangle = 2 \cdot b_i \cdot \frac{m}{|X|} (\sigma_1 - \sigma_{i+1})$.

In [18, Theorem 3.3], P. Terwilliger proved that $\Gamma$ is $Q$-polynomial if and only if $\Gamma$ satisfies $B(2)$- and $C(3)$- balanced conditions under a condition that $\sigma_i \neq 1$ for every $i \geq 1$.

Let $x$ and $y$ be vertices with $\partial(x, y) = i$. Let 

$$u \in \{E\gamma(x, y), E\alpha(x, y), E\beta(x, y)\} \quad \text{and} \quad v = E(\hat{x} - \hat{y})$$

Suppose $\sigma_i \neq 1$. Then by Lemma 5.2 (i), $v \neq 0$. Now by the Cauchy-Schwartz inequality, 

$$\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \geq 0,$$  \hspace{1cm} (14) 

and equality holds if and only if $u \in \text{Span}(v)$. As in [18], the corresponding balanced conditions can be expressed in terms of the parameters of $\Gamma$ by taking the sum over all pairs of vertices $x, y$ with $\partial(x, y) = i$. Since $\|v\|^2$ and
\((u, v)\) are expressed in terms of the parameters of \(\Gamma\) in general by Lemma 5.2, the condition may have a simpler form if the quantity \(\|u\|^2\) is expressed only by the parameters of \(\Gamma\). The following lemma gives formulas when \(\Gamma\) is parallelogram-free.

**Lemma 5.3** Suppose for some integer \(i < D\), \(\Gamma\) satisfies the condition \(SS_j\) for all \(j\) with \(1 \leq j \leq i\).

(i) Let \(u\) and \(v\) be vertices in \(\Gamma\) with \(\partial(u, v) = i\). Then

\[
\|E\beta(u, v)\|^2 = 2 \cdot b_i \cdot \frac{m}{|X|} (\sigma_0 + a_1 \sigma_1 + (b_i - a_1 - 1)\sigma_2 - (c_{i+1} - c_i)\sigma_i - (a_{i+1} - a_i)\sigma_{i+1} - b_{i+1}\sigma_{i+2})
\]

\[
= 2 \cdot b_i \cdot \frac{m}{|X|} (\theta(\sigma_1 - \sigma_{i+1}) - c_i(\sigma_2 - \sigma_i) - a_i(\sigma_2 - \sigma_{i+1})).
\]

(ii) Let \(u\) and \(v\) be vertices in \(\Gamma\) with \(\partial(u, v) = i + 1\). Then

\[
\|E\gamma(u, v)\|^2 = 2 \cdot c_{i+1} \cdot \frac{m}{|X|} (\sigma_0 + (c_{i+1} - 1)\sigma_2 + c_i \sigma_{i-1} - (c_{i+1} - c_i)\sigma_{i+1}).
\]

**Proof.** By Corollary 3.3, there is no kite of length \(j \in \{2, 3, \ldots, i+1\}\).

(i) We have

\[
\|E\beta(u, v)\|^2 = \| \sum_{z \in B(y, x)} E\hat{z}\|^2 + \| \sum_{w \in B(x, y)} E\hat{w}\|^2 - 2(\sum_{z \in B(y, x)} E\hat{z}, \sum_{w \in B(x, y)} E\hat{w}).
\]

Since there is no kite of length 2 nor \(i+1\), both \(B(y, x)\) and \(B(x, y)\) are disjoint unions of \(b_i/(a_1 + 1)\) cliques of size \(a_1 + 1\). Hence

\[
\| \sum_{z \in B(y, x)} E\hat{z}\|^2 = \| \sum_{w \in B(x, y)} E\hat{w}\|^2 = b_i \cdot \frac{m}{|X|} (\sigma_0 + a_1 \sigma_1 + (b_i - a_1 - 1)\sigma_2).
\]

Let \(z \in B(y, x)\). Because of the property \(SS_i\), there is no edge between \(\Gamma_i(x) \cap \Gamma_i(z)\) and \(\Gamma_i(x) \cap \Gamma_{i+1}(z)\). Hence we have the following.

\[
|B(x, y) \cap \Gamma_i(z)| = c_{i+1} - c_i,
\]

\[
|B(x, y) \cap \Gamma_{i+1}(z)| = a_{i+1} - a_i, \text{ and}
\]

\[
|B(x, y) \cap \Gamma_{i+2}(z)| = b_{i+1}.
\]
Therefore we have
\[
\langle \sum_{z \in B(x,y)} E \hat{z}, \sum_{w \in B(x,y)} E \hat{w} \rangle = b_i \cdot \frac{m}{|X|} ( (c_{i+1} - c_i) \sigma_i + (a_{i+1} - a_i) \sigma_{i+1} + b_{i+1} \sigma_{i+2} ).
\]

The second equality is obtained by Lemma 2.1 (i).

(ii) In this case we have
\[
\| \sum_{z \in C(x,y)} E \hat{z} \|^2 = \| \sum_{w \in C(x,y)} E \hat{w} \|^2 = c_{i+1} \cdot \frac{m}{|X|} ( \sigma_0 + (c_{i+1} - 1) \sigma_2 ).
\]

For \( z \in C(y,x) \),
\[
|C(x,y) \cap \Gamma_{i-1}(z)| = c_i, \\
|C(x,y) \cap \Gamma_i(z)| = 0, \\
|C(x,y) \cap \Gamma_{i+1}(z)| = c_{i+1} - c_i.
\]

Therefore we obtain the formula.

**Proposition 5.4** Suppose for some integer \( i < D \), \( \Gamma \) satisfies the condition \( SS_j \) for all \( j \) with \( 1 \leq j \leq i \).

(i) We have the following.
\[
(\theta(\sigma_1 - \sigma_{i+1}) - c_i (\sigma_2 - \sigma_i) - a_i (\sigma_2 - \sigma_{i+1})) (\sigma_0 - \sigma_i) \geq b_i (\sigma_1 - \sigma_{i+1})^2. \quad (15)
\]
\[
(\sigma_0 + (c_{i+1} - 1) \sigma_2 + c_i \sigma_{i-1} - (c_{i+1} - c_i) \sigma_{i+1}) (\sigma_0 - \sigma_i) \geq c_{i+1} (\sigma_1 - \sigma_i)^2. \quad (16)
\]

(ii) If \( \sigma_i \neq 1 \), then the following are equivalent.

(a) \( \Gamma \) satisfies the condition \( B(i) \).

(b) \( E \beta(u,v) \in \text{Span}(E(\hat{u} - \hat{v})) \) for some vertices \( u, v \) with \( \partial(u,v) = i \).

(c) Equality holds in (15).

(iii) If \( \sigma_{i+1} \neq 1 \), then the following are equivalent.

(a) \( \Gamma \) satisfies the condition \( C(i + 1) \).

(b) \( E \gamma(u,v) \in \text{Span}(E(\hat{u} - \hat{v})) \) for some vertices \( u, v \) with \( \partial(u,v) = i + 1 \).

(c) Equality holds in (16).
Proof. The formulas in (i) are obtained from the Cauchy-Schwartz inequality.

(ii) That (a) implies (b) follows from the definition of the condition. The equivalence of (b) and (c) is from the equality condition of the Cauchy-Schwartz. (b) and (c) imply (a) as the formula in (15) does not depend on the choices of vertices \( x, y \) with \( \partial(x, y) = i \).

(ii) This is similar to (i). 

Proposition 5.5 Suppose \( \Gamma \) satisfies the conditions \( \text{SS}_1 \) and \( \text{SS}_2 \).

(i) \( a_2((b_2 - b_1)\theta^2 + (2b_2 - c_2b_1 + a_1b_1)\theta + kb_1 - c_2b_1 + b_2) \geq 0 \).

(ii) Suppose \( a_2 \neq 0 \). Then the following conditions are equivalent.

(a) \( (b_2 - b_1)\theta^2 + (2b_2 - c_2b_1 + a_1b_1)\theta + kb_1 - c_2b_1 + b_2 = 0 \).

(b) \( \Gamma \) satisfies the condition \( B(2) \) with respect to the idempotent associated with \( \theta \).

Proof. (i) We evaluate the quantity below using Lemma 2.1.

\[
\frac{|X|^2}{4 \cdot b_2 \cdot m^2} (\|E\beta(u, v)\|^2 \|E(\hat{u} - \hat{v})\|^2 - \|E\beta(u, v), E(\hat{u} - \hat{v})\|^2)
= (\theta(\sigma_1 - \sigma_3) - a_2(\sigma_2 - \sigma_3))(\sigma_0 - \sigma_2) - b_2(\sigma_1 - \sigma_3)^2
= \frac{(k - \theta)^2}{k^2b_1^2b_2} \cdot (\theta(\theta^2 + (k - a_1 - a - 2)\theta - a_2) - a_2(\theta^2 + (c_2 - a_2)\theta + c_2 - k))
\times (\theta^2 + (k - a_1 - a_2)\theta - a_2^2)
= \frac{(k - \theta)^2}{k^2b_1^2b_2} \cdot ((k - a_1 - a_2)(1 + b_1) - a_2(c_2 - a_2 + 1) + 2a_2 - (k - a_1 - a_2)^2)\theta^2
+ (-a_2(c_2 - a_2 + 1)(1 + b_1) - a_2(c_2 - k) + 2a_2(k - a_1 - a_2))\theta
+ (-a_2(c_2 - k)(1 + b_1) - a_2^2)
= a_2 \cdot \frac{(k - \theta)^2}{k^2b_1^2b_2} \cdot ((b_2 - b_1)\theta^2 + (2b_2 - c_2b_1 + a_1b_1)\theta + kb_1 - c_2b_1 + b_2).
\]

Now the result follows from Proposition 5.4 (i).

(ii) Since \( a_2 \neq 0 \), we have \( \theta + 1 + b_1 > 0 \) by Lemma 2.2. Hence by Lemma 2.1 (iv), \( \sigma_2 \neq 1 \). Therefore \( E(\hat{u} - \hat{v}) \neq 0 \) by Lemma 5.3. Now the assertion follows from (i) and Proposition 5.4 (ii). 

Proposition 5.6 Suppose \( \Gamma \) satisfies the conditions \( \text{SS}_1 \) and \( \text{SS}_2 \). Assume \( \sigma_2 \neq 1 \). Then the following are equivalent.
(i) For a pair of vertices \(u\) and \(v\) in \(\Gamma\) with \(\partial(u,v) = 2\),
\[E\beta(u,v) \in \text{Span}(E\hat{u} - E\hat{v}).\]

(ii) For every pair of vertices \(u\) and \(v\) in \(\Gamma\) with \(\partial(u,v) = 2\).
\[E\alpha(u,v) = \xi(E\hat{u} - E\hat{v}),\]
where \(\xi = \frac{a_2(\sigma_1 - \sigma_2)}{\sigma_0 - \sigma_2} = \frac{a_2(\theta + 1)}{\theta + 1 + b_1} \cdot \frac{\sigma_0 - \sigma_2}{\theta + 1 + b_1} = a_2(\theta + 1)(\theta + 1 + b_1).

(iii) \(a_2((b_2 - b_1)\theta^2 + (2b_2 - c_2b_1 + a_1b_1)\theta + kb_1 - c_2b_1 + b_2) = 0\).

Proof. All assertions follow from Proposition 5.4 except the value of \(\xi\), which can be evaluated by taking the inner products with \(E\hat{u}\).

Corollary 5.7 Let \(\Gamma\) be a distance-regular graph of diameter \(D \geq 3\) with parameters \(b_1 = b_2\) and \(a_2 > 0\). If \(\Gamma\) satisfies the \(B(2)\)-balanced condition with respect to a primitive idempotent \(E_i\), then \(i = D\) and the eigenvalue associated with \(E_D\) is \(-k_1/(1 + a_1)\).

Proof. Since \(b_1 = b_2\), \(c_2 = 1\) by Proposition 5.4.3 in [3]. It is easy to see that \(\Gamma\) satisfies the conditions \(SS_1\) and \(SS_2\). We apply Proposition 5.5 to get \((a_1 + 1)\theta - k \geq 0\), and equality holds if and only if \(\Gamma\) satisfies the \(B(2)\)-balanced condition. Since the value \((a_1 + 1)\theta - k\) is decreasing as \(\theta\), equality holds only when \(\theta = \theta_D\).

6 Parallelogram-Free Distance-Regular Graphs

In this section we investigate distance-regular graphs \(\Gamma = (X,R)\), satisfying the conditions \(SS_j\) for all \(j\) \((1 \leq j \leq D - 1)\). We call \(\Gamma\) a parallelogram-free distance-regular graph. Parallelogram-free distance-regular graphs were extensively studied in [10, 19, 21, 22, 23, 24, 20].

Definition 6.1 Let \(\Gamma\) denote a distance-regular graph with diameter \(D \geq 3\). We say \(\Gamma\) has classical parameters \((D,q,\alpha,\beta)\) whenever the intersection numbers are given by
\[c_i = \begin{bmatrix} i \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i - 1 \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq D),\]
\[b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq D),\]
where \(\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}\).
All distance-regular graphs with classical parameters are $Q$-polynomial. See [3] for detail. The following theorem will be used in the proof of our results.

**Theorem 6.1 ([18, Theorem 4.1])** Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \mathbb{R} \setminus \{0, -1\}$. Then the following conditions (i), (ii) are equivalent.

(i) $\Gamma$ has a nontrivial cosine sequence $\sigma_0, \sigma_1, \ldots, \sigma_D$ such that

$$\sigma_{i-1} - q \sigma_i \text{ is independent of } i \ (1 \leq i \leq D).$$

(ii) The intersection numbers of $\Gamma$ are such that

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \text{ is independent of } i \ (1 \leq i \leq D).$$

Furthermore, if (i), (ii) hold, then

$$c_3 \geq (c_2 - q)(1 + q + q^2).$$

**Theorem 6.2 ([18, Theorem 4.2])** Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \mathbb{R} \setminus \{0, -1\}$. Then the following conditions (i), (ii) are equivalent.

(i) Statements (i), (ii) hold in Theorem 6.1, and

$$c_3 = (c_2 - q)(1 + q + q^2).$$

(ii) There exists $\alpha, \beta \in \mathbb{R}$ such that $\Gamma$ has classical parameters $(D, q, \alpha, \beta)$.

**Lemma 6.3** Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \mathbb{R}$. Suppose the intersection numbers of $\Gamma$ satisfy the following condition.

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \text{ is independent of } i \ (1 \leq i \leq D). \quad (17)$$

Then the following hold.

(i) $(q + 1 + a_1)(q + 1 - c_2) = a_2 - a_1c_2$.

(ii) $(q + 1 + a_1)(q^2 + q + 1 - c_3) = a_3 - c_3a_3$. 

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Then \( \theta \) with diameter \( D \) is a distance-regular graph with idempotent associated with an eigenvalue \( \theta \).

Proof. (i) Set the quantity with \( i = 1 \) equals the one with \( i = 2 \) in (17). Then we have

\[
q - b_1 + q b_0 = q c_2 - b_2 - q(q - b_1).
\]

By (5), we have

\[
q - k + 1 + a_1 + qk = q c_2 - k + c_2 + a_2 + qk - q(q + a_1 + 1).
\]

By simplifying the equation we have (i).

(ii) Set the quantity with \( i = 2 \) equals the one with \( i = 3 \) in (17). Then we have

\[
q c_2 - b_2 - q(qc_1 - b_1) = q c_3 - b_3 - q(qc_2 - b_2).
\]

By (5), we have

\[
(q+1)c_2 - k + a_2 - q^2 + qk - q(1 + a_1) = (q+1)c_3 - k + a_3 - q^2 c_2 + qk - q c_2 - qa_2.
\]

Subtract the right hand side from the left, we have

\[
0 = (q+1)c_2 + a_2 - q^2 - q(1 + a_1) - (q+1)c_3 - a_3 + q^2 c_2 + q c_2 + qa_2
\]

\[
= (q+1)(q + 1)c_2 - q(q + 1) - qa_1 + a_2(q + 1) - (q+1)c_3 - a_3
\]

\[
= (q+1)(q + 1)c_2 - q(q + 1) - qa_1 + a_2(q + 1) - (q+1)c_3
\]

\[
- c_3 a_1 - c_3 a_1 - c_2 a_1(q + 1) + c_2 a_1(q + 1)
\]

\[
= (q+1)(q + 1 + a_1)c_2 - q(q + 1 + a_1) - (q+1 + a_1)c_3
\]

\[
- (a_3 - c_3 a_1) + (a_2 - c_2 a_1)(q + 1)
\]

\[
= (q + 1 + a_1) c_2 (q + 1) - q - c_3) - (a_3 - c_3 a_1) + (a_2 - c_2 a_1)(q + 1)
\]

\[
= (q + 1 + a_1) c_2 (q + 1) - q - c_3 + (q + 1)(q + 1 - c_2) - (a_3 - c_3 a_1)
\]

\[
= (q + 1 + a_1) (q^2 + q + 1 - c_3) - (a_3 - c_3 a_1),
\]

as desired. We used the formula in (i). \( \blacksquare \)

We consider the case (ii) in Proposition 5.6 when \( \Gamma \) is a parallelogram-free distance-regular graph with \( a_2 > 0 \). By Lemma 2.2 and Lemma 2.1, \( \sigma_2 \neq 1 \).

Lemma 6.4 Let \( \Gamma = (X, R) \) be a parallelogram-free distance-regular graph with diameter \( D \geq 3 \) and intersection number \( a_2 > 0 \). Let \( E \) be the primitive idempotent associated with an eigenvalue \( \theta \neq \theta_0 \), and let \( \sigma_0, \sigma_1, \ldots, \sigma_D \) denote the corresponding cosine sequence. Let \( u \) and \( v \) be vertices with \( \partial(u, v) = 2 \).

Suppose

\[
E \alpha(u, v) = \xi(E \hat{u} - E \hat{v}), \text{ where } \xi = \frac{a_2(\sigma_1 - \sigma_2)}{\sigma_0 - \sigma_2} = \frac{a_2(\theta + 1)}{\theta + 1 + b_1}.
\]

Then \( \theta = \theta_1 \) or \( \theta_D \), \( \xi \neq 0 \), and the following hold.
(i) $\sigma_{i-1} - q \sigma_i$ is independent of $i$ ($1 \leq i \leq D$), where $q = a_2/\xi - 1$.

(ii) If $\theta = \theta_1$, then $q + 1 \geq c_2$ and $q^2 + q + 1 \geq c_3$, and if $\theta = \theta_D$, then $b + 1 \leq -a_1$.

Proof. We first claim that $\theta = \theta_1$ or $\theta_D$. If $b_1 > b_2$, then by Proposition 3.1 there is a strongly closed strongly regular subgraph $Y$. Since $\alpha(u, v) - \xi(\hat{u} - \hat{v})$ is a tight vector in $E^*V$, $\theta = \theta_1$ or $\theta_D$ by Theorem 1.1. If $b_1 = b_2$, then by Corollary 5.7 we have the claim.

Now we also have $\xi \neq 0$ by Lemma 2.2.

Let $x \in \Gamma_{i-2}(u) \cap \Gamma_i(v)$ for $2 \leq i \leq D$. Then by Corollary 3.3 the inner product of $E\hat{x}$ with $E\alpha(u, v) = \xi(E\hat{u} - E\hat{v})$ yields

$$a_2(\sigma_{i-1} - \sigma_i) = \xi(\sigma_{i-2} - \sigma_i).$$

Hence with $a_2/\xi = q + 1$ we have

$$(q + 1)(\sigma_{i-1} - \sigma_i) = \sigma_{i-2} - \sigma_i$$

and

$$\sigma_{i-2} - q \sigma_{i-1} = \sigma_{i-1} - \sigma_i.$$

Therefore we have (i).

(ii) Observe that $a_2 \geq a_1c_2$ and $a_3 \geq a_1c_3$. We apply Lemma 6.3 to have

$$(q + 1 + a_1)(q + 1 - c_2) \geq 0, \quad \text{and} \quad (q + 1 + a_1)(q^2 + q + 1 - c_3) \geq 0.$$

Suppose $\theta = \theta_1$. Then $q + 1 + a_1 > 0$. Hence $q + 1 - c_2 \geq 0$ and $q^2 + q + 1 - c_3 \geq 0$.

If $\theta = \theta_D$. Then $q + 1 - c_2 < 0$. Hence $q + 1 + a_1 \leq 0$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose $\Gamma = (X, R)$ is a parallelogram-free distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_2 > 0$ and $b_1 > b_2$. Then by Proposition 3.1, $\Gamma$ has a strongly closed regular subgraph of diameter 2, and $\Gamma$ satisfies Hypothesis 4.1. Hence we have (i).

Suppose $\theta \in \{\theta_1, \theta_D\}$ attains one of the bounds in (i). Then by Theorem 1.1, $A(2)$-balanced condition is satisfied. Now we apply Lemma 6.4. Note that

$$q = \frac{a_2}{\xi} - 1 = \frac{\theta + 1 + b_1}{\theta + 1} - 1 = \frac{b_1}{\theta + 1}.$$

Hence we have all assertions.
Proposition 6.5 Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_2 = s - 1 > 0$, $b_1 = b_2$. Suppose $B(2)$-condition holds with respect to a primitive idempotent $E_i$. Then $\Gamma$ is a regular near $2D$-gon and $c_3 \geq 1 - q^3$, where $q = -s = -(a_1 + 1)$. If equality holds, then $\Gamma$ is a classical distance-regular graph with parameters

$$(D, q, \alpha, \beta) = (D, -s, s, k(1 + s) \frac{k(1 + s)}{1 - s} 1 - (-s)^2)$$

Proof. We apply Corollary 5.7. Now the number $q$ in Lemma 6.3 is $-s$ as we set $a_1 = a_2 = s - 1$. By Theorem 6.2, $\Gamma$ is a classical distance-regular graph with parameters $(D, -s, \alpha, \beta)$ for some reals $\alpha$ and $\beta$. Now the values of $\alpha$ and $\beta$ can be computed by Proposition 6.2.1 in [3].

7 Examples

1. If $\Gamma$ contain a strongly closed subgraph isomorphic to (the collinearity graph of) a generalized quadrangle, $\theta_D$ attains the second bound in (2) if and only if $\theta_D = -k/(a_1 + 1)$.

2. Dual polar graphs and Hamming graphs are the only $Q$-polynomial regular near polygons of diameter $D \geq 4$ with intersection numbers $c_2 > 1$ and $a_1 \neq 0$ by Corollary 5.7 in [22] and these are distance-regular graphs having classical parameters with $\alpha = 0$ and $a_1 \neq 0$. These graphs are $Q$-polynomial with respect to $\theta_1$ and attain both of the bounds in (2).

3. Let $\Gamma$ be a parallelogram-free $Q$-polynomial distance-regular graph of diameter $D \geq 4$ with $a_2 \neq 0$. Then $\Gamma$ has classical parameters $(D, q, \alpha, \beta)$ and $\Gamma$ is either a regular near polygon or $q < -1$. Distance-regular graphs having classical parameters $(D, q, \alpha, \beta)$ with $q < -1$ are said to be of negative type. These graphs satisfy the second bound in (2). [10, 19, 21, 22, 23, 24].

For a list of negative type distance-regular graphs among others with classical parameters, see [3, Table 6.1].

The author does not know any parallelogram free distance-regular graph with intersection number $a_2 \neq 0$ of diameter $D \geq 3$ which is not $Q$-polynomial.

In a subsequent paper [8] we study distance-regular graphs with a subset $Y$ of width two such that $E^*V \cap 1^*_V$ is spanned by tight vectors.
Acknowledgments

The author would like to give thanks to Professor Paul Terwilliger and Professor Chih-Wen Weng for sending him preprints and giving valuable suggestions and comments. I must admit that all motivations and ideas were taken from their papers.

References


