A Few Topics in micropump simulations

Chin-Tien Wu

Institute of Mathematical Modeling and Scientific Computation
Department of Applied mathematics
National Chiao-Tung University

05/14/2010
台東大學綠色科技博覽會
Motivations

For designing a more reliable and efficient micro pump, it is important to understand:

1. How the fluid interacts with the solid structure?
2. How the composite membrane (a diaphragm coated with PZT ceramics) deform?
3. Can we have a good frequency control on the diaphragm vibration such that the flux through the pump can be maximized?
4. Can we detect or predict impurity of the PZT membrane to prevent possible malfunction of the pump?

We hope that numerical simulations can help in answering these questions.
(1) and (2) involve computational mechanics (including fluids and structures), solving PDEs with interfaces and solving PDEs on evolution surface.

(3) and (4) involve solving polynomial eigenvalue problems especially for seeking resonance modes.

For crack or intrusion detection in (4), one needs to solve inverse problems. For fracture and crack detection, singular solutions need to be computed.
Outline

• Incompressible Fluid simulations
• Nonlinear elasticity
• Piezoelectric material
• Potential problem on surfaces
• Interface problems
• Immersed finite element method for interface problems
• Intrusion detection problems
• conclusion
Incompressible fluid solver

\[ \frac{\partial u}{\partial t} + u \nabla u - \nu \Delta u + \nabla p = 0 \quad \text{in } \Omega \times [0, T] \]
\[ \nabla \cdot u = 0 \quad \text{in } \Omega \times [0, T], \]
\[ u(x, t) = g(x, t) \quad \text{on } \partial \Omega \times [0, T] \]
\[ u(x, 0) = u_0(x) \quad \text{in } \Omega \]

Time discretization (CN)

\[ \frac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+\theta} + u^* \nabla u^{n+\theta} + \nabla p^{n+\theta} = 0 \]
\[ \nabla \cdot u^{n+\theta} = 0 \quad \text{and } u^{n+\theta} = g \quad \text{on } \partial \Omega \]

Space discretization

\[ B_\delta \left( \langle (u^{n+1}, p), (v, q) \rangle \right) = \left( \frac{u^{n+1} - u^n}{\Delta t} - \varepsilon \Delta u^{n+\theta} + (u^* \cdot \nabla) u^{n+\theta} + \nabla p^{n+\theta}, v \right) + \]
\[ \sum_T \left( \frac{u^{n+1} - u^n}{\Delta t} + (u^* \cdot \nabla) u^{n+\theta} + \nabla p^{n+\theta}, \tau_T \left[ (u^* \cdot \nabla) v - \nabla q \right] \right) - \]
\[ \sum_T \left( \nabla \cdot u^{n+\theta}, q - \delta_T \nabla \cdot v \right) = \langle f, v \rangle + \sum_T \langle f, \tau_T \left[ (u^* \cdot \nabla) v - \nabla q \right] \rangle \]

Reynold’s number:
flow passing around a cylinder

Re=20

Re=200
(B) **Accuracy is achieved:**

In the following, $C_D =$ drag coefficient $= \frac{2F_d}{\rho U_\infty^2 L}$, $C_L =$ lift coefficient $= \frac{2F_l}{\rho U_\infty^2 L}$

$ST =$ Strouhal number $= \frac{freq \times L}{\nu}$

<table>
<thead>
<tr>
<th>Re</th>
<th>20</th>
<th>50</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_D$</td>
<td>2.11</td>
<td>2.22(1)</td>
<td>1.46(1)</td>
<td>1.38(3)</td>
</tr>
<tr>
<td></td>
<td>2.19(2)</td>
<td>1.41(1)</td>
<td>1.38 ± 0.01</td>
<td>1.24(1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.35 ± 0.012(4)</td>
<td>1.38 0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.16(1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.31 ± 0.049(4)</td>
</tr>
<tr>
<td>$C_L$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.339(4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.69(4)</td>
</tr>
<tr>
<td>ST</td>
<td>-</td>
<td>0.125</td>
<td>0.12~0.13(1)</td>
<td>0.167</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.139(3)</td>
<td></td>
<td>0.167(1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.164(4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.19(4)</td>
</tr>
</tbody>
</table>

(1) for $Re=100$ and $Re=200$ are obtained experimentally by Clift and Reshko respectively.
(2) is computed on a 640x320 grid by Donna Calhoun, Courant Institute of mathematics science, in 2002.
(3) is computed on a 267x147 grid by Saki and Biringen, Dept. of Aerespace Engineering, Univ. of Colorado, in 1996.
(4) is computed on a 256x256 grid by Liu, Zheng and Sung in 1998.
Simulation of flow, Re=10^6, around the airfoil NACA0012 with 20° angle of attack

Chin-Tien’s simulation on a grid with 4720 nodes

Kunio Kuwahara’s simulation on a 128x256 grid (JSCFD 2000)
Cp of Naca0012 at 0° angle of attack with Re=3e+06

Fusen He and Tsung-Chow Su,

Chin Tien’s simulation result

Number of points = 4984
400 time steps at \( \Delta t=0.02 \) and
400 time steps at \( \Delta t=0.01 \)
Mathematical Modeling for large deformation of a elastic beam
Linear Euler-Bernoulli Beam

\[ \epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z \frac{\partial \theta_y}{\partial x} + y \frac{\partial \theta_z}{\partial x} \]

\[ \delta_{xx} = E \epsilon_{xx} \]

\[ \delta \epsilon_{xx} = \frac{\partial \delta u}{\partial x} = \frac{\partial \delta u_0}{\partial x} + z \frac{\partial \delta \theta_y}{\partial x} + y \frac{\partial \delta \theta_z}{\partial x} \]

\[ \delta U = \int \delta \epsilon_{xx} \sigma_{xx} \, dv \]

\[ = \int_{X=0}^{L} EA \frac{\partial u_0}{\partial x} \frac{\partial \delta u_0}{\partial x} \, dx + \int_{X=0}^{L} EI_y \frac{\partial \theta_y}{\partial x} \frac{\partial \delta \theta_y}{\partial x} \, dx + \int_{X=0}^{L} EI_z \frac{\partial \theta_z}{\partial x} \frac{\partial \delta \theta_z}{\partial x} \, dx \]

\[ \int EI_y \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \, dx \]

\[ \rightarrow \quad 4th \text{ order differential equation (Bi-harmonic equation)} \]
Nonlinear Euler Beam

Geometry nonlinearity

\[ E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right) \]

\[ = \frac{\partial}{\partial x} \left( u^{(i)} + \Delta u \right) + \frac{1}{2} \left[ \left( \frac{\partial u^{(i)}}{\partial x} + \frac{\partial \Delta u}{\partial x} \right)^2 + \left( \frac{\partial v^{(i)}}{\partial x} + \frac{\partial \Delta v}{\partial x} \right)^2 + \left( \frac{\partial w^{(i)}}{\partial x} + \frac{\partial \Delta w}{\partial x} \right)^2 \right] \]

\[ = \frac{\partial u^{(i)}}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u^{(i)}}{\partial x} \right)^2 + \left( \frac{\partial v^{(i)}}{\partial x} \right)^2 + \left( \frac{\partial w^{(i)}}{\partial x} \right)^2 \right] \]

\[ + \frac{\partial \Delta u}{\partial x} + \frac{\partial u^{(i)}}{\partial x} \frac{\partial \Delta u}{\partial x} + \frac{\partial v^{(i)}}{\partial x} \frac{\partial \Delta v}{\partial x} + \frac{\partial w^{(i)}}{\partial x} \frac{\partial \Delta w}{\partial x} + O \left( (\Delta u)^2, (\Delta v)^2, (\Delta w)^2 \right) \]

\[ = E_{xx}^{(i)} + \Delta \varepsilon_{xx} + O \left( (\Delta u)^2, (\Delta v)^2, (\Delta w)^2 \right) \]
\[
\delta E_{xx} = \frac{\partial \delta u}{\partial x} + \frac{\partial u^{(i)}}{\partial x} \frac{\partial \delta u}{\partial x} + \frac{\partial v^{(i)}}{\partial x} \frac{\partial \delta v}{\partial x} + \frac{\partial w^{(i)}}{\partial x} \frac{\partial \delta w}{\partial x}
\]
\[
+ \frac{\partial \Delta u}{\partial x} \frac{\partial \delta u}{\partial x} + \frac{\partial \Delta v}{\partial x} \frac{\partial \delta v}{\partial x} + \frac{\partial \Delta w}{\partial x} \frac{\partial \delta w}{\partial x}
\]
\[
= \delta \varepsilon_{xx} + \delta \eta_{xx}
\]

\[
\delta U = \int \left( \delta \varepsilon + \delta \eta \right)^T \left( S^{(i)} + \Delta S \right) dv
\]
\[
= \int \delta \varepsilon^T S^{(i)} + \int \delta \eta^T S^{(i)} + \int \delta \varepsilon^T \Delta S + \int \delta \eta^T \Delta S
\]
\[
= \left\{ \begin{array}{l}
\int \delta \varepsilon^T CE^{(i)} + \int \delta \eta^T CE^{(i)} \\
\text{virtual work at i-th step}
\end{array} \right\} + \left\{ \begin{array}{l}
\int \delta \varepsilon^T C\Delta E + \int \delta \eta^T C\Delta E \\
\text{increment in Newton-Rapson}
\end{array} \right\}
\]
Full circle benchmark problem:

(1) bar length = 12, width and height of the cross section=1,

(2) end moment $M=mf$, here $f$ varies from 0 to 2 and $m=654761.9$

(3) Young’s modulus = $3.0e+07$, Poisson ratio=0,
Results from our codes

Original configuration

Configuration under 1M moment

Configuration under 2M moment
4th order bi-harmonic equation

Accuracy check for bi-harmonic solver (using BICZ element):

- Simply supported plate with uniform load $P=1$
  - max($u$) = 4.378e-07 and exact max($u$) = 4.5427e-07

- Clamped cantilever plate with end moment $M_y=1$
  - max displacement = 1.60
  - $v=0.3$, $E=1e+10$, $h=0.01$
PZT simulation on saw filter

\[ \sigma = C^E \varepsilon - e^T E \]
\[ D = e\varepsilon + K^\varepsilon E \]

\[
C^E = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12})
\end{bmatrix}
\]

\[
K^\varepsilon = \begin{bmatrix}
k_{11} & 0 & 0 \\
0 & k_{22} & 0 \\
0 & 0 & k_{33}
\end{bmatrix}
\]

\[
e = \begin{bmatrix}
0 & 0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{15} & 0 & 0 \\
e_{31} & e_{31} & e_{13} & 0 & 0 & 0
\end{bmatrix}
\]

where \(\sigma\) and \(\varepsilon\) are the stress and strain tensors respectively, \(C^E\), \(K^\varepsilon\) and \(e\) are the elasticity constant, dielectric constant and piezoelectric constant matrices measured at constant strain and constant temperature, \(D\) is the electric displacement and \(E\) is the electric field.
The unit cell problem

\[ - \text{div}(c \frac{1}{2}((\nabla u)^T + \nabla u) + e^T \nabla \phi) = \omega^2 \rho u \text{ in } \Omega_P \]
\[ - \text{div}(e \frac{1}{2}((\nabla u)^T + \nabla u) - \epsilon \Phi) = 0 \text{ in } \Omega_P \]
\[ T.n = 0, D.n = 0 \text{ on } \Gamma_N \]
\[ \Phi = 0 \text{ on } \Gamma_{EI} \]

Quasi-periodic boundary conditions on \( \Gamma_L, \Gamma_R \)

\[ \tilde{u}(p, x_2) = \gamma \tilde{u}(0, x_2) \]
\[ \frac{\partial \tilde{u}}{\partial N_r}(p, x_2) = -\gamma \frac{\partial \tilde{u}}{\partial N_l}(0, x_2) \text{ with } \tilde{u} = (u_1, u_2, u_3, \Phi) \]

with propagation parameter \( \gamma := e^{(\alpha + i\beta)p} \).
The frequency-dependent eigenvalue problem

For given parameters $\omega^2$ search for $(\gamma, (u, \lambda)^T)$ such that

\[
\begin{align*}
\frac{k(\omega)(u,v)}{a(u,v) + i\omega c(u,v) - \omega^2 m(u,v)} + <(tr_l^* - \gamma tr_r^*)\lambda, v> &= 0 \quad \forall v \in [H^1(\Omega)]^4 \\
< (tr_r - \gamma tr_l)u, \mu > &= 0 \quad \forall \mu \in [H^{-0.5}(\Gamma)]^4
\end{align*}
\]

The FE-discretized problem is of the form

\[
\begin{pmatrix}
\overline{K}(\omega) & Tr_l^T \\
Tr_r & 0
\end{pmatrix}
\begin{pmatrix}
u_h \\
\lambda_h
\end{pmatrix} = \gamma
\begin{pmatrix}
0 & Tr_r^T \\
Tr_l & 0
\end{pmatrix}
\begin{pmatrix}
u_h \\
\lambda_h
\end{pmatrix}
\]

with $\overline{K} := K + i\omega C - \omega^2 M$ complex-symmetric and indefinite (of saddle point structure).
Simulation results from 36° YX-Cut LiTaO₃

Simulation from our code

Potential=100  Potential=-100

SAW in fluid transfer

Surface Acoustic Wave-Induced Atomization

Metering and dosing in Advalytix labelhip 
first filling with blue detergent solution
Solving Laplace on evolution surface

\[
\frac{d}{dt} \int_{\mathcal{M}(t)} u = \int_{\partial \mathcal{M}(t)} q \cdot \mu, \quad \rightarrow \quad \dot{u} + u \nabla \cdot \nu + \nabla \cdot q = 0.
\]

FEM accuracy check for surface laplacian: \( u(x,y) = x*y \)

<table>
<thead>
<tr>
<th>mesh points</th>
<th>max err</th>
<th>relative L2 err</th>
</tr>
</thead>
<tbody>
<tr>
<td>162</td>
<td>0.03164</td>
<td>0.0597</td>
</tr>
<tr>
<td>642</td>
<td>0.00885</td>
<td>0.0154</td>
</tr>
<tr>
<td>2562</td>
<td>0.0024</td>
<td>0.0038</td>
</tr>
</tbody>
</table>


Fig. 6. Level lines of the stationary solution of Example 7.5: Front side of the sphere (left) and backside (right).
Interface problems

• 2\textsuperscript{nd} order elliptic interface problems: heat diffusion, electric potential and plane elasticity for composite materials

\[ \nabla \cdot \sigma + F = 0 \quad \text{in } \Omega^- \cup \Omega^+, \]
\[ u = G \quad \text{on } \partial \Omega. \]
\[ \sigma = \alpha \nabla u \]

• 4\textsuperscript{th} order elliptic interface problems: composite membrane bending

\[ \nabla^2 (\alpha \nabla^2 u) = f \quad \text{in } \Omega \subset \mathbb{R}^2, \]
\[ u = \partial_n u = 0 \quad \text{on } \partial \Omega. \]
2\textsuperscript{nd} coupled with 4\textsuperscript{th} order interface problems:

- Structure-Structure interactions:

\[
\begin{align*}
\Gamma_1 & : \text{Rigid baffle (plate)} \\
\Gamma_2 & : \text{Rigid baffle (plate)}
\end{align*}
\]

\[
\begin{align*}
-\Delta u_1 + \alpha u_1 &= f \quad \text{in } \Omega_1 \\
u_1 &= 0 \quad \text{on } \Gamma_1
\end{align*}
\]

\[
\begin{align*}
\sigma^2 \Delta^2 u_2 - \Delta u_2 + \alpha u_2 &= f \quad \text{in } \Omega_2, \\
u_2 &= \frac{\partial u_2}{\partial n} = 0 \quad \text{on } \Gamma_2,
\end{align*}
\]

**Plate-membrane coupling: modeling the acoustic**

characteristics of baffled membranes and the surrounding sound fields (sound insulation performance of building elements).

Interface conditions

- 2\textsuperscript{nd} order elliptic interface problems

\[ [u]_\Gamma = W, \]
\[ [\sigma n]_\Gamma = Q, \]
\[ [u] = 0 \]
\[ [u_x] = [u_y] = 0 \]
\[ [\alpha \Delta u] = 0 \]
\[ \left[ \alpha \frac{\partial}{\partial n} \Delta u \right] = 0 \]

- 4\textsuperscript{th} order elliptic interface problems

- 2\textsuperscript{nd} coupled with 4\textsuperscript{th} order interface problems:
  - Structure-Structure interactions:

\[ u_2 - u_1 = g_1, \]
\[ \sigma \Delta u_2 = g_2, \]
\[ n \cdot (-\nabla (\sigma \Delta u_2)) - Tn \cdot \nabla u_1 = g_3 \quad \text{on} \quad \Gamma. \]
Lopatinski–Shapiro conditions

Given an elliptic interface problem, the interface conditions will lead to a well-posed problem, if and only if, they satisfy the so called covering conditions or Lopatinski–Shapiro conditions.

**Definition**

The system of interface operators $B_{ij-1}(x; D)$, $i = 1, 2; j = 1, \ldots, m_1 + m_2$ cover the pair of elliptic operators $P_1(x; D), P_2(x; D)$ at the interface $\Gamma$, if

$$d(x_0; \xi) = \det \begin{pmatrix} b_{11}^1(\xi) & b_{21}^1(\xi) & \cdots & b_{m_1+m_2}^1(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ b_{11}^{m_1}(\xi) & b_{21}^{m_1}(\xi) & \cdots & b_{m_1+m_2}^{m_1}(\xi) \\ b_{11}^{m_1+m_2}(\xi) & b_{21}^{m_1+m_2}(\xi) & \cdots & b_{m_1+m_2}^{m_1+m_2}(\xi) \end{pmatrix} \neq 0$$

for any $x_0 \in \Gamma, \xi \neq 0$.

Fluid-Structure interactions:

\[
\rho^f \left( \frac{\partial}{\partial t} u^f + u^f \cdot \nabla u^f \right) + \nabla p^f = \text{div}(\sigma^f) + g_b^f \\
\nabla \cdot u^f = 0 \\
u^f \big|_{\Gamma_1} = u^s \big|_{\Gamma_1} \quad \text{and} \quad u^f \big|_{\Gamma_2} = (\bar{u}_\text{mean}^f, 0),
\]

Here \( \sigma^f = -p^f I + \rho^f \varepsilon \left( \nabla u^f + \nabla u^f^T \right) \)

\[
\rho^s \left( \frac{\partial}{\partial t} u^s + (\nabla u^s) u^s \right) = \nabla \cdot \sigma^s + g_b^s
\]

\( \sigma^s \cdot \bar{n} = \sigma^f \cdot \bar{n} \) and \( u^s \big|_{\Gamma_1} = u^f \big|_{\Gamma_1} \)

Here \( \sigma^s \) is the Cauchy stress tensor defined as

\[
\sigma^s = \frac{1}{\det(F)} F \left( \lambda^s \cdot tr(E) I + 2\mu^s E \right) F^T,
\]

\( F = I + \nabla u^s \) and \( E = \frac{1}{2} \left( F^T F - I \right). \)
Moving mesh in FSI simulations


Simulation result from our code
What is troubling us?

• Accuracy on the interface degrades seriously as the jump ratio of the coefficient $\sigma$ becomes large.

• Convergence rate of the linear solver such as multigrid iterative method decreases.

• For FSI dynamic simulation, moving mesh strategies are expansive and failure occurs sometimes due to improper time step size.
Toward to overcome the difficulties arising from interfaces

Immersed finite element method

In IFEM, the basis functions of the interface element are constructed to satisfy the interface conditions!

Fig. 2.2. (a): A standard domain of six triangles with an interface cutting through. (b): A global basis function on its support in the non-conforming immersed finite element space. The basis function has small jump across some edges.
A simple 1-d 2\textsuperscript{nd} order problem

\[ (\beta u_x)_x = 12x^2, \quad 0 \leq x \leq 1, \quad \beta = \begin{cases} \beta^- & \text{if } x < \alpha, \\ \beta^+ & \text{if } x > \alpha, \end{cases} \]

\[ u(0) = 0, \quad u(1) = 1/\beta^+ + (1/\beta^- - 1/\beta^+)\alpha^4. \]

\[ u(x) = \begin{cases} x^4/\beta^- & \text{if } x < \alpha, \\ x^4/\beta^+ + (1/\beta^- - 1/\beta^+)\alpha^4 & \text{if } x > \alpha, \end{cases} \]
A simple 4\textsuperscript{th} order problem

\[ \Delta(\beta \Delta u) = 24 \]

\[ u(0) = u'(0) = 0; u(1) = \frac{1 - 2\alpha^2}{\beta^+} - \alpha^4 \left( \frac{1}{\beta^-} - \frac{1}{\beta^+} \right), u'(1) = 4 \frac{(1 - \alpha^2)}{\beta^+} \]

rigid connection interface conditions:

\[ [u]_{\alpha} = 0; [u']_{\alpha} = 0; [\beta u'']_{\alpha} = 0; [\beta u''']_{\alpha} = 0; \]

Finite element basis:

\[ N_{w1} = 1 - 3s^2 + 2s^3 \]

\[ N_{\theta1} = (-s + 2s^2 - s^3)h \]

\[ N_{w2} = 3s^2 - 2s^3 \]

\[ N_{\theta2} = (s^2 - s^3)h \]

\textit{Figure 12.10.} Cubic shape functions of plane beam element.
IFEM for bi-harmonic equations

Assume the basis functions of the interface elements are

\[ \phi^-(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]

\[ \phi^+(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \]

Enforce the interface conditions and nodal continuity, to determine the coefficients, one simply need to solve the following linear system:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \\
1 & s & s^2 & s^3 & -1 & -s & -s^2 & -s^3 & \\
0 & \frac{1}{h} & \frac{2s}{h} & \frac{3s^2}{h} & 0 & -\frac{1}{h} & -\frac{2s}{h} & -\frac{3s^2}{h} & \\
0 & 0 & \frac{2\beta^-}{h^2} & \frac{6s\beta^-}{h^2} & 0 & 0 & -\frac{2\beta^+}{h^2} & -\frac{6s\beta^+}{h^2} & \\
0 & 0 & \frac{6\beta^-}{h^3} & 0 & 0 & 0 & -\frac{6\beta^-}{h^3} & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
b_0 \\
b_1 \\
b_2 \\
b_3
\end{bmatrix} =
\begin{bmatrix}
\phi_a \\
\phi'_a \\
\phi_b \\
\phi'_b \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
Results from our code
BVP for the beam equation

\[(\beta(x)u''(x))'' = f(x), \, x \in \Omega = (0,1)\]

\[u(0) = u'(0) = u(1) = u'(1) = 0\]

Consider a model interface problem with the exact solution:

\[u(x) = \begin{cases} 
    a \cos(x) + b \sin(x), & 0 < x < \alpha \\
    c e^x + d x^3 + 5, & \alpha \leq x < 1 
\end{cases}\]

in which the coefficients \(a, b, c, d\) are chosen so that \(u\) satisfies:

\[[u]_\alpha = [u']_\alpha = [\beta u']_\alpha = [(\beta' u'')]_\alpha = 0\]

The interface is \(\alpha = \frac{\pi}{6}\) and the following configurations of the values for \(\beta(x)\) are considered:

- **Case 1:** \(B^- = 2, B^+ = 5\)
- **Case 2:** \(B^- = 2, B^+ = 500\)
- **Case 3:** \(B^- = 2, B^+ = 50000\)
IFEM for a 2d elliptic problem

\[-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega\]

\[u |_{\partial \Omega} = g,\]

\[[u] |_{\Gamma} = 0,\]

\[[\beta u_n] |_{\Gamma} = 0.\]

\[\Omega^- = \{(x, y) : x^2 + y^2 \leq r_0^2\}.\]

Exact solution:

\[u(x, y) = \begin{cases} \frac{r^\alpha}{\beta^-}, & \text{if } r \leq r_0, \\ \frac{r^\alpha}{\beta^-} + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right)r_0^\alpha, & \text{otherwise,} \end{cases}\]
Basis functions

\[ \phi^+ = a_0 + a_1 x + a_2 y \quad \text{for } (x, y) \in T^+ \]
\[ \phi^- = b_0 + b_1 x + b_2 y \quad \text{for } (x, y) \in T^- \]

Interface conditions:

\[ \phi^+(D) = \phi^-(D) \]
\[ \begin{aligned}
\vec{n}(\Phi^{-1})^T \nabla \phi^+ &= \rho \vec{n}(\Phi^{-1})^T \nabla \phi^- \\
\vec{t}(\Phi^{-1})^T \nabla \phi^+ &= \vec{t}(\Phi^{-1})^T \nabla \phi^- ,
\end{aligned} \]

To find the basis functions, one only needs to solve a 3x3 linear system:

\[
\begin{aligned}
(-1 + \hat{y}_1) a_2 - \hat{y}_1 b_2 &= \phi_1 - \phi_3. \\
 m_1 a_1 + m_2 a_2 - \rho m_2 b_2 &= -\rho m_1 \phi_1 + \rho m_1 \phi_2 \\
 m_3 a_1 + m_4 a_2 &= m_3 (\phi_2 - \phi_1) + m_4 b_2.
\end{aligned}
\]

\[
\begin{aligned}
a_0 &= \phi_3 - a_2 \\
 b_0 &= \phi_1 \\
 b_1 &= \phi_2 - \phi_1
\end{aligned}
\]
Error estimation for IFEM

Interpolation error estimation:

**Theorem 3.1** Let $T$ be a triangle in a uniform mesh $\mathcal{S}_h$ and the interface $\bar{\Gamma}$ satisfies the hypothesis $(H1)$, $(H2)$ and $(H3)$. Let $\Gamma_T$ denote the line segment that approximates $\bar{\Gamma}_T$. Let $\phi$ be an arbitrary function in $C^2(T)$ and $\phi_I \in S_h^I(T)$ be the IFE interpolant of $\phi$. The following error estimates hold.

\[
\| \nabla \phi(x, y) - \nabla \phi_I(x, y) \|_{\infty, T} \leq \begin{cases} 
ch \| D^2 \phi \|_{\infty, T} & \text{when } (x, y) \in \Omega \setminus T^* \\
c \| D^2 \phi \|_{\infty, T} & \text{when } (x, y) \in T^*
\end{cases}
\]  \tag{19}

\[
\| \phi(x, y) - \phi_I(x, y) \|_{\infty, T} \leq ch^2 \| D^2 \phi \|_{\infty, T}.
\]  \tag{20}

where $c = O(\max\{1/\rho, \rho\})$ and $T^*$ is the region enclosed by $\bar{\Gamma}_T$ and $\Gamma_T$.

**Theorem 3.2** The following interpolation error estimates hold. For function $\phi \in H^2(\Omega)$, if $\phi$ is a piecewise $C^2$ function on any interface element $\tau$, for all $\tau \in \mathcal{S}_h$, then there exist constants $c_0$ and $c_1$ such that

\[
\| \phi - \phi_I \|_0 < c_0 h^2 \| \phi \|_2
\]  \tag{39}

\[
\| \phi - \phi_I \|_1 < c_1 h \| \phi \|_2,
\]  \tag{40}

where $c_0$ and $c_1$ are $O(\max\{1/\rho, \rho\})$. 

• A priori error estimation: rigorous proof: one needs the 2nd Strange Lemma. For detail, please see


\textbf{Theorem 4.8} Let \( u \in H^2(\Omega), \) \( \hat{u}_h \in S_h(\Omega) \) be the solutions of (2.3) and (4.1) respectively. Then there exists a constant \( C > 0 \) such that

\[
\|u - \hat{u}_h\|_{1,h} \leq C h \|u\|_{H^2(\Omega)}.
\]  

\textbf{Theorem 5.1} Let \( u \in \tilde{H}^2(\Omega), \) \( \hat{u}_h \in \tilde{S}_h(\Omega) \) be the solutions of (2.3) and (4.1) respectively. Then there exists a constant \( C > 0 \) such that

\[
\|u - \hat{u}_h\|_{L^2(\Omega)} \leq C h^2 \|u\|_{\tilde{H}^2(\Omega)}.
\]
Numerical tests for 2\textsuperscript{nd} order elliptic interface problem:

**A priori error check:**

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-1}$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-2}$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{8}$</td>
<td>3.689e-03</td>
<td>3.676e-03</td>
<td>4.164e-03</td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>9.897e-04</td>
<td>9.998e-04</td>
<td>1.110e-03</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>2.700e-04</td>
<td>2.673e-04</td>
<td>3.370e-04</td>
</tr>
<tr>
<td>$\frac{1}{64}$</td>
<td>6.766e-05</td>
<td>6.318e-05</td>
<td>7.567e-05</td>
</tr>
</tbody>
</table>

Error in L2 norm

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-1}$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-2}$</th>
<th>$\frac{\beta^-}{\beta^+} = 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{8}$</td>
<td>1.922e-01</td>
<td>4.677e-01</td>
<td>1.471e-00</td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>8.314e-02</td>
<td>1.439e-01</td>
<td>4.390e-01</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>4.526e-02</td>
<td>8.726e-02</td>
<td>2.686e-01</td>
</tr>
<tr>
<td>$\frac{1}{64}$</td>
<td>2.222e-02</td>
<td>2.942e-02</td>
<td>8.394e-02</td>
</tr>
</tbody>
</table>

Error in H1 norm

Linear regression shows that:

\[
\| u - u^I_h \|_0 \approx 0.25h^{1.97}, \quad \| u - u^I_h \|_0 \approx 0.27h^{2.00} \quad \text{and}, \quad \| u - u^I_h \|_0 \approx 0.28h^{1.96},
\]

\[
\| u - u^I_h \|_{\beta_1} \approx 1.71h^{1.05}, \quad \| u - u^I_h \|_{\beta_2} \approx 6.89h^{1.30}, \quad \text{and} \quad \| u - u^I_h \|_{\beta_3} \approx 6.75h^{1.00}.
\]
• A posteriori estimation: We follow Verfurth’s frame works.

\[ \| u - u_h^I \|_\beta \leq c_p \left\{ \sum \left[ \eta^2 + h^2 \beta^{-1} \| f - f_\tau \|_{0,\tau} \right] \right\}^{1/2}, \]

\[ \eta_\tau = \left\{ h^2 \beta^{-1} \| f_h + \text{div} \beta \nabla u_h^I \|_{0,\tau}^2 + \frac{1}{2} \sum_{e \in \partial \tau} h_e \beta^{-1} \| \beta \left[ \partial_{n_e} u_h^I \right] \|_{0,e}^2 \right\}^{1/2}, \]

\[ \eta_\tau = \left\{ \max \left\{ \rho, \frac{1}{\rho} \right\} \left( \sum_{\tau' \in \{\tau^+, \tau^-\}} h_{\tau'}^2 \beta^{-1} \| f_h + \text{div} \beta \nabla u_h^I \|_{0,\tau'}^2 + \frac{1}{2} \sum_{e' \in \{\partial^+ \tau, \partial^- \tau\}} h_{e'} \beta^{-1} \| \beta_{e'} \left[ \partial_{n_{e'}} u_h^I \right] \|_{0,e'}^2 \right) \right\}^{1/2}, \]


A posteriori error check:

| $|N_h|$ | $\| u - u_h^I \|_\beta$ | $(\sum_{r \in \Omega_h} \eta_r^2)^{1/2}$ |
|-------|----------------|----------------------------------|
| 324   | 1.922e-01      | 4.117e-00                        |
| 557   | 1.338e-01      | 2.316e-00                        |
| 899   | 1.217e-01      | 1.756e-00                        |
| 2516  | 6.281e-02      | 1.054e-00                        |
| 3527  | 6.116e-02      | 7.515e-01                        |
| 10482 | 3.097e-02      | 3.842e-01                        |

$\beta^- = 1, \beta^+ = 10$

\[
\beta^- = 1, \beta^+ = 100
\]
\( \beta^- = 1, \beta^+ = 1000 \)

| \( |N_h| \) | \( \| u - u_h^\beta \| \) | \( \left( \sum_{r \in \Theta_h} \eta_r^2 \right)^{1/2} \) |
|---|---|---|
| 324 | 1.471e-00 | 1.592e+02 |
| 410 | 6.378e-01 | 1.659e+02 |
| 626 | 5.332e-01 | 8.339e-00 |
| 1066 | 2.057e-01 | 3.673e-00 |
| 1923 | 1.251e-01 | 1.572e-00 |
| 4021 | 4.105e-02 | 6.986e-01 |
IFEM for a 2d elastic system

\[
\begin{align*}
\int_{\Omega} \left( \begin{array}{c} \partial_1 v_1 \\ \partial_2 v_2 \\ \partial_2 v_1 + \partial_1 v_2 
\end{array} \right)^T \left( \begin{array}{ccc} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu 
\end{array} \right) \left( \begin{array}{c} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 
\end{array} \right) dx \\
- \int_{\partial\Omega} v_1 (\lambda \nabla \cdot u + 2\mu \partial_1 u_1, \mu (\partial_2 u_1 + \partial_1 u_2)) \cdot n \ ds \\
- \int_{\partial\Omega} v_2 (\mu (\partial_2 u_1 + \partial_1 u_2), \lambda \nabla \cdot u + 2\mu \partial_2 u_2) \cdot n \ ds = 0
\end{align*}
\]

Interface conditions:

\[
\left[ \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) n_1 + 2\mu \frac{\partial u_1}{\partial x_1} n_1 + \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) n_2 \right] \bigg|_{\Gamma} = q_1,
\]

\[
\left[ \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) n_2 + 2\mu \frac{\partial u_2}{\partial x_2} n_2 + \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) n_1 \right] \bigg|_{\Gamma} = q_2,
\]

\[
[u_1]_{\Gamma} = w_1, \quad [u_2]_{\Gamma} = w_2.
\]

Flux-jump conditions

Interface continuity conditions
Assume basis functions $\phi$ and $\psi$ of the displacement vector $(u_1, u_2)$ are both linear in $T^-$ and $T^+$ of the interface element $T$

There are total 12 coefficients of these basis functions to be determined! By solving the linear system given by

- 6 nodal continuity constrains
- 4 interface continuity constrains
- 2 interface jump conditions

One can construct the basis functions of the interface element!

Theorems, implementation and numerical results can be seen in Yan Gong’s PhD thesis 2007 (Thesis advisor: Zhilin Li)

We are working on IFEM for bi-harmonic equation in 2D for the PZT composite diaphragm and the fluid-structure Interface.
Intrusion detection problem

EIT (Electrical Impedance Tomography)

Different materials have different electric conductivity $\sigma$ and permittivity $\varepsilon$

<table>
<thead>
<tr>
<th>tissue</th>
<th>$1/\sigma$ (Ohm-cm)</th>
<th>$\varepsilon$ ($\mu$F/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>lung</td>
<td>950</td>
<td>0.22</td>
</tr>
<tr>
<td>muscle</td>
<td>760</td>
<td>0.49</td>
</tr>
<tr>
<td>liver</td>
<td>685</td>
<td>0.49</td>
</tr>
<tr>
<td>heart</td>
<td>600</td>
<td>0.88</td>
</tr>
<tr>
<td>fat</td>
<td>$&gt; 1000$</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Table 1: Electrical properties of biological tissue measured at frequency 10kHz [10, 131]

<table>
<thead>
<tr>
<th>rock or fluid</th>
<th>$1/\sigma$ (Ohm-cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>marine sand, shale</td>
<td>1 - 10</td>
</tr>
<tr>
<td>terrestrial sands, claystone</td>
<td>15 - 50</td>
</tr>
<tr>
<td>volcanic rocks, basalt</td>
<td>10 - 200</td>
</tr>
<tr>
<td>granite</td>
<td>500 - 2000</td>
</tr>
<tr>
<td>limestone dolomite, anhydrite</td>
<td>50 - 5000</td>
</tr>
<tr>
<td>chloride water from oil fields</td>
<td>0.16</td>
</tr>
<tr>
<td>sulfate water from oil fields</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Table 2: Resistivity of rocks and fluids [99]
EIT is the inverse problem of determining the impedance in the interior of a domain, given simultaneous measurements of the direct or alternating electric currents and voltages at the boundary of the domain.

Some good introductions can be found in


Mathematical Model

\[(*)\] \[
\begin{align*}
\nabla \cdot \left[ \gamma(x, \omega) \nabla \phi(x, \omega) \right] &= 0 \text{ in } \Omega \\
\phi(x, \omega) &= V(x, \omega) \text{ on } \partial \Omega 
\end{align*}
\]

Here \( \phi \) is the electrical potential and \( \gamma(x, \omega) \) is the admittivity.

Q: How to reconstruct \( \gamma(x, \omega) \) by measuring the current \( I = \frac{\partial \phi}{\partial n} \)?
Maxwell Equation:

\[ \nabla \times E(x, \omega) = -i\omega \mu(x) H(x, \omega), \text{ here } \mu \text{ is the magnetic permeability} \]

\[ \omega \mu [\sigma][x]^2 \ll 1 \text{ (typical parameter } \omega=28.8 \text{ KHZ, } [\sigma] \leq 1, \text{ and } [x] \leq 1) \]

\[ E = -\nabla \phi(x, \omega) \]

\[ \nabla \times H(x, \omega) = \left[ \sigma(x) + i\omega \varepsilon(x) \right] E(x, \omega) \]

\[ \nabla \cdot \left( \nabla \times H(x, \omega) \right) = \nabla \left( -\gamma \nabla \phi(x, \omega) \right) = 0 \]

Apply current \( J_b \) on the boundary

\[ \nabla \cdot ( -\gamma \nabla \phi) = 0 \quad \Rightarrow \quad \int_{\partial \Omega} \gamma \frac{\partial \phi}{\partial n} \, ds = \int_{\partial \Omega} J_b \, ds \]
Theorems

Theorem 1

Consider $U_{N,t,h} = \chi_{N,t} e^{-1/t} U_{N,h}$ here $U_{N,h} = e^{C_N X^N}$, $X^N = \varphi(x) + i \psi(x)$ is a complex conformal mapping (the complex geometrical optical solution), and $\chi_{N,t}$ is the cut off function on the conic region contains level curves of $\varphi(x) = \frac{1}{t}$.

Consider $W_{N,t,h}$ be the solution of (*) with Dirichlet data $U_{N,t,h}$ on $\partial \Omega$. There exists constants $c$ and $\varepsilon$ such that

$$\|U_{N,t,h} - W_{N,t,h}\|_2 < ce^{-\varepsilon}$$

for $h \leq 1$. 
Probing Level Curves

Figure 4.1. Some level curves of $\phi_N$.

A probing sequence
Theorem 2

Assume for any \( p \in \partial D \), there exist \( B_\varepsilon (p) \) such that \( \delta \gamma > \varepsilon \), here \( \delta \gamma = \gamma_D - \gamma_\Omega > 0 \) and \( \gamma = \gamma_\Omega + \chi_D \delta \gamma \).

Let \( \Lambda_0 : V \to \gamma_\Omega \frac{\partial \phi}{\partial n} \) be the associated (DtN map) without intrusion and \( \Lambda_D : V \to \gamma \frac{\partial \phi}{\partial n} \) be the DtN map with intrusion \( D \).

The following inequalities hold.

\[
\int_{\partial \Omega} (\Lambda_D - \Lambda_0) \overline{V} \cdot V \, ds \leq \int_{D} \delta \gamma |\nabla \phi|^2 \, dx
\]

\[
\int_{\partial \Omega} (\Lambda_D - \Lambda_0) \overline{V} \cdot V \, ds \geq \int_{D} \frac{\gamma_\Omega \delta \gamma}{\gamma_D} |\nabla \phi|^2 \, dx
\]
Theorem 3

Let $\ell_t$ be the level curve of $\varphi(x) = \frac{1}{t}$ and

$$
E(N, t, h) = \int_{\partial \Omega} (\Lambda_D - \Lambda_0) \vec{V}_{N,t,h} \cdot V_{N,t,h} \, ds.
$$

We have

(i) if $\ell_t \cap \overline{D} = \emptyset$, then $\exists \varepsilon_1$ and $h_1$, $E(N, t, h) \prec e^{-\varepsilon_1/h}$ for all $h \geq h_1$

(ii) if $\ell_t \cap \overline{D} \neq \emptyset$, then $\exists \varepsilon_2$ and $h_2$, $E(N, t, h) \succ e^{-\varepsilon_2/h}$ for all $h \geq h_2$

Remarks:

1. The constant $c$ in Theorem 1 depends on the phase angle of the CGO solution. The estimations in Theorem 1 and 3 are sharp only for small phase angles.
Numerical results
Reconstruction in Elastic System

Mathematical Model:

\[
\begin{align*}
\nabla \cdot \left( (\lambda \nabla u) I + 2\mu S(\nabla u) \right) &= 0, \\
\left. u \right|_{\partial \Omega} &= g,
\end{align*}
\]

here \( S(A) = \frac{A + A^T}{2} \) and \( \sigma(u) = (\lambda \nabla u) I + 2\mu S(\nabla u) \)

is the stress tensor

\( \text{DtN map: } \Lambda_D : g \rightarrow \sigma(u) \cdot \bar{n} \)

Energy quantity:

\[
E(N, t, h) = \int_{\partial \Omega} (\Lambda_D - \Lambda_0) \bar{g}_{N,t,h} \cdot g_{N,t,h} \, ds.
\]
Through direct computing, the weak form of the system PDE can be rewritten as

$$\int_{\Omega} \left[ \frac{\partial v_1}{\partial x} \frac{\partial v_2}{\partial y} + \frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right] C \left[ \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} + \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right]^T \, dx - $$

$$\int_{\partial \Omega} \left[ \begin{array}{cc} \lambda \nabla \cdot u + 2\mu \frac{\partial u_1}{\partial x} & \mu \left( \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\
\mu \left( \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & \lambda \nabla \cdot u + 2\mu \frac{\partial u_2}{\partial y} \end{array} \right] \cdot \vec{n} \, ds, \text{ here,}$$

$$C = \left[ \begin{array}{ccc} \lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu \end{array} \right]$$

is the stress-strain relationship and, \(\lambda \nabla \cdot u + 2\mu \frac{\partial u_1}{\partial x}\) and \(\mu \left( \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)\) are the normal stress and shear stress.
Numerical Studies

Consider $\lambda$ and $\mu$ are constant, here $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$

where $(E, \nu) = \begin{cases} (6 \times 10^6, 0.45) & \text{in } \Omega \setminus D \\ (6 \times 10^7, 0.45) & \text{in } D \end{cases}$

The special solution can be given by

$$u_{N,t,h} = \chi_{N,t} e^{\frac{-1}{t}} U_{N,h}$$

here $U_{N,h} = e^{C_N X^N} \nabla X^N, X^N = \phi(x) + i \psi(x)$

Displacement field near the boundary
Figure 4.3. The first column represents the actual location of the inclusion. The second column is the numerical reconstruction with noiseless simulated data. The third column is the numerical reconstruction with noisy data with $A = 0.01\%$. All gray areas are inclusion-free regions.

Figure 4.5. The first column is the actual location of the inclusion. The second column is the numerical reconstruction with noiseless simulated data when $\Omega$ is a rectangle. The third column is the numerical reconstruction with noiseless simulated data when $\Omega$ is a strip. All gray areas are inclusion-free regions.
Conclusion

More works to be done!!!

• Immersed finite element method for PZT Interface and fluid-structure interface
• Accurate elliptic solver on evolving surface
• Intrusion and crack detection for PZT
• Fracture and fatigue analysis of PZT membrane
• Resonance of fluid and composite structure
Thanks for your attention