Solutions to Exercises on Le Gall’s Book: Brownian Motion, Martingales, and Stochastic Calculus

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Chapter 1
Gaussian Variables and Gaussian Processes

1.1 Exercise 1.15

Let \((X_t)_{t \in [0,1]}\) be a centered Gaussian process. We assume that the mapping \((t, w) \mapsto X_t(w)\) from \([0,1] \times \Omega\) into \(\mathbb{R}\) is measurable. We denote the covariance function of \(X\) by \(K(u, v)\).

1. Show that the mapping \(t \mapsto X_t\) from \([0,1]\) into \(L^2(\Omega)\) is continuous if and only if \(K(u, v)\) is continuous on \([0,1]^2\). In what follows, we assume that this condition holds.

2. Let \(h : [0,1] \rightarrow \mathbb{R}\) be a measurable function such that
\[
\int_0^1 |h(t)| \sqrt{K(t,t)} dt < \infty.
\]
Show that the integral, for a.e., the integral
\[
\int_0^1 h(t)X_t(w) dt
\]
is absolutely integral. We set \(Z(w) = \int_0^1 h(t)X_t(w) dt\).

3. We now make the stronger assumption
\[
\int_0^1 |h(t)| dt < \infty.
\]
Show that \(Z\) is the \(L^2\) limit of the variables
\[
Z_n = \sum_{i=1}^n X_{\frac{i}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} h(t) dt
\]
when \(n \rightarrow \infty\) and infer that \(Z\) is a Gaussian random variable.

4. We assume that \(K(u, v)\) is twice continuously differentiable. Show that, for every \(t \in [0,1]\), the limit
\[
\tilde{X}_t = \lim_{s \rightarrow t} \frac{X_s - X_t}{s - t}
\]
exists in \(L^2\). Verify that \((\tilde{X}_t)_{t \in [0,1]}\) is a centered Gaussian process and compute its covariance function.

Proof.

1. First, we assume that \(K(u, v)\) is continuous. Note that
\[
\|X_{t+h} - X_t\|_{L^2(\Omega)}^2 = E[|X_{t+h} - X_t|^2] = K(t+h, t+h) - 2K(t+h, t) + K(t, t).
\]
By letting \(h \downarrow 0\), we see that the mapping \(t \mapsto X_t\) is continuous.

Conversely, we assume that the mapping \(t \mapsto X_t\) is continuous. By using Cauchy Schwarz inequality, we get
\[
|K(u + t, v + s) - K(u, v)|
\leq |K(u + t, v + s) - K(u, v + s)| + |K(u, v + s) - K(u, v)|
= E[||(X_{u+t} - X_u)X_{v+s}|| + E[||X_{v+s} - X_v||L^2||X_u||L^2]]
\]
Since \(||X_{v+s}||L^2\) is bounded for small \(s\), we see that \(K(u, v)\) is continuous.
2. It’s clear that
\[
\int_0^1 \int_0^1 |X_t(w)||h(t)|dt \mathcal{P}(dw)
\]
\[
= \int_0^1 \int_0^1 |X_t(w)||h(t)| \mathcal{P}(dw)dt
\]
\[
= \int_0^1 ||X_t||_{L^1} |h(t)|dt
\]
\[
\leq \int_0^1 ||X_t||_{L^2} |h(t)|dt
\]
\[
= \int_0^1 \sqrt{K(t,t)} |h(t)|dt < \infty
\]
Thus, the integral, for a.e., the integral
\[
\int_0^1 h(t)X_t(w)dt
\]
is absolutely integral.

3. It suffices to show that $Z_n \to Z$ in $L^2$. Indeed, since $\{Z_n\}_{n \geq 1}$ are Gaussian random variables and $Z_n \to Z$ in $L^2$, we see that $Z$ is a Gaussian random variable. Note that
\[
Z_n(w) = \int_0^1 \sum_{i=1}^n X_{\frac{n}{i+1}}(w)1_{[\frac{i}{n}, \frac{i+1}{n})}(t)h(t)dt.
\]
Thus,
\[
E[|Z - Z_n|^2]^{\frac{1}{2}}
\]
\[
= \left( \int \left| \int_0^1 h(t)(X_t(w) - \sum_{i=1}^n X_{\frac{n}{i+1}}(w)1_{[\frac{i}{n}, \frac{i+1}{n})}(t))dt \right|^2 \mathcal{P}(dw) \right)^{\frac{1}{2}}
\]
\[
\leq \int_0^1 \left( \int \left|h(t)\right|^2 |(X_t(w) - \sum_{i=1}^n X_{\frac{n}{i+1}}(w)1_{[\frac{i}{n}, \frac{i+1}{n})}(t))|^2 \mathcal{P}(dw) \right)^{\frac{1}{2}}dt
\]
\[
= \int_0^1 |h(t)| \left( \int \left| (X_t(w) - \sum_{i=1}^n X_{\frac{n}{i+1}}1_{[\frac{i}{n}, \frac{i+1}{n})}(t)) \right|^2 \mathcal{P}(dw) \right)^{\frac{1}{2}}dt
\]
\[
= \int_0^1 |h(t)| \times \left\| (X_t - \sum_{i=1}^n X_{\frac{n}{i+1}}1_{[\frac{i}{n}, \frac{i+1}{n})}(t)) \right\|_{L^2} dt.
\]
For each $t \in [0, 1]$ and $n \geq 1$ such that $\frac{k-1}{n} \leq t < \frac{k}{n}$, we get
\[
\left\| (X_t - \sum_{i=1}^n X_{\frac{n}{i+1}}1_{[\frac{i}{n}, \frac{i+1}{n})}(t)) \right\|_{L^2} = \left\| X_t - X_{\frac{k}{n}} \right\|_{L^2} \leq \left\| X_t \right\|_{L^2} + \left\| X_{\frac{k}{n}} \right\|_{L^2} \leq 2 \sup_{t \in [0,1]} \sqrt{K(t,t)} < \infty.
\]
and therefore
\[
|h(t)| \times \left\| (X_t - \sum_{i=1}^n X_{\frac{n}{i+1}}1_{[\frac{i}{n}, \frac{i+1}{n})}(t)) \right\|_{L^2} \leq C|h(t)|
\]
for each $t \in [0, 1)$ and some $0 < C < \infty$.

Fix $t \in [0, 1)$. Choose $\{k_n\}$ such that $\frac{k_n-1}{n} \leq t < \frac{k_n}{n}$ for each $n \geq 1$. Since $t \mapsto X_t$ is continuous, we have
\[
\left\| (X_t - \sum_{i=1}^n X_{\frac{n}{i+1}}1_{[\frac{i}{n}, \frac{i+1}{n})}(t)) \right\|_{L^2} = \left\| X_t - X_{\frac{k_n}{n}} \right\|_{L^2} \to 0 \text{ as } n \to \infty.
\]
By using dominated convergence theorem, we have
\[
\limsup_{n \to \infty} E[|Z - Z_n|^2]^{\frac{1}{2}} \leq \lim_{n \to \infty} \int_0^1 |h(t)| \times ||(X_t - \sum_{i=1}^n X_{\frac{t}{n}}1_{\frac{i-1}{n}, \frac{i}{n}})(t)||_{L^2} dt = 0
\]
and, hence, \( Z_n \to Z \) in \( L^2 \).

4. To show that \( \lim_{s \to t} \frac{X_s - X_t}{s-t} \) exists in \( L^2 \), it suffices to show that
\[
||\frac{X_{t+h_1} - X_t}{h_1} - \frac{X_{t+h_2} - X_t}{h_2}||_{L^2} \to 0 \text{ as } h_1, h_2 \to 0.
\]
Note that
\[
||\frac{X_{t+h_1} - X_t}{h_1} - \frac{X_{t+h_2} - X_t}{h_2}||_{L^2}^2 = A + B - 2C,
\]
where
\[
A = \frac{1}{|h_1|^2} E[(X_{t+h_1} - X_t)^2] = \frac{1}{|h_1|^2} (E[X_{t+h_1}^2] + E[X_t^2] - 2E[X_{t+h_1}X_t]),
\]
\[
B = \frac{1}{|h_2|^2} E[(X_{t+h_2} - X_t)^2] = \frac{1}{|h_2|^2} (E[X_{t+h_2}^2] + E[X_t^2] - 2E[X_{t+h_2}X_t]),
\]
and
\[
C = \frac{1}{|h_1||h_2|} E[(X_{t+h_2} - X_t)(X_{t+h_1} - X_t)]
\]
\[
= \frac{1}{|h_2||h_1|} (E[X_{t+h_2}X_{t+h_1}] + E[X_t^2] - E[X_{t+h_2}X_t] - E[X_{t+h_1}X_t]).
\]
First, we show that \( C \to \frac{\partial^2 K}{\partial u \partial v}(t,t) \) as \( h_1, h_2 \to 0 \). Without loss of generality, we may suppose \( h_1, h_2 > 0 \). Set
\[
g(z) = K(t+h_1, z) - K(t, z).
\]
Then
\[
C = \frac{1}{h_1 h_2} (g(t+h_2) - g(t)).
\]
Since \( K \in C^2([0,1]^2) \), there exist \( t_1^*, t_2^* \) such that
\[
C = \frac{1}{h_1} g'(t_2^*) = \frac{1}{h_1} \left( \frac{\partial K(t+h_1, t_2^*)}{\partial v} - \frac{\partial K(t, t_2^*)}{\partial v} \right) = \frac{\partial^2 K(t_1^*, t_2^*)}{\partial u \partial v}
\]
By using the continuity of \( \frac{\partial^2 K}{\partial u \partial v} \), we see that \( C \to \frac{\partial^2 K}{\partial u \partial v}(t,t) \) as \( h_1, h_2 \to 0 \).
Similarly, we have \( A \to \frac{\partial^2 K}{\partial u \partial v}(t,t) \) and \( B \to \frac{\partial^2 K}{\partial u \partial v}(t,t) \) as \( h_1, h_2 \to 0 \). Therefore,
\[
||\frac{X_{t+h_1} - X_t}{h_1} - \frac{X_{t+h_2} - X_t}{h_2}||_{L^2} \to 0 \text{ as } h_1, h_2 \to 0
\]
and, hence, \( \lim_{s \to t} \frac{X_s - X_t}{s-t} \) exists in \( L^2 \). Since \( \frac{X_s - X_t}{s-t} \) is a centered Gaussian random variable for all \( s \neq t \), we see that \( \tilde{X}_t \equiv \lim_{s \to t} \frac{X_s - X_t}{s-t} \) is a centered Gaussian random variable. Moreover, since any linear combination
\[
\sum_{k=1}^n c_k \frac{X_{s_k} - X_{t_k}}{s_k - t_k}
\]
is a centered Gaussian random, we see that \( (\tilde{X}_t)_{t \in [0,1]} \) is a centered Gaussian process.
Finally, we show that
\[
\tilde{K}(t, s) = \frac{\partial^2 K}{\partial u \partial v}(t, s),
\]
where $\tilde{K}(t, s)$ is the covariance function of $(\tilde{X}_t)_{t \in [0, 1]}$. By using similar argument as in (3), there exist $t_h, s_h$ such that
\[
E\left[\frac{X_{t+h} - X_t}{h} \frac{X_{s+h} - X_s}{h}\right] = \frac{\partial^2 K}{\partial u \partial v}(t_h, s_h)
\]
for each $h \neq 0$ and $t_h \to t$ and $s_h \to s$ as $h \to 0$. Since $K(u, v) \in C^2([0, 1]^2)$, there exist $0 < C < \infty$ such that
\[
|E\left[\frac{X_{t+h} - X_t}{h} \frac{X_{s+h} - X_s}{h}\right]| = \left|\frac{\partial^2 K}{\partial u \partial v}(t_h, s_h)\right| \leq C
\]
for all $h \neq 0$. By using dominated convergence theorem and the continuity of $\frac{\partial^2 K}{\partial u \partial v}$, we have
\[
\tilde{K}(t, s) = E[\tilde{X}_t \tilde{X}_s] = \lim_{h \to 0} E\left[\frac{X_{t+h} - X_t}{h} \frac{X_{s+h} - X_s}{h}\right] = \lim_{h \to 0} \frac{\partial^2 K}{\partial u \partial v}(t_h, s_h) = \frac{\partial^2 K}{\partial u \partial v}(t, s).
\]

### 1.2 Exercise 1.16 (Kalman filtering)

Let $(\epsilon_n)_{n \geq 0}$ and $(\eta_n)_{n \geq 0}$ be two independent sequences of independent Gaussian random variables such that, for every $n$, $\epsilon_n$ is distributed according to $N(0, \sigma^2)$ and $\eta_n$ is distributed according to $N(0, \delta^2)$, where $\sigma > 0$ and $\delta > 0$. We consider two other sequences $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ defined by the properties
\[
X_0 = 0, \quad X_{n+1} = a_n X_n + \epsilon_{n+1} \quad \text{and} \quad Y_n = c X_n + \eta_n,
\]
where $c$ and $a_n$ are positive constants. We set
\[
\hat{X}_{n/n} = E[X_n | Y_0, ..., Y_n]
\]
and
\[
\hat{X}_{n+1/n} = E[X_{n+1} | Y_0, ..., Y_n].
\]
The goal of the exercise is to find a recursive formula allowing one to compute these conditional expectations.

1. Verify that $\hat{X}_{n+1/n} = a_n \hat{X}_{n/n}$, for every $n \geq 0$.
2. Show that, for every $n \geq 1$,
\[
\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{E[X_n Z_n]}{E[Z_n]} Z_n,
\]
where $Z_n = Y_n - c \hat{X}_{n/n-1}$.
3. Evaluate $E[X_n Z_n]$ and $E[Z_n^2]$ in terms of $P_n \equiv E[(X_n - \hat{X}_{n/n-1})^2]$ and infer that, for every $n \geq 1$,
\[
\hat{X}_{n+1/n} = a_n (\hat{X}_{n/n-1} + \frac{c P_n}{c^2 P_n + \delta^2} Z_n)
\]
4. Verify that $P_1 = \sigma^2$ and that, for every $n \geq 1$, the following induction formula holds:
\[
P_{n+1} = \sigma^2 + a_n^2 \frac{\delta^2 P_n}{c^2 P_n + \delta^2}.
\]

**Proof.**
1. By observing the construction of $X_n$ and $Y_n$, we see that $Y_0 = \eta_0$ and for every $n \geq 1$, $X_n$ is a $\sigma(\epsilon_k, k = 0, \ldots, n)$-measurable centered Gaussian random variable and $Y_n$ is a $\sigma(\eta_k, \epsilon_k, k = 0, \ldots, n)$-measurable centered Gaussian random variable. Since $\sigma(Y_0) = \sigma(\eta_0)$ and for each $n \geq 1$, $\sigma(Y_0, \ldots, Y_n) \subseteq \sigma(\epsilon_k, \eta_k, k = 0, \ldots, n)$, we have

$$
\hat{X}_{n+1/n} = E[X_{n+1}|Y_0, \ldots, Y_n] = a_n E[X_n|Y_0, \ldots, Y_n] + E[\epsilon_{n+1}|Y_0, \ldots, Y_n] = a_n \hat{X}_{n/n} + E[\epsilon_{n+1}]
$$

and therefore

$$
\hat{X}_{n/n} = a_n \hat{X}_{n/n}.
$$

2. Given $n \geq 1$. Set $K_n = \text{span}\{Y_0, \ldots, Y_n\}$. Then, for each centered Gaussian random variable $X \in L^2(\Omega, \mathcal{F}, P)$,

$$
E[X|Y_0, \ldots, Y_n] = p_{K_n}(X),
$$

where $p_{K_n}$ is the orthogonal projection onto $K_n$ in the Hilbert space $L^2(\Omega, \mathcal{F}, P)$. Observe that

$$
Z_n = Y_n - c \hat{X}_{n/n-1}
= Y_n - c E[X_n|Y_0, \ldots, Y_{n-1}]
= Y_n + E[\eta_n - Y_n|Y_0, \ldots, Y_{n-1}]
= Y_n + E[\eta_n] - E[Y_n|Y_0, \ldots, Y_{n-1}]
= Y_n - P_{K_{n-1}}(Y_n)
$$

Set $V_n = \text{span}\{Z_n\}$. Then $K_n = \text{span}\{Y_0, \ldots, Y_{n-1}, Z_n\} = K_{n-1} \oplus V_n$. Thus,

$$
\hat{X}_{n/n} = E[X_n|Y_0, \ldots, Y_n] = p_{K_n}(X_n) = p_{K_{n-1}}(X_n) + p_{V_n}(X_n)
= E[X_n|Y_0, \ldots, Y_{n-1}] + E[Z_n|\|Z_n\|_{L^2(\Omega)} > L^2(\Omega)]
= \hat{X}_{n-1/n} + \frac{E[X_n Z_n]}{\|Z_n\|_{L^2(\Omega)}} Z_n
$$

3. First, we show that

$$
E[Z_n^2] = c^2 P_n + \delta^2.
$$

Note that

$$
E[Z_n^2] = E[(Y_n - c \hat{X}_{n/n-1})^2]
= E[(Y_n - cX_n + cX_n - c \hat{X}_{n/n-1})^2]
= E[(\eta_n + cX_n - c \hat{X}_{n/n-1})^2]
= c^2 P_n + E[\eta_n^2] + 2c E[\eta_n(X_n - \hat{X}_{n/n-1})]
= c^2 P_n + \delta^2 + 2c E[\eta_n(X_n - \hat{X}_{n/n-1})]
$$

Since $X_n$ is $\sigma(\epsilon_k, k = 0, \ldots, n)$-measurable, $\hat{X}_{n/n-1}$ is $\sigma(Y_k, k = 0, \ldots, n-1)$-measurable, and $\sigma(Y_k, k = 0, \ldots, n-1) \subseteq \sigma(\eta_k, \epsilon_k, k = 0, \ldots, n-1)$, we see that

$$
E[\eta_n(X_n - \hat{X}_{n/n-1})] = E[\eta_n] E[X_n - \hat{X}_{n/n-1}] = 0
$$

and therefore

$$
E[Z_n^2] = c^2 P_n + \delta^2.
$$
Next, we show that

\[ E[X_n Z_n] = cP_n. \]

Observe that

\[
E[\hat{X}_{n-1} (X_n - \hat{X}_{n-1})]
= E[p_{K_{n-1}}(X_n) (X_n - p_{K_{n-1}}(X_n))].
\]

Since \( X_n \) is \( \sigma(\epsilon_k, k = 0, \ldots, n) \)-measurable, we have \( E[X_n \eta_n] = 0 \) and therefore

\[
E[X_n Z_n] = E[X_n (Y_n - c\hat{X}_{n-1})]
= E[X_n (Y_n - cX_n + cX_n - c\hat{X}_{n-1})]
= E[X_n (\eta_n + cX_n - c\hat{X}_{n-1})]
= cE[X_n (X_n - \hat{X}_{n-1})]
= cE[X_n (X_n - \hat{X}_{n-1}) - cE[\hat{X}_{n-1}(X_n - \hat{X}_{n-1})]]
= cE[X_n Z_n].
\]

Finally, we have

\[
\hat{X}_{n+1/n} = a_n \hat{X}_{n/n}
= a_n (\hat{X}_{n-1/n} + \frac{E[X_n Z_n]}{E[Z_n^2]} Z_n)
= a_n (\hat{X}_{n-1/n} + \frac{cP_n}{c^2 P_n + \sigma^2} Z_n).
\]

4. Note that

\[
P_1 = E[(X_1 - E[X_1 | \eta_0]^2] = E[(\epsilon_1 - E[\epsilon_1 | \eta_0])^2] = E[(\epsilon_1 - E[\epsilon_1])^2] = \sigma^2
\]

and

\[
P_{n+1} = E[(X_{n+1} - \hat{X}_{n+1/n})^2]
= E[(a_n X_n + \epsilon_{n+1} - a_n \hat{X}_{n/n})^2]
= E[(\epsilon_{n+1} - a_n (X_n - \hat{X}_{n/n}))^2]
= E[\epsilon_{n+1}^2] + a_n^2 E[(X_n - \hat{X}_{n/n})^2] - 2a_n E[\epsilon_{n+1}(X_n - \hat{X}_{n/n})]
\]

Since \( X_n \) is \( \sigma(\epsilon_k, k = 0, \ldots, n) \)-measurable, \( \hat{X}_{n/n} \) is \( \sigma(Y_k, k = 0, \ldots, n) \)-measurable, and \( \sigma(Y_k, k = 0, \ldots, n) \subseteq \sigma(\eta_k, \epsilon_k, k = 0, \ldots, n) \), we see that

\[ E[\epsilon_{n+1}(X_n - \hat{X}_{n/n})] = 0 \]

and therefore

\[
P_{n+1} = E[\epsilon_{n+1}^2] + a_n^2 E[(X_n - \hat{X}_{n/n})^2] = \sigma^2 + a_n^2 E[(X_n - \hat{X}_{n/n})^2].
\]
Because $Z_n$ and $\hat{X}_{n/2}$ are orthogonal and $Z_n$ is centered Gaussian, we get $E[Z_n \hat{X}_{n/2}] = 0$ and, hence,

$$
P_{n+1} = \sigma^2 + \alpha_n^2 E[(X_n - \hat{X}_{n/2})^2]
= \sigma^2 + \alpha_n^2 E[(X_n - \hat{X}_{n/2} + \hat{X}_{n/2} - \hat{X}_{n/2})^2]
= \sigma^2 + \alpha_n^2 E[(X_n - \hat{X}_{n/2} - \frac{E[Z_n]}{E[Z_n^2]} Z_n)^2]
= \sigma^2 + \alpha_n^2 (p_n + (\frac{E[X_n Z_n]}{E[Z_n^2]})^2 E[Z_n^2] - 2 \frac{E[X_n Z_n]}{E[Z_n^2]} E[Z_n(X_n - \hat{X}_{n/2}))])
= \sigma^2 + \alpha_n^2 (p_n - \frac{E[X_n Z_n]}{E[Z_n^2]})^2
= \sigma^2 + \alpha_n^2 (p_n - \frac{c^2 P_n}{c^2 P_n + \delta^2})
= \sigma^2 + \alpha_n^2 \frac{\delta^2 P_n}{c^2 P_n + \delta^2}
$$

1.3 Exercise 1.17

Let $H$ be a (centered) Gaussian space and let $H_1$ and $H_2$ be linear subspaces of $H$. Let $K$ be a closed linear subspace of $H$. We write $p_K$ for the orthogonal projection onto $K$. Show that the condition

$$\forall X_1 \in H_1, \forall X_2 \in H_2, \ E[X_1 X_2] = E[p_K(X_1)p_K(X_2)] \tag{1}$$

implies that the $\sigma$-fields $\sigma(H_1)$ and $\sigma(H_2)$ are conditionally independent given $\sigma(K)$. (This means that, for every nonnegative $\sigma(H_1)$-measurable random variable $X_1$, and for every nonnegative $\sigma(H_2)$-measurable random variable $X_2$, one has

$$E[X_1 X_2 | \sigma(K)] = E[X_1 | \sigma(K)] E[X_2 | \sigma(K)] \tag{2}$$

Hint: Via monotone class arguments explained in Appendix A1, it is enough to consider the case where $X_1$, resp. $X_2$, is the indicator function of an event depending only on finitely many variables in $H_1$, resp. in $H_2$.

Proof.

To show (2), it suffices to show that

$$E[1_{\{X_1 \in \Gamma_1\} \cap \{X_2 \in \Gamma_2\} | \sigma(K)] = E[1_{\{X_1 \in \Gamma_1\}} | \sigma(K)] E[1_{\{X_2 \in \Gamma_2\}} | \sigma(K)] \tag{3}$$

for each $\Gamma_1 \in \mathcal{B}_{\mathbb{R}}$, $\mathcal{B}_s \subset \mathcal{B}_{\mathbb{R}}$, $s = 1, 2$.

Let $\{X^*_i : i = 1, 2, ..., m_s\}$ be an orthonormal basis of linear subspace space $M_s$ of $L^2$ spanned by $\{X^*_i : i = 1, 2, ..., m_s\}$. Then $\{Z_s, Z^*_s, ..., Z^*_m_s\} \subset H_s$ are independent centered Gaussians. To show (3), it suffices to show that

$$E[1_{\{Z_1 \in \Gamma_1\} \cap \{Z_2 \in \Gamma_2\} | \sigma(K)] = E[1_{\{Z_1 \in \Gamma_1\}} | \sigma(K)] E[1_{\{Z_2 \in \Gamma_2\}} | \sigma(K)] \tag{4}$$

for each $\Gamma_1 \in \mathcal{B}_{\mathbb{R}}$. Indeed, by the theorem of monotone class, we get

$$E[1_{\{E_1\}} | \sigma(K)] = E[1_{\{E_1\}} | \sigma(K)] E[1_{\{E_2\}} | \sigma(K)] \ \forall E_1 \in \sigma(M_s) \text{ and } s = 1, 2.$$
and so
\[
E[1_{\{X_1^i \in r_1^i\}}...1_{\{X_{i1}^i \in r_{i1}^i\}} \times 1_{\{X_{j1}^j \in r_{j1}^j\}}...1_{\{X_{m2}^j \in r_{m2}^j\}} | \sigma(K)]
= E[1_{\{X_1^i \in r_1^i\}}...1_{\{X_{i1}^i \in r_{i1}^i\}} | \sigma(K)] \times E[1_{\{X_{j1}^j \in r_{j1}^j\}}...1_{\{X_{m2}^j \in r_{m2}^j\}} | \sigma(K)]
\]
for each $\Gamma_i^a \in B_r$.

By independence of $\{Z_1^i, Z_2^i, ..., Z_{m_a}^i\}$, we have
\[
E[(Z_s^i - p_K(Z_s^i))(Z_j^j - p_K(Z_j^j))] = 0 \quad \forall i \neq j, \forall s = 1, 2.
\] (5)

By (1) and Corollary 1.10, we get
\[
E[(Z_s^i - p_K(Z_s^i))(Z_j^j - p_K(Z_j^j))]
= E[Z_1^i Z_2^j] + E[p_K(Z_1^i)p_K(Z_2^j)] - E[Z_1^ip_K(Z_2^j)] - E[p_K(Z_1^i)Z_2^j]
= E[p_K(Z_1^i)p_K(Z_2^j)] + E[p_K(Z_1^i)p_K(Z_2^j)] - E[E[Z_1^i]p_K(Z_2^j)] - E[p_K(Z_1^i)E[Z_2^j]\sigma(K)]
\]
and
\[
P(Z_i^* \in \Gamma_i^a | \sigma(K)) = \frac{1}{\sigma^a_i \sqrt{2\pi}} \int_{\Gamma_i^a} \exp\left(-\frac{(-y - p_K(Z_i^*))^2}{2(\sigma^a_i)^2}\right)dy,
\]
where $(\sigma^a_i)^2 = E[(Z_s^i - p_K(Z_s^i))^2]$. Set
\[
Y_i^* = Z_i^* - p_K(Z_i^*).
\]

By (5) and (6), $\{Y_i^*: s = 1, 2 \text{ and } i = 1, 2, ..., m_s\}$ are independent centered Gaussians. Set
\[
F(z_{11}, ..., z_{s1}, z_{12}, ..., z_{s2}) = 1_{\{r_1^i \subseteq Z_i^*\}} 1_{\{r_{i1} \subseteq Z_i^*\}} 1_{\{r_{i2} \subseteq Z_i^*\}}
\]
Since $\{Y_i^*: s = 1, 2 \text{ and } i = 1, 2, ..., n_s\}$ is independent of $\sigma(K)$, we get
\[
E[1_{\{Z_i^* \in r_i^a\}}...1_{\{Z_{i1}^* \in r_{i1}^a\}} \times 1_{\{Z_{j1}^* \in r_{j1}^a\}}...1_{\{Z_{m2}^j \in r_{m2}^j\}} | \sigma(K)]
= E[F(Z_1^*, ..., Z_{m1}^*, Z^2, ..., Z_{m2}^j) | \sigma(K)]
= E[F(Y_1^1 + p_K(Z_1^1), ..., Y_{m1}^1 + p_K(Z_{m1}^1), Y_1^2 + p_K(Z_1^2), ..., Y_{m2}^j + p_K(Z_{m2}^j)) | \sigma(K)]
= \int F(y_1^1 + p_K(Z_1^1), ..., y_{m1}^1 + p_K(Z_{m1}^1), y_1^2 + p_K(Z_1^2), ..., y_{m2}^j + p_K(Z_{m2}^j))
\]
and
\[
P_{Y_1^1}...P_{Y_{m1}^1}(dy_1^1) \times ... \times dy_{m1}^1 \times dy_1^2 \times ... \times dy_{m2}^j
\]
Since $\{Y_i^*: s = 1, 2 \text{ and } i = 1, 2, ..., n_s\}$ is independent of $\sigma(K)$, we get
\[
E[F(y_1^1 + p_K(Z_1^1)), ..., y_{m1}^1 + p_K(Z_{m1}^1), y_1^2 + p_K(Z_1^2), ..., y_{m2}^j + p_K(Z_{m2}^j))]
= \prod_{1 \leq s \leq 2, 1 \leq i \leq m_s} \int 1_{\{r_i^a\}}(y_i^s + p_K(Z_i^*))P_{Y_i^s}(dy_i^s)
\]

1.4 Exercise 1.18 (Levy’s construction of Brownian motion)

For each $t \in [0, 1]$, we set $h_0(t) = 1$, and then, for every integer $n \geq 0$ and every $k \in \{0, 1, ..., 2^n - 1\}$,
\[
h_{n,k}(t) = 2^n 1_{\frac{1}{2^{n+1}} \leq t < \frac{1}{2^n}} - 2^n 1_{\frac{1}{2^n} \leq t < \frac{1}{2^{n+1}}}.
\]
1. Verify that the functions (Haar system) \( H := \{h_{n,k} | n \geq 0 \text{ and } k = 0, 1, \ldots, 2^n - 1\} \cup \{h_0\} \) form an orthonormal basis of \( L^2([0,1], B_{[0,1]}, dt) \). (Hint: Observe that, for every fixed \( n \geq 0 \), any function \( f : [0,1] \to \mathbb{R} \) that is constant on every interval of the form \( \left[ \frac{j-1}{2^n}, \frac{j}{2^n} \right) \), for every \( 1 \leq j \leq 2^n \), is a linear combination of the functions in \( H \)).

2. Suppose that \( \{N_0\} \cup \{N_{n,k}\} \) are independent \( \mathcal{N}(0,1) \) random variables. Justify the existence of the (unique) Gaussian white noise \( G \) on \([0,1]\) with intensity \( dt \), such that \( G(h_0) = N_0 \) and \( G(h_{n,k}^2) = N_{n,k}^2 \) for every \( n \geq 0 \) and \( 0 \leq k \leq 2^n - 1 \).

3. For every \( t \in [0,1] \), set \( B_t = G([0,t]) \). Show that
\[
B_t = tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n - 1} g_{n,k}(t)N_{n,k},
\]
where the series converges in \( L^2 \), and the functions \( g_{n,k} : [0,1] \to [0,\infty) \) are given by
\[
g_{n,k}(t) = \int_0^t h_{n,k}(s)ds.
\]
Note that the functions \( g_{n,k} \) are continuous and satisfy the following property: For every fixed \( n \geq 0 \), the functions \( g_{n,k}, \quad 0 \leq k \leq 2^n - 1 \), have disjoint supports and are bounded above by \( 2^{-\frac{n}{2}} \).

4. For every integer \( m \geq 0 \) and every \( t \in [0,1] \) set
\[
B_t^m = tN_0 + \sum_{n=0}^{m-1} \sum_{k=0}^{2^n - 1} g_{n,k}(t)N_{n,k}.
\]
Verify that the continuous functions \( t \to B_t^m \) converge uniformly on \([0,1]\) as \( m \to \infty \) (a.s.) (Hint: If \( N \) is \( \mathcal{N}(0,1) \) distributed, prove the bound \( P(|N| \geq a) \leq \exp(-\frac{a^2}{2}) \) for every \( a \geq 1 \), and use this estimate to bound the probability of the event \( \{\sup_{0 \leq k \leq 2^n - 1} |N_{n,k}| > 2^{\frac{n}{2}}\} \), for every fixed \( n \geq 0 \).)

5. Conclude that we can, for every \( t \geq 0 \), select a random variable \( W_t \) which is a.s. equal to \( B_t \), in such a way that the mapping \( t \to W_t \) is continuous for every \( w \in \Omega \).

**Proof.**

1. It’s clear that \( H \) is an orthonormal system in \( L^2([0,1], B_{[0,1]}, dt) \). Now, we show that \( H \) is complete. Since
\[
V = L^2([0,1], B_{[0,1]}, dt),
\]
where \( V := \text{span}(S) \), \( S = \bigcup_{n=0}^{\infty} S_n \), and
\[
S_n := \{ f : [0,1] \to \mathbb{R} : f(x) = \sum_{k=0}^{2^n - 1} c_k 1_{\left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right]} \} \quad \forall n \geq 0,
\]
it suffices to show that \( S \subseteq \text{span}(H) \).

Fix \( f \in S_m \) such that
\[
f(x) = \sum_{k=0}^{2^m - 1} c_m 1_{\left[ \frac{k}{2^m}, \frac{k+1}{2^m} \right]}(x) \text{ for some } m \geq 0.
\]
It’s clear that \( f \in \text{span}(H) \) if \( m = 0 \). Now, we assume that \( m \geq 1 \). To show that \( f \in \text{span}(H) \), it suffices to show that there exists real numbers \( \alpha_0, \ldots, \alpha_{2^m-1} \) such that
\[
f(x) = \sum_{k=0}^{2^m - 1} \alpha_k h_{m-1,k}(x) \in S_{m-1}
\]
Set\[\alpha_k = \frac{1}{2^{m-1}} (c_{2k} - c_{2k+1}) \quad \forall 0 \leq k \leq 2^{m-1} - 1.\]

Then
\[c_{2k+1} \frac{1_{[\frac{k}{2^m}, \frac{k+1}{2^m})}}{2^m} (x) + c_{2k+1} \frac{1_{[\frac{k}{2^m}, \frac{k+1}{2^m})}}{2^m} (x) - \alpha_k h_{m-1,k}(x)\]
\[= \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{k}{2^m}, \frac{k+1}{2^m})} (x) + \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{k}{2^m}, \frac{k+1}{2^m})} (x)\]
\[= \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{k}{2^m}, \frac{k+1}{2^m})} \quad \forall 0 \leq k \leq 2^{m-1} - 1\]

and so \(f(x) - \sum_{k=0}^{2^m-1} \alpha_k h_{m-1,k}(x) \in \mathcal{S}_{m-1}\).

2. Let \(\{N_0\} \cup \{N_{n,k}\}\) be independent \(\mathcal{N}(0,1)\) random variables. Define
\[G(c_0h_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k}h_{n,k}) = c_0N_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k}N_{n,k}.\]

It’s clear that \(G\) is a Gaussian white noise with intensity \(dt\).

3. It’s clear that
\[B_t := G(1_{[0,t]}(x) = tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k},\]

where
\[g_{n,k}(t) = 1_{[0,t]}(x) = \int_0^t h_{n,k}(s)ds.\]

By the definition of \(h_{n,k}\), we get \(g_{n,k}(t)\) is continuous, \(0 \leq g_{n,k}(t) \leq 2^\frac{m}{2}\), and \(supp(g_{n,k}) \subseteq [\frac{k}{2^m}, \frac{k+1}{2^m})\) for \(n \geq 0\) and \(k = 0, 1, ..., 2^n - 1\).

4. Note that
\[\sum_{n=0}^{\infty} P(\sup_{0 \leq k \leq 2^n-1} |N_{n,k}| > 2^\frac{m}{2}) \leq \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} P(|N_{n,k}| > 2^\frac{m}{2}) \leq \sum_{n=0}^{\infty} 2^n \exp(-2^\frac{m}{2}-1) < \infty.\]

By Borel Cantelli lemma, we have \(P(E) = 1\), where
\[E := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{ \sup_{0 \leq k \leq 2^n-1} |N_{n,k}| \leq 2^\frac{m}{2} \}.\]

Fix \(w \in E\). By problem 3, we get
\[\sup_{t \in [0,1]} | \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k} | \leq \sup_{t \in [0,1]} \sum_{k=0}^{2^n-1} g_{n,k}(t)|N_{n,k}| = \sup_{0 \leq k \leq 2^n-1} \sup_{t \in [0,1]} (g_{n,k}(t)|N_{n,k}|)\]
\[\leq (2^{-\frac{m}{2}}) \sup_{0 \leq k \leq 2^n-1} |N_{n,k}| \leq 2^{-\frac{m}{2}} \times 2^\frac{m}{2} = 2^{-\frac{m}{2}} \text{ for large } n\]

and so
\[\sup_{t \in [0,1]} \sum_{n=m_1}^{m_2} \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k} \leq \sum_{n=m_1}^{m_2} \sup_{t \in [0,1]} | \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k} | \leq \sum_{n=m_1}^{m_2} 2^{-\frac{m}{2}} \xrightarrow{m_1,m_2 \to \infty} 0.\]
Thus, \( \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k} N_{n,k}(w) \) converge uniformly on \([0, 1]\) and so
\[
t \in [0, 1] \mapsto B_t := tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k}
\]
is continuous (a.s.).

Moreover, since
\[
E[(B_t - B_s)^2] = E[G(1_{(s,t)}]^2] = t - s \quad \forall 0 \leq s \leq t \leq 1
\]
and
\[
E[(B_t - B_s)B_r] = E[G(1_{(s,t)}G(1_{[0,r]}]) = 0 \quad \forall 0 \leq r \leq s \leq t \leq 1,
\]
we see that \( B_t - B_s \sim \mathcal{N}(0, t-s) \) and \( B_t - B_s \perp \sigma(B_r, 0 \leq r \leq s) \) for every \( 0 \leq s \leq t \leq 1 \).

5. Let \( \{N_0^m : m \geq 1\} \cup \{N_{n,k}^m : m \geq 1, n \geq 0, 0 \leq k \leq 2^n - 1\} \) be independent \( \mathcal{N}(0, 1) \). Define Gaussian white noises
\[
G^m(c_0h_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k}h_{n,k}) := c_0N_0^m + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k}N_{n,k}^m \quad \forall m \geq 1
\]
and
\[
B_t^m := G^m(1_{[0,t)}) = tN_0^m + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k}^m \quad \forall m \geq 1, t \in [0, 1].
\]
Then \( B^1, B^2, \ldots \) are independent. Define
\[
W_t := \sum_{k=1}^{m-1} B_k^m + B_1^m \quad \text{if } m - 1 \leq t < m.
\]
Since \( (B_t^m)_{t \in [0, 1]} \) is continuous for every \( m \geq 1 \), we see that \( (W_t)_{t \geq 0} \) has continuous sample path. Moreover, since
\[
W_t - W_s = B_t^m - B_s^m + B_1^{m-1} + \ldots + B_1^{n+1} + B_1^n - B_s^n \sim \mathcal{N}(0, t-s) \quad \forall 0 \leq s < t, n-1 \leq s < n, m-1 \leq t < m
\]
and
\[
E[(W_t - W_s)W_r] = 0 \quad \forall 0 \leq r \leq s \leq t,
\]
we see that we see that \( W_t - W_s \perp \sigma(W_r, 0 \leq r \leq s) \) for every \( 0 \leq s \leq t \) and so \( (W_t)_{t \geq 0} \) is a Brownian motion.

\( \square \)
Chapter 2
Brownian Motion

2.1 Exercise 2.25 (Time inversion)

Show that the process \((W_t)_{t \geq 0}\) defined by

\[ W_t = \begin{cases} \frac{t}{2} B_1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases} \]

is indistinguishable of a real Brownian motion started from 0.

Proof.

First, we show that \((W_t)_{t \geq 0}\) is a pre-Brownian motion. That is \((W_t)_{t \geq 0}\) is a centered Gaussian with covariance function \(K(t, s) = s \wedge t\). Since \((B_t)_{t \geq 0}\) is a centered Gaussian process, we see that \((W_t)_{t \geq 0}\) is a centered Gaussian process. Let \(t > 0\) and \(s > 0\). Then

\[ E[W_t W_s] = E[tsB_{\frac{1}{s}} B_{\frac{1}{t}}] = ts(\frac{1}{s} \wedge \frac{1}{t}) = t \wedge s \]

and

\[ E[W_t W_0] = 0 \]

Thus, \((W_t)_{t \geq 0}\) is a pre-Brownian motion.

Next, we show that

\[ \lim_{t \to \infty} W_t = \lim_{t \to \infty} \frac{B_t}{t} = 0 \text{ a.s.} \]

By considering \((B_{k+1} - B_k)_{k \geq 0}\) and using the strong law of large number, we get

\[ \frac{B_n}{n} \to 0 \text{ a.s.} \]

Let \(m, n \geq 0\). By using Kolmogorov’s inequality, we see that

\[ P(\max_{0 \leq k \leq 2^m} |B_{n + \frac{k}{2^n}} - B_n| \geq n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{2}{3}}} E[(B_{n+1} - B_n)^2] = \frac{1}{n^{\frac{2}{3}}} \]

By letting \(m \to \infty\), we get

\[ P(\sup_{t \in [n, n+1]} |B_t - B_n| \geq n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{2}{3}}} \]

By using Borel-Cantelli is lemma, we have a.s.

\[ \left| \frac{B_t}{t} \right| \leq \frac{1}{n^{\frac{2}{3}}} + \frac{B_n}{n} \text{ for large } n \text{ and } n \leq t \leq n + 1 \]

and, hence,

\[ \lim_{t \to \infty} \frac{B_t}{t} = 0 \text{ a.s.} \]

Therefore, \(W_t\) is continuous at \(t = 0\) a.s.

Finally, we set \(E = \{\lim_{t \to \infty} \frac{B_t}{t} = 0\}\) and

\[ \widetilde{W}_t(w) = \begin{cases} W_t(w), & \text{if } w \in E \\ 0, & \text{otherwise} \end{cases} \]

for all \(t \geq 0\). Then \((\widetilde{W}_t)_{t \geq 0}\) and \((W_t)_{t \geq 0}\) are indistinguishable. Since \((\widetilde{W}_t)_{t \geq 0}\) has continuous sample path, we see that \((\widetilde{W}_t)_{t \geq 0}\) is the Brownian motion. Thus, \((W_t)_{t \geq 0}\) is indistinguishable of a real Brownian motion \((\widetilde{W}_t)_{t \geq 0}\) started from 0. \(\square\)
2.2 Exercise 2.26

For each real $a \geq 0$, we set $T_a = \inf\{t \geq 0 | B_t = a\}$. Show that the process $(T_a)_{a \geq 0}$ has stationary independent increments, in the sense that, for every $0 \leq a \leq b$, the variable $T_b - T_a$ is independent of the $\sigma$-field $\sigma(T_c, 0 \leq c \leq a)$ and has the same distribution as $T_{b-a}$.

Proof.

1. First, we show that $T_b - T_a \overset{D}{=} T_{b-a}$ for each $0 \leq a < b$. Given $0 \leq a < b$. Set

   $$\tilde{B}_t = 1_{T_a \leq \infty}(B_{T_a} + t - B_{T_a}).$$

   Since $T_a < \infty$ a.s., we see that $(\tilde{B}_t)_{t \geq 0}$ is a Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. Set

   $$\tilde{T}_c = \inf\{t \geq 0 | \tilde{B}_t = c\}$$

   for each $c \in \mathbb{R}$. Then we see that $\tilde{T}_{b-a} \overset{D}{=} T_{b-a}$. Since $T_a < \infty$ a.s., we have a.s. $s \geq T_a$ if $B_s = b$. Thus, we see that a.s.

   $$\tilde{T}_{b-a} = \inf\{t \geq 0 | \tilde{B}_t = b-a\}$$

   $$= \inf\{t + T_a | B_{T_a} + t = b \text{ and } t \geq 0\} - T_a$$

   $$= \inf\{s | B_s = b \text{ and } s \geq T_a\} - T_a$$

   $$= \inf\{s | B_s = b\} - T_a = T_b - T_a$$

   and therefore

   $$T_b - T_a \overset{D}{=} T_{b-a}.$$

2. Next, we show that $T_b - T_a$ is independent of the $\sigma$-field $\sigma(T_c, 0 \leq c \leq a)$. Given $0 \leq a < b$. By using strong Markov property, we see that $\tilde{B}_t$ is independent of $\mathcal{F}_{T_a}$. Since $T_c \leq T_a$ for $0 \leq c \leq a$, we have $\mathcal{F}_{T_c} \subseteq \mathcal{F}_{T_a}$ for each $0 \leq c \leq a$. Indeed, if $A \in \mathcal{F}_{T_c}$, then

   $$A \cap \{T_a \leq t\} = (A \cap \{T_c \leq t\}) \cap \{T_a \leq t\} \in \mathcal{F}_t.$$

   Therefore

   $$\{T_{c_1} \leq t_1, ..., T_{c_n} \leq t_n\} \in \mathcal{F}_{T_a}$$

   for each $n \geq 1$, $0 \leq c_1 \leq ... \leq c_n \leq a$, and non-negative real number $t_1, ..., t_n$. By using monotone class theorem, we have

   $$\sigma(T_c, 0 \leq c \leq a) \subseteq \mathcal{F}_{T_a}.$$

   Note that $T_b - T_a = \tilde{T}_{b-a}$ a.s. To show $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$, it suffices to show that $\tilde{T}_{b-a}$ is independent of $\sigma(T_c, 0 \leq c \leq a)$. Since $\{\tilde{T}_{b-a} \leq t\} = \{\inf_{s \in \mathbb{Q} \cap [0,t]} |\tilde{B}_s - (b-a)| = 0\}$ and $\tilde{B}_t$ is independent of $\mathcal{F}_{T_a}$, we see that $\tilde{T}_{b-a}$ is independent of $\mathcal{F}_{T_a}$. Because $\sigma(T_c, 0 \leq c \leq a) \subseteq \mathcal{F}_{T_a}$, we see that $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$.

$$\square$$

2.3 Exercise 2.27 (Brownian bridge)

We set $W_t = B_t - tB_1 \quad \forall t \in [0,1]$.

1. Show that $(W_t)_{t \in [0,1]}$ is a centered Gaussian process and give its covariance function.
2. Let $0 < t_1 < t_2 < \ldots < t_m < 1$. Show that the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_m})$ has density
\[
g(x_1, x_2, \ldots, x_m) = \sqrt{2\pi} p_t(x_1)p_{t_2-t_1}(x_2-x_1)\ldots p_{t_m-t_{m-1}}(x_m-x_{m-1})p_{1-t_p}(-x_m),
\]
where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$. Explain why the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_m})$ can be interpreted as the conditional law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ knowing that $B_1 = 0$.

3. Verify that the two processes $(W_t)_{t \in [0,1]}$ and $(W_{1-t})_{t \in [0,1]}$ have the same distribution (similarly as in the definition of Wiener measure, this law is a probability measure on the space of all continuous functions from $[0,1]$ into $\mathbb{R}$).

\[ \text{Proof.} \]

1. Let $0 < t_1 < t_2 < \ldots < t_m < 1$, $Q := \sum_{i=1}^{m} t_i c_i$, and $R_j := \sum_{i=j}^{m} c_i$ for each $1 \leq j \leq m$. Then
\[
\sum_{i=1}^{m} c_i W_{t_i} = -Q(B_1 - B_{t_m}) + (Q + R_m)(B_{t_m} - B_{t_{m-1}}) + \ldots + (Q + R_2)(B_{t_2} - B_{t_1}) + (Q + R_1)B_1
\]
is a centered Gaussian and so $(W_t)_{t \in [0,1]}$ is a centered Gaussian process. Moreover, the its covariance function
\[
E[W(t)W(s)] = E[(B_t - tB_1)(B_s - sB_1)] = t \wedge s - ts + ts = t \wedge s - ts ~ \forall t, s \in [0,1].
\]

2. Let $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$ and $F(x_1, \ldots, x_m)$ be nonmeasurable measurable on $\mathbb{R}^m$. Then
\[
E[F(W_{t_1}, W_{t_2}, \ldots, W_{t_m})] = E[F(B_{t_1} - t_1B_1, B_{t_2} - t_2B_1, \ldots, B_{t_m} - t_mB_1)]
\]
\[
= \int_{\mathbb{R}^{m+1}} F(x_1 - t_1x_{m+1}, x_2 - t_2x_{m+1}, \ldots, x_m - t_mx_{m+1}) \prod_{i=1}^{m+1} p_{t_i - t_{i-1}}(x_i - x_{i-1})dx_1dx_2\ldots dx_{m+1}(x_0 = 0)
\]
\[
= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, \ldots, y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1})y_{m+1})p_{1-t_m}(y_{m+1} - y_m - t_my_{m+1})dy_1\ldots dy_{m+1}
\]
(Set $y_0 = 0, y_i = x_i - t_i x_{m+1}$, and $y_{m+1} = x_{m+1}$).

Note that
\[
p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1})y_{m+1}) = p_{t_i - t_{i-1}}(y_i - y_{i-1}) \exp(-y_{m+1}(y_i - y_{i-1})) \exp(-\frac{1}{2}(t_i - t_{i-1})y_{m+1}^2)
\]
for each $1 \leq i \leq m$ and
\[
p_{1-t_m}(y_{m+1} - y_m - t_my_{m+1}) = p_{1-t_m}(-y_m) \exp(y_my_{m+1}) \exp(-\frac{1}{2}y_{m+1}^2).
\]

Then
\[
\prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1})y_{m+1})p_{1-t_m}(y_{m+1} - y_m - t_my_{m+1}) = \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1})p_{1-t_m}(-y_m) \exp(-\frac{1}{2}y_{m+1}^2)
\]
and so
\[
E[F(W_{t_1}, W_{t_2}, \ldots, W_{t_m})]
\]
\[
= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, \ldots, y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1})y_{m+1})p_{1-t_m}(y_{m+1} - y_m - t_my_{m+1})dy_1\ldots dy_{m+1}
\]
\[
= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, \ldots, y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1})p_{1-t_m}(-y_m)\sqrt{2\pi}dy_1\ldots dy_{m}.
\]

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3. We have two ways to explain why the law of Brownian bridge \((W_t)_{t \in [0,1]}\) can be interpreted as the conditional law of \((B_t)_{t \in [0,1]}\) knowing that \(B_1 = 0\).

(a) First, we show that, if \(B_1(w) = 0\), then

\[
E[F(B_{t_1}, \ldots, B_{t_m})|B_1](w) = \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) g(x_1, \ldots, x_m) dx_1 \ldots dx_m
\]

for every \(0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1\) and \(F(x_1, \ldots, x_m)\) be nonnegative measurable function on \(\mathbb{R}^m\). Observe that

\[
E[F(B_{t_1}, \ldots, B_{t_m})|B_1] = \varphi(B_1),
\]

where \(x_0 = 0\),

\[
q(x_{m+1}) = \int_{\mathbb{R}^m} f_{B_{t_1}, \ldots, B_{t_m}, B_1}(x_1, \ldots, x_m, x_{m+1}) dx_1 \ldots dx_m = \int_{\mathbb{R}^m} \prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1}) dx_1 \ldots dx_m,
\]

and

\[
\varphi(x_{m+1}) = \frac{1}{q(x_{m+1})} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) f_{B_{t_1}, \ldots, B_{t_m}, B_1}(x_1, \ldots, x_m, x_{m+1}) dx_1 \ldots dx_m
\]

\[
= \frac{1}{q(x_{m+1})} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1}) dx_1 \ldots dx_m.
\]

Note that

\[
q(0) = \int_{\mathbb{R}^m} \prod_{i=1}^m p_{t_i-t_{i-1}}(x_i - x_{i-1}) p_{1-t_m}(-x_m) dx_1 \ldots dx_m = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^m} g(x_1, \ldots, x_m) dx_1 \ldots dx_m = \frac{1}{\sqrt{2\pi}}
\]

and

\[
\varphi(0) = \frac{1}{q(0)} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1}) dx_1 \ldots dx_m
\]

\[
= \sqrt{2\pi} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \prod_{i=1}^m p_{t_i-t_{i-1}}(x_i - x_{i-1}) p_{1-t_m}(-x_m) dx_1 \ldots dx_m
\]

\[
= \sqrt{2\pi} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \frac{1}{\sqrt{2\pi}} g(x_1, \ldots, x_m) dx_1 \ldots dx_m
\]

\[
= \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) g(x_1, \ldots, x_m) dx_1 \ldots dx_m.
\]

Thus, if \(w \in \{B_1 = 0\}\), then

\[
E[F(B_{t_1}, \ldots, B_{t_m})|B_1](w) = \varphi(0) = \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) g(x_1, \ldots, x_m) dx_1 \ldots dx_m.
\]

(b) Next, we show that

\[
((B_{t_1}, \ldots, B_{t_m})||B_1 \leq \epsilon) \quad \overset{d}{\to} \quad (W_{t_1}, \ldots, W_{t_m})
\]

for every \(0 < t_1 < t_2 < \ldots < t_m < 1\) and so the conditional law of \((B_t)_{t \in [0,1]}\) knowing that \(|B_1| \leq \epsilon\) converges weakly to the law of \((W_t)_{t \in [0,1]}\). Given \(0 < t_1 < t_2 < \ldots < t_m < 1\) and \(F(x_1, \ldots, x_m)\) be nonnegative measurable function on \(\mathbb{R}^m\). Set

\[
\mu_\epsilon(dx_1 \ldots dx_m) := P((B_{t_1}, \ldots, B_{t_m} \in dx_1 \ldots dx_m)||B_1 \leq \epsilon) \quad \forall \epsilon > 0.
\]
Then
\[
\int F(x_1, \ldots, x_m) \mu_t(dx_1 \ldots dx_m) = P(\{B_t | \leq \epsilon\}^{-1} E[F(B_{t_1}, \ldots, B_{t_m})1_{\{B_1 | \leq \epsilon\}}]
= P(\{B_t | \leq \epsilon\}^{-1} E[E[F(B_{t_1}, \ldots, B_{t_m}) | B_1]1_{\{B_1 | \leq \epsilon\}}]
= P(\{B_t | \leq \epsilon\}^{-1} E[\varphi(B_1)1_{\{B_1 | \leq \epsilon\}}]
= \int \varphi(x) \times (P(\{B_t | \leq \epsilon\}^{-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} 1_{\{|x| \leq \epsilon\}})dx.
\]

It’s clear that \(\varphi(x)\) is continuous and so
\[
\int F(x_1, \ldots, x_m) \mu_t(dx_1 \ldots dx_m) \to \varphi(0) = \int_{\mathbb{R}^m} F(x_1, \ldots, x_m)g(x_1, \ldots, x_m)dx_1 \ldots dx_m \text{ as } \epsilon \to 0.
\]

4. Let \(0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1\) and \(F(x_1, \ldots, x_m)\) be nonnegative measurable function on \(\mathbb{R}^m\). Set \(s_i = 1 - t_{m+1-i}\) for every \(0 \leq i \leq m + 1\). Then
\[
E[F(W_{t_1}, \ldots, W_{t_m})] = E[F(W_{s_1}, \ldots, W_{s_m})]
= \int_{\mathbb{R}^m} F(y_{s_1}, y_{s_2}, \ldots, y_{s_m}) \prod_{i=1}^m p_{s_i-s_{i-1}}(y_i - y_{i-1})p_{1-s_m}(y_m)\sqrt{2\pi}dy_1 \ldots dy_m
= \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \prod_{i=1}^m p_{s_i-s_{i-1}}(x_i - x_{i-1})p_{1-s_m}(x_m)\sqrt{2\pi}dx_1 \ldots dx_m
= \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) \prod_{i=1}^m p_{t_i-t_{i-1}}(x_i - x_{i-1})p_{1-t_m}(x_m)\sqrt{2\pi}dx_1 \ldots dx_m
= E[F(W_{t_1}, \ldots, W_{t_m})]
\]
and so \((W_t)_{t \in [0,1]}\) and \((W_{t-})_{t \in [0,1]}\) have the same distribution.

\[\square\]

2.4 Exercise 2.28 (Local maxima of Brownian paths)
Show that, a.s., the local maxima of Brownian motion are distinct: a.s., for any choice of the rational numbers
\(0 \leq p < q < r < s\), we have
\[
\sup_{p \leq t \leq q} B_t \neq \sup_{r \leq t \leq s} B_t.
\]

Proof.
Fixed any rational numbers \(0 \leq p < q < r < s\). We show that
\[
P(\sup_{p \leq t \leq q} B_t = \sup_{r \leq t \leq s} B_t) = 0.
\]
Set
\[
X = \sup_{p \leq t \leq q} B_t - B_r
\]
and
\[
Y = \sup_{r \leq t \leq s} B_t - B_r.
\]
Since \(\{B_r - B_t | p \leq t \leq q\}\) and \(\{B_t - B_r | r \leq t \leq s\}\) are independent, we see that \(X\) and \(Y\) are independent.
By using simple Markov property, we see that \((B_t - B_r)_{t \geq r}\) is a Brownian motion. Set \(S_t = \sup_{t \geq r} (B_t - B_r)\). By using reflection principle, we have

\[
P(S_t \geq a) = P(\sup_{t \geq r} B_t - B_r \geq a) = P(\sup_{t \geq r} B_{t-r} \geq a) = P(|B_{t-r}| \geq a)
\]

and, hence, \(S_t\) is a continuous random variable for each \(t \geq r\). Therefore,

\[
P(\sup_{p \leq t \leq q} B_t = \sup_{r \leq t \leq s} B_t) = P(\sup_{p \leq t \leq q} B_t - B_r = \sup_{r \leq t \leq s} B_t - B_r) = P(X - Y = 0) = \int_{\mathbb{R}^2} 1_{(0)}(x + y)P_{(X,Y)}(dx \times dy) = \int_{\mathbb{R}^2} 1_{(0)}(x + y)P_{X,Y}(dx \times dy) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(0)}(y)P_{Y}(dy)P_X(dx) = \int_{\mathbb{R}} P(-Y = -x)P_X(dx) = 0
\]

Thus, we have

\[
P(\bigcup_{0 \leq p < q < r < s \text{ are rational}} \sup_{p \leq t \leq q} B_t = \sup_{r \leq t \leq s} B_t) = 0
\]

\(\square\)

2.5 Exercise 2.29 (Non-differentiability)

Show that, a.s.,

\[
\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty,
\]

and infer that, for each \(s \geq 0\), the function \(t \mapsto B_t\) has a.s. no right derivative at \(s\).

Proof.

1. First, we show that a.s.,

\[
\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.
\]

Given \(M > 0\). Since

\[
\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \limsup_{t \downarrow 0} \sup_{0 \leq t < s} \frac{B_t}{\sqrt{t}} \in \mathcal{F}_0^+
\]

and therefore

\[
\{\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} \geq M\} \in \mathcal{F}_0^+.
\]
Now, by Fatou’s lemma, we have
\[
\mathbb{P}(\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} \geq M) \\
\geq \mathbb{P}(\limsup_{n \to \infty} \frac{B_{n-1}}{\sqrt{n-1}} \geq M) \\
= \mathbb{P}(\frac{B_{n-1}}{\sqrt{n-1}} \geq M \ i.o) \\
= \mathbb{P}(\limsup_{n \to \infty} \{ \frac{B_{n-1}}{\sqrt{n-1}} \geq M \}) \\
\geq \limsup_{n \to \infty} \mathbb{P}(\frac{B_{n-1}}{\sqrt{n-1}} \geq M) \\
= \int_{M}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx > 0
\]

Therefore, by zero-one law, we have a.s.
\[
\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} \geq M.
\]

Since M is arbitrary, we get
\[
\mathbb{P}(\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty) = \lim_{n \to \infty} \mathbb{P}(\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} \geq n) = 1.
\]

Because \((-B_t)_{t \geq 0}\) is a Brownian motion, we see that
\[
\mathbb{P}(\liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty) = \mathbb{P}(\limsup_{t \downarrow 0} -\frac{B_t}{\sqrt{t}} = \infty) = 1.
\]

2. We show that, for each \(s \geq 0\), the function \(t \mapsto B_t\) has a.s. no right derivative at \(s\). Given \(s \geq 0\). Observe that
\[
\mathbb{P}(\limsup_{t \downarrow s} \frac{B_t - B_s}{t-s} = \infty) \\
= \mathbb{P}(\limsup_{t \downarrow s} \frac{B_t - B_s}{\sqrt{t-s}} \times \frac{1}{\sqrt{t-s}} = \infty) \\
= \mathbb{P}(\limsup_{t \downarrow s} \frac{B_{t-s}}{\sqrt{t-s}} = \infty) = 1
\]
and
\[
\mathbb{P}(\liminf_{t \downarrow s} \frac{B_t - B_s}{t-s} = -\infty) \\
= \mathbb{P}(\liminf_{t \downarrow s} \frac{B_t - B_s}{\sqrt{t-s}} \times \frac{1}{\sqrt{t-s}} = -\infty) \\
= \mathbb{P}(\liminf_{t \downarrow s} \frac{B_{t-s}}{\sqrt{t-s}} = -\infty) = 1
\]

Then the function \(t \mapsto B_t\) has a.s. no right derivative at \(s\).
2.6 Exercise 2.30 (Zero set of Brownian motion)
Let \( H = \{ t \in [0,1] | B_t = 0 \} \). Show that \( H \) is a.s. a compact subset of \([0,1]\) with no isolated point and zero Lebesgue measure.

Proof.
Since \((B_t)_{t \in [0,1]}\) is continuous, we see that \( H \) is closed and so \( H \) is compact. Observe that
\[
E[\lambda_R(H)] = \int_\Omega \int_0^1 1_{\{s \in [0,1]: B_s = 0\}}(t)dt P(dw) = \int_0^1 \int_\Omega 1_{\{s \in [0,1]: B_s = 0\}}(t)P(dw)dt = \int_0^1 P(B_t = 0)dt = 0
\]
and so \( \lambda_R(H) = 0 \) (a.s.).
Now, we show that \( H \) has no isolated points (a.s.). Define
\[
T_q := \inf \{ t \geq q : B_t = 0 \} \quad \forall q \in [0,1) \cap \mathbb{Q}.
\]
Observe that
\[
P( \sup_{0 \leq s \leq \epsilon} B_{T_q+s} > 0 \text{ and } \inf_{0 \leq s \leq \epsilon} B_{T_q+s} < 0 \quad \forall \epsilon \in (0,1-q) \cap \mathbb{Q}, \quad \forall q \in [0,1) \cap \mathbb{Q} ) = 1.
\]
Indeed, by proposition 2.14 and the strong Markov property, we get
\[
P( \sup_{0 \leq s \leq \epsilon} B_s > 0 \text{ and } \inf_{0 \leq s \leq \epsilon} B_s < 0 \quad \forall \epsilon \in (0,1-q) \cap \mathbb{Q} ) = 1 \quad \forall q \in [0,1) \cap \mathbb{Q}.
\]
Set
\[
E := \bigcap_{q \in (0,1) \cap \mathbb{Q}} \bigcap_{\epsilon \in (0,1-q) \cap \mathbb{Q}} \{ \exists p \in (0,1) \cap \mathbb{Q} : T_q < T_p < T_q + \epsilon \}.
\]
Then \( P(E) = 1 \) and so \( T_q \) is not an isolated point for every \( q \in [0,1) \cap \mathbb{Q} \) (a.s.). Fix \( w \in E \). Let \( t \in H \setminus \{ T_q : q \in [0,1) \cap \mathbb{Q} \} \). Choose \( q_n \in [0,1) \cap \mathbb{Q} \) such that \( q_n \uparrow t \). Since \( q_n < t \) and \( B_t = 0 \), we have
\[
q_n \leq T_{q_n} \leq t \quad \forall n \geq 1
\]
and so \( T_{q_n} \uparrow t \). Thus, \( t \) is not an isolated. Therefore, \( H \) has no isolated points (a.s.).

2.7 Exercise 2.31 (Time reversal)
We set \( B'_t = B_1 - B_{1-t} \) for every \( t \in [0,1] \). Show that the two processes \((B_t)_{t \in [0,1]}\) and \((B'_t)_{t \in [0,1]}\) have the same law (as in the definition of Wiener measure, this law is a probability measure on the space of all continuous functions from \([0,1]\) into \( \mathbb{R} \)).

Proof.
Let \( 0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1 \) and \( F(x_1, ..., x_m) \) be nonnegative measurable function on \( \mathbb{R}^m \). Set
\[ s_i = 1 - t_{m+1-i} \text{ for every } 0 \leq i \leq m+1 \text{ and } p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right). \]

Then

\[ E[F(B_{t_1}', ..., B_{t_m}')] = E[F(B_1 - B_{s_1}, ..., B_1 - B_{s_t})] \]

\[ = \int_{R^{m+1}} F(x_{m+1} - x_m, x_{m+1} - x_{m-1}, ..., x_{m+1} - x_1) \prod_{i=1}^{m+1} p_{s_i-s_{i-1}}(x_i - x_{i-1}) dx_1 ... dx_{m+1} (x_0 = 0) \]

\[ = \int_{R^{m+1}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m+1} p_{t_{m+1-(i-1)}-t_{m+1-i}}(y_{m+1-(i-1)} - y_{m+1-i}) dy_1 ... dy_{m+1} \]

\[ = \int_{R^m} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_{i-1}}(y_i - y_{i-1}) dy_1 ... dy_m \]

\[ \int_{R^m} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_{i-1}}(y_i - y_{i-1}) \times (\int_R p_{t_{m+1}}(y_{m+1} - y_m) dy_{m+1}) dy_1 ... dy_m = E[F(B_{t_1}, ..., B_{t_m})] \]

and so \((B_t)_{t \in [0,1]}\) and \((B_t')_{t \in [0,1]}\) have the same distribution. \(\square\)

2.8 Exercise 2.32 (Arcsine law)

Set \(T := \inf\{t \geq 0 : B_t = S_1\}\).

1. Show that \(T < 1\) a.s. (one may use the result of the previous exercise) and then that \(T\) is not a stopping time.
2. Verify that the three variables \(S_t, S_t - B_t\) and \(|B_t|\) have the same law.
3. Show that \(T\) is distributed according to the so-called arcsine law, whose density is

\[ g(t) = \frac{1}{\pi \sqrt{t(1-t)}} 1_{(0,1)}(t). \]

4. Show that the results of questions 1. and 3. remain valid if \(T\) is replaced by

\[ L := \sup\{t \leq 1 : B_t = 0\}. \]

Proof.

1. It’s clear that \(P(T \leq 1) = 1\). Suppose that \(P(T = 1) > 0\). By exercise 2.31 and proposition 2.14, we get

\[ P(\inf_{0 \leq s \leq \epsilon} B'_s < 0 \quad \forall \epsilon \in (0,1)) = P(\inf_{0 \leq s \leq \epsilon} B_s < 0 \quad \forall \epsilon \in (0,1)) = 1, \]

where \(B'_t = B_1 - B_{1-t}\) for every \(t \in [0,1]\). On the other hand,

\[ 0 < P(T = 1) \leq P(B'_s \geq 0 \quad \forall s \in [0,1]) \]

which is a contradiction. Thus, we have \(P(T < 1) = 1\).

Now, we show that \(T\) is not a stopping time by contradiction. Assume that \(T\) is a stopping time. By theorem 2.20 (strong Markov property), we see that \(B'_T = B_{T+t} - B_T\) is a Brownian motion. Since \(P(T < 1) = 1\), we get

\[ P(\sup_{0 \leq s \leq \epsilon} B'_s \leq 0 \text{ for some } \epsilon > 0) = 1, \]

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which contradiction to (proposition 2.14)
\[ P(\sup_{0 \leq s \leq t} B_s^T > 0 \quad \forall \epsilon > 0) = 1. \]

Thus, we see that \( T \) is not a topping time.

2. Fix \( t > 0 \). By theorem 2.21, we have \( S_t \overset{d}{=} |B_t| \). Now, we show that \( S_t \overset{d}{=} S_t - B_t \). By similar argument as the proof of exercise 2.31, we get \((B_s')_{s \in [0,t]} \overset{d}{=} (B_s)_{s \in [0,t]}\), where \( B'_s = B_t - B_{t-s} \) for every \( s \in [0,t] \). It’s clear that \((B'_s)_{s \in [0,t]} \overset{d}{=} (-B'_s)_{s \in [0,t]}\). Thus, we have

\[ S_t = \sup_{0 \leq s \leq t} B_s = \sup_{0 \leq s \leq t} -B'_s = \sup_{0 \leq s \leq t} B_{t-s} - B_t = \sup_{0 \leq s \leq t} B_s - B_t = S_t - B_t. \]

3. Since
\[ P(\sup_{p_1 \leq s \leq q_1} B_s \neq \sup_{p_2 \leq s \leq q_2} B_s \text{ for all rational numbers } p_1 < q_1 < p_2 < q_2) = 1, \]
we see that the global maximum of \((B_t)_{t \in [0,1]}\) is attained at a unique time (a.s.). That is,
\[ P(\exists t \in [0,1] \quad B_t = S_t) = 1. \]

Let \( r \in (0,1) \) and \( Z_1, Z_2 \overset{i.i.d}{\sim} \mathcal{N}(0,1) \). Then
\[ P(T < r) = P(\max_{0 \leq t \leq r} B_t > \max_{r \leq s \leq 1} B_s) = P(\max_{0 \leq t \leq r} B_t - B_r > \max_{r \leq s \leq 1} B_s - B_r). \]

Since
\[ \max_{0 \leq t \leq r} B_t - B_r \overset{d}{=} \max_{r \leq s \leq 1} B_s - B_r, \]
and
\[ \max_{0 \leq s \leq 1} B_s - B_r = \max_{r \leq s \leq 1} (B_{s-r} - B_r) \overset{d}{=} \max_{0 \leq s \leq 1-r} B_s = S_{1-r} \overset{d}{=} \sqrt{1-r} |Z_2|, \]
we get
\[ P(T < r) = P(\sqrt{r}|Z_1| > \sqrt{1-r}|Z_2|) = P(\frac{|Z_2|^2}{|Z_1|^2 + |Z_2|^2} < r) \]
and so \( T \overset{d}{=} \frac{|Z_2|^2}{|Z_1|^2 + |Z_2|^2} \). Since
\[
E[f(\frac{|Z_2|^2}{|Z_1|^2 + |Z_2|^2})] = \int_{\mathbb{R}^2} f(\frac{y^2}{x^2 + y^2}) \frac{1}{2\pi} \exp(-\frac{x^2 + y^2}{2}) dxdy = 4 \int_0^\infty \int_0^\infty f(\frac{y^2}{x^2 + y^2}) \frac{1}{2\pi} \exp(-\frac{x^2 + y^2}{2}) dxdy = 4 \int_0^\infty \int_0^\frac{\pi}{2} f(\sin(\theta)^2) \frac{1}{2\pi} \exp(-\frac{x^2 + y^2}{2}) rdrd\theta = \frac{2}{\pi} \int_0^1 f(t) \frac{1}{2\sqrt{1-t}t^2} dt \]
\[ = \int_\mathbb{R} \frac{1}{\pi \sqrt{t(1-t)}} 1_{(0,1)}(t) dt, \]
we see that
\[ g(t) = \frac{1}{\pi \sqrt{t(1-t)}} 1_{(0,1)}(t) \]
is the density function of \( T \).
4. We redefine $L(f)$ as the latest time of $f \in C([0, 1])$ such that $f(t) = f(0)$. That is,

$$L(f) = \sup\{t \leq 1 : f(t) = f(0)\}.$$

Then $L = L((B_t)_{t \in [0, 1]})$. Since the global maximum of $(B_t)_{t \in [0, 1]}$ is attained at a unique time (a.s.), we see that $T = L((S_t - B_t)_{t \in [0, 1]})$ (a.s.). Since $(S_t - B_t)_{t \geq 0} \overset{d}{=} (|B_t|)_{t \geq 0}$ and so $L \overset{d}{=} T$. Thus, $g(t)$ is the density function of $L$, $L < 1$ (a.s.), and $L$ is not a stopping time. Indeed, if $L$ is a stopping time,

$$B'_t := B_{L+t} - B_L \overset{(a.s.)}{=} B_{L+t} \quad \forall t \geq 0$$

is a Brownian motion with 0 is an isolated point of $\{t \in [0, 1] : B'_t = 0\}$ (a.s.) which contradict to Exercise 2.30.

\[\square\]

2.9 Exercise 2.33 (Law of the iterated logarithm)

The goal of the exercise is to prove that

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \text{ a.s.}$$

We set $h(t) = \sqrt{2t \log \log t}$.

1. Show that, for every $t > 0$,

$$P(S_t > u \sqrt{t}) \sim \frac{2}{\sqrt{\pi}u} \exp\left(-\frac{u^2}{2}\right),$$

when $u \to \infty$.

2. Let $r$ and $c$ be two real numbers such that $1 < r < c^2$ and set $S_t = \sup_{s \leq t} B_s$. From the behavior of the probabilities $P(S_r > c h(r^{n-1}))$ when $n \to \infty$, infer that, a.s.,

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log 2t}} \leq 1.$$  

3. Show that a.s. there are infinitely many values of $n$ such that

$$B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}} h(r^n).$$

Conclude that the statement given at the beginning of the exercise holds.

4. What is the value of

$$\liminf_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} ?$$

Proof.

1. Given $t > 0$. By using the reflection principle, we have

$$P(S_t > u \sqrt{t})$$

$$= P(S_t > u \sqrt{t}, B_t > u \sqrt{t}) + P(S_t > u \sqrt{t}, B_t \leq u \sqrt{t})$$

$$= P(B_t > u \sqrt{t}) + P(B_t \geq u \sqrt{t})$$

$$= 2P(B_t \geq u \sqrt{t})$$

$$= 2 \int_{u \sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_u^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$$

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Note that, for $x > 0$, 

$$
\left( \frac{1}{x} - \frac{1}{x^3} \right) \exp\left(-\frac{x^2}{2}\right) \leq \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy \leq \frac{1}{x^3} \exp\left(-\frac{x^2}{2}\right),
$$

Indeed, since $\exp(-\frac{y^2}{2}) \leq 1$ and

$$
\int_x^\infty \exp(-\frac{y^2}{2}) dy = \int_0^\infty \exp(-\frac{(z + x)^2}{2}) dz \leq \exp(-\frac{x^2}{2}) \int_0^\infty \exp(-xz) dz = \frac{1}{x} \exp\left(-\frac{x^2}{2}\right),
$$

we have

$$
\int_x^\infty \exp(-\frac{y^2}{2}) dy = \int_0^\infty \exp(-\frac{y^2}{2}) dy \leq \int_x^\infty \exp(-\frac{y^2}{2}) dy.
$$

Thus,

$$
\frac{2}{\sqrt{2\pi}} \left( \frac{1}{x} - \frac{1}{x^3} \right) \exp\left(-\frac{x^2}{2}\right) \leq P(S_t > \sqrt{t}) \leq \frac{2}{\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{u^2}{2}\right)
$$

and therefore

$$
P(S_t > \sqrt{t}) \sim \frac{2}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right),
$$

when $u \to \infty$.

2. Given $1 < r < \infty$. By using similar argument, we have

$$
P(S_{r^n} > ch(r^{n-1})) = 2 \int_{ch(r^{n-1})}^\infty \frac{1}{\sqrt{2\pi} r^n} \exp\left(-\frac{x^2}{2 r^n}\right) dx = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{r^n}}{ch(r^{n-1})} \int_{ch(r^{n-1})}^\infty \exp\left(-\frac{y^2}{2}\right) dy.
$$

Because

$$
h(r^{n-1}) \sqrt{r^n} \to \infty \text{ as } n \to \infty
$$

and

$$
\int_x^\infty \exp(-\frac{y^2}{2}) dy \leq \frac{1}{x} \exp\left(-\frac{x^2}{2}\right),
$$

we get

$$
\lim_{n \to \infty} P(S_{r^n} > ch(r^{n-1})) \leq \lim_{n \to \infty} \frac{2}{\sqrt{2\pi}} \frac{\sqrt{r^n}}{ch(r^{n-1})} \exp\left(-\frac{1}{2} \frac{c^2 h(r^{n-1})^2}{r^n}\right) = 0.
$$

Choose $\{n_k\}$ such that

$$
\sum_{k=1}^\infty P(S_{r^{n_k}} > ch(r^{n_k-1})) < \infty.
$$

By using Borel-Cantelli lemma, we get

$$
P\left( \frac{S_{r^{n_k}}}{h(r^{n_k})} > \frac{c h(r^{n_k-1})}{h(r^{n_k})} \text{ i.o. } \right) = P(S_{r^{n_k}} > ch(r^{n_k-1}) \text{ i.o. }) = 0.
$$

Observe that

$$
\lim_{k \to \infty} \frac{h(r^{n_k-1})}{h(r^{n_k})} = 1.
$$

Then

$$
P(\limsup_{t \to \infty} \frac{S_t}{h(t)} \geq \frac{c}{\sqrt{r}}) = 0.
$$
and, hence,
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} \leq \frac{c}{\sqrt{r}}) \geq P(\limsup_{t \to \infty} \frac{S_t}{h(t)} \leq \frac{c}{\sqrt{r}}) = 1. \]

Fixed \( r > 1 \). Choose \( \{c_n\} \) such that \( 1 < r < c_n^2 \) and \( c_n^2 \downarrow r \). Then
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} \leq \frac{c_n}{\sqrt{r}}) = 1 \]
for each \( n \geq 1 \). By letting \( n \to \infty \), we have
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} \leq 1) = 1 \]

3. Given \( r > 1 \). Set \( d \) to be the positive number such that \( d = \log(r) \). By using the fact that the increments of Brownian motion are Gaussian random variables, we have
\[
P(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r - 1}{r} h(r^n)})
= P\left(\frac{B_{r^n} - B_{r^{n-1}}}{\sqrt{\log r^n - r^{n-1}}} \geq \sqrt{2 \log \log r^n}\right)
= \int_{2 \log \log r^n}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx
\geq \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{(2 \log \log r^n)^{1/2}}\right) \frac{1}{dn}
\]
Because \( \sum_{n=2}^{\infty} \frac{1}{n \sqrt{\log n}} = \infty \) and \( \sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty \), we see that
\[ \sum_{n=1}^{\infty} P(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r - 1}{r} h(r^n)}) = \infty. \]

Note that \( \{B_{r^n} - B_{r^{n-1}}\}_{n \geq 1} \) are independent. By using Borel-Cantelli lemma, we have
\[ P(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r - 1}{r} h(r^n)} \text{ i.o.}) = 1. \]

Now, we show that
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} = 1) = 1. \]

It remain to show that
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} \geq 1) = 1. \]

Given \( r > 1 \). Since
\[ P(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r - 1}{r} h(r^n)} \text{ i.o.}) = 1, \]
we have
\[ P\left(\frac{B_{r^n}}{h(r^n)} \geq \sqrt{\frac{r - 1}{r}} + \sqrt{\frac{\log \log r^n - 1}{\log \log r^n}} \sqrt{\frac{1}{r h(r^{n-1})}} \text{ i.o.}\right) = 1, \]
and, hence, we have a.s.
\[ \limsup_{t \to \infty} \frac{B_t}{h(t)} \geq \frac{r - 1}{r} + \sqrt{\frac{1}{r}} \limsup_{t \to \infty} \frac{B_t}{h(t)}. \]
Thus,
\[ P((\limsup_{t \to \infty} \frac{B_t}{h(t)})^2 \geq \frac{r - 1}{r - 2\sqrt{r} + 1}) = 1 \text{ for each } r > 1. \]
Choose \( \{r_n | r_n > 1\} \) such that \( r_n \downarrow 1 \). Since \( \frac{r - 1}{r - 2\sqrt{r} + 1} \to 1 \) as \( r \downarrow 1 \), we see that
\[ P((\limsup_{t \to \infty} \frac{B_t}{h(t)})^2 \geq 1) = \lim_{n \to \infty} P((\limsup_{t \to \infty} \frac{B_t}{h(t)})^2 \geq \frac{r_n - 1}{r_n - 2\sqrt{r_n} + 1}) = 1 \]
and, hence,
\[ P(\limsup_{t \to \infty} \frac{B_t}{h(t)} \geq 1) = 1. \]
4. Since \((-B_t)_{t \geq 0}\) is a Brownian motion, we see that
\[ P(\liminf_{t \to \infty} \frac{B_t}{h(t)} = -1) = P(\limsup_{t \to \infty} \frac{-B_t}{h(t)} = 1) = 1 \]
and, hence, we have a.s.
\[ \liminf_{t \to \infty} \frac{B_t}{h(t)} = -1. \]
Chapter 3
Filtrations and Martingales

3.1 Exercise 3.26

1. Let $M$ be a martingale with continuous sample paths such that $M_0 = x \in \mathbb{R}_+$. We assume that $M_t \geq 0$ for each $t \geq 0$, and that $M_t \to 0$ as when $t \to \infty$, a.s. Show that, for each $y > x$,

$$
P(\sup_{t \geq 0} M_t \geq y) = \frac{x}{y}.
$$

2. Give the law of

$$
\sup_{t \leq T_0} B_t
$$

when $B$ is a Brownian motion started from $x > 0$ and $T_0 = \inf\{ t \geq 0 | B_t = 0 \}$.

3. Assume now that $B$ is a Brownian motion started from 0, and let $\mu > 0$. Using an appropriate exponential martingale, show that

$$
\sup_{t \geq 0} (B_t - \mu t)
$$

is exponentially distributed with parameter $2\mu$.

Proof.

1. Given $y > x > 0$. First, we suppose $(M_t)_{t \geq 0}$ is uniformly integrable. Then $(M_t)_{t \geq 0}$ is bounded in $L^1$ and, hence,

$$
M_\infty = \lim_{t \to \infty} M_t = 0 \text{ a.s.}
$$

Set $T = \inf\{ t \geq 0 | M_t = y \}$. Then $T$ is a stopping time. By optional stopping times, we have

$$
E[M_T] = E[M_0] = x.
$$

Observe that

$$
E[M_T] = yP(T < \infty) + P(T = \infty) \times 0 = yP(T < \infty)
$$

and

$$
P(T < \infty) = P(\sup_{t \geq 0} M_t \geq y).
$$

Thus, we have

$$
P(\sup_{t \geq 0} M_t \geq y) = \frac{x}{y}.
$$

Next, we consider a general martingale $(M_t)_{t \geq 0}$. For each $n \geq 1$, we set

$$
N_t^{(n)} = M_{t \wedge n}.
$$

Then $(N_t^{(n)})_{t \geq 0}$ is an uniformly integrable martingale for each $n \geq 1$ and therefore

$$
P(\sup_{0 \leq t \leq n} M_t \geq y) = P(\sup_{t \geq 0} N_t^{(n)} \geq y) = \frac{x}{y}.
$$

Letting $n \to \infty$, gives

$$
P(\sup_{t \geq 0} M_t \geq y) = \frac{x}{y}.
$$
2. If \( y \leq x \), it’s clear that
\[
P(\sup_{t \leq T_0} B_t \geq y) = 1.
\]
Now we consider \( y > x \). Set
\[
N_t = B_{t \land T_0}
\]
for each \( t \geq 0 \). Then \((N_t)_{t \geq 0}\) is a martingale. Since \( T_0 < \infty \) a.s., we get \( N_t \to 0 \) when \( t \to \infty \). Thus,
\[
P(\sup_{t \leq T_0} B_t \geq y) = P(\sup_{t \geq 0} N_t \geq y) = \frac{x}{y}.
\]

3. Given \( \mu > 0 \). If \( y \leq 0 \), it’s clear that
\[
P(\sup_{t \geq 0} (B_t - \mu t) \geq y) = 1.
\]
Now we suppose \( y > 0 \). Observe that
\[
P(\sup_{t \geq 0} (B_t - \mu t) \geq y) = P(\sup_{t \geq 0} (\frac{B_t}{\sqrt{2\mu}})^2 \geq \frac{y}{2\mu})
\]
\[
= P(\sup_{t \geq 0} B_t - \frac{1}{2} t \geq 2\mu y)
\]
\[
= P(\sup_{t \geq 0} e^{B_t - \frac{1}{2} t} \geq e^{2\mu y})
\]
Set \( M_t = e^{B_t - \frac{1}{2} t} \) for each \( t \geq 0 \). Then \((M_t)_{t \geq 0}\) is a nonnegative martingale with continuous simple path. Since \( \lim_{t \to \infty} \frac{B_t}{\sqrt{2\mu}} = 0 \) a.s., we get
\[
\lim_{t \to \infty} (B_t - \frac{1}{2} t) = \lim_{t \to \infty} t(\frac{B_t}{t} - \frac{1}{2}) = -\infty \text{ a.s.}
\]
and, hence, \( \lim_{t \to \infty} M_t = 0 \) a.s. Because \( e^{2\mu y} > 1 = M_0 \), we get
\[
P(\sup_{t \geq 0} (B_t - \mu t) \geq y) = P(\sup_{t \geq 0} M_t \geq e^{2\mu y}) = e^{-2\mu y}.
\]
Therefore, we have
\[
P(\sup_{t \geq 0} (B_t - \mu t) \leq y) = \begin{cases} 1 - e^{-2\mu y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\]
and, hence, \( \sup_{t \geq 0} (B_t - \mu t) \) has exponentially distributed with parameter \( 2\mu \).

3.2 Exercise 3.27
Let \( B \) be an \( \mathcal{F}_t \)-Brownian motion started from 0. Recall the notation \( T_x = \inf\{t \geq 0 | B_t = x\} \), for each \( x \in \mathbb{R} \). We fix two real numbers \( a \) and \( b \) with \( a < 0 < b \), and we set
\[
T = T_a \land T_b.
\]
1. Show that, for every \( \lambda > 0 \),
\[
E[e^{-\lambda T}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.
\]
2. Show similarly that, for every $\lambda > 0$,

$$\mathbb{E}[e^{-\lambda T 1_{\{T_n\}}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$ 

3. Show that

$$P(T_a < T_b) = \frac{b}{b-a}.$$ 

**Proof.**

1. Set $\alpha = \frac{b+a}{2}$ and

$$M_t = e^{\sqrt{2\lambda}(B_t - \alpha) - \lambda t} + e^{-\sqrt{2\lambda}(B_t - \alpha) - \lambda t}$$

for each $t \geq 0$.

Since

$$(U_t)_{t \geq 0} = (e^{\sqrt{2\lambda}(B_t - \frac{\lambda t}{2})})_{t \geq 0}$$

and

$$(V_t)_{t \geq 0} = (e^{-\sqrt{2\lambda}(B_t - \frac{\lambda t}{2})})_{t \geq 0}$$

are martingales, we see that

$$M_t = e^{-\sqrt{2\lambda}a}U_t + e^{\sqrt{2\lambda}a}V_t$$

is a martingale. Because

$$0 \leq U_{t\wedge T} \leq e^{\sqrt{2\lambda}b}$$

and

$$0 \leq V_{t\wedge T} \leq e^{\sqrt{2\lambda}(-a)}$$

for each $t \geq 0$, we see that $((U_{t\wedge T}))_{t \geq 0}$ and $((V_{t\wedge T}))_{t \geq 0}$ are uniformly integrable martingales and, hence, $(M_{t\wedge T})_{t \geq 0}$ is a uniformly integrable martingale. Thus, by optional stopping theorem, we get

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = 2 \cosh(\sqrt{2\lambda} \frac{b+a}{2}).$$

Observe that

$$\mathbb{E}[M_T] = e^{-\sqrt{2\lambda}a} \mathbb{E}[e^{-\lambda T 1_{T_n \leq T_b}}] + e^{\sqrt{2\lambda}a} \mathbb{E}[e^{-\lambda T 1_{T_n \leq T_b}}]$$

$$+ e^{\sqrt{2\lambda}a} \mathbb{E}[e^{-\lambda T 1_{T_n > T_b}}] + e^{-\sqrt{2\lambda}a} \mathbb{E}[e^{-\lambda T 1_{T_n > T_b}}]$$

$$= \mathbb{E}[e^{-\lambda T}](e^{\sqrt{2\lambda}a} + e^{-\sqrt{2\lambda}a})$$

$$= \mathbb{E}[e^{-\lambda T}]2 \cosh(\sqrt{2\lambda} \frac{b-a}{2})$$

and therefore

$$\mathbb{E}[e^{-\lambda T}] = \frac{\cosh(\frac{b+a}{2} \sqrt{2\lambda})}{\cosh(\frac{b-a}{2} \sqrt{2\lambda})}.$$

2. Set $\alpha = \frac{b+a}{2}$ and

$$N_t = e^{\sqrt{2\lambda}(B_t - \alpha) - \lambda t} - e^{-\sqrt{2\lambda}(B_t - \alpha) - \lambda t}$$

for each $t \geq 0$. By using similar arguments as above, we get

$$\mathbb{E}[N_T] = \mathbb{E}[N_0] = -2 \sinh(\sqrt{2\lambda} \frac{a+b}{2})$$

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and

\[ E[N_T] = e^{-\sqrt{2\lambda b-a}} E[e^{-\lambda T 1_{T_a \leq T_b}}] - e^{\sqrt{2\lambda b-a}} E[e^{-\lambda T 1_{T_a \leq T_b}}] + e^{\sqrt{2\lambda b-a}} E[e^{-\lambda T 1_{T_a > T_b}}] - e^{-\sqrt{2\lambda b-a}} E[e^{-\lambda T 1_{T_a > T_b}}]. \]

Observe that

\[ 2 \cosh(\sqrt{2\lambda b-a}) = E[M_T] \]

\[ = 2 \cosh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a \leq T_b}}] + 2 \cosh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a > T_b}}]. \]

Thus, we have

\[
\begin{aligned}
&\left\{ \cosh(\sqrt{2\lambda b+a}) = \cosh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a \leq T_b}}] + \cosh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a > T_b}}] \\
&- \sinh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a \leq T_b}}] - \sinh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a > T_b}}] \right. \\
&= -\sinh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a \leq T_b}}] + \sinh(\sqrt{2\lambda b-a}) E[e^{-\lambda T 1_{T_a > T_b}}]
\end{aligned}
\]

By using the formula

\[ \sinh(x+y) = \sinh(x) \cosh(y) + \sinh(y) \cosh(x), \]

we get

\[ E[e^{-\lambda T 1_{T_a \leq T_b}}] = \frac{\sinh(b \sqrt{2\lambda})}{\sinh((b-a) \sqrt{2\lambda})}. \]

3. By using dominated convergence theorem and the result in problem 2, we have

\[ P(T_a < T_b) = E[1_{T=a}] \]

\[ = \lim_{\lambda \to 0} E[e^{-\lambda T 1_{T_a \leq T_b}}] \]

\[ = \lim_{\lambda \to 0} \frac{\sinh(b \sqrt{2\lambda})}{\sinh((b-a) \sqrt{2\lambda})} \]

\[ = \frac{b}{b-a}. \]

3.3 Exercise 3.28

Let B be an \((\mathcal{F}_t)\)-Brownian motion started from 0. Let \(a > 0\) and

\[ \sigma_a = \inf \{ t \geq 0 \mid B_t \leq t - a \}. \]

1. Show that \(\sigma_a\) is a stopping time and that \(\sigma_a < \infty\) a.s.

2. Using an appropriate exponential martingale, show that, for every \(\lambda \geq 0\),

\[ E[e^{-\lambda \sigma_a}] = e^{-a(\sqrt{1+2\lambda}-1)}. \]

The fact that this formula remains valid for \(\lambda \in [-\frac{1}{2}, 0]\) can be obtained via an argument of analytic continuation.
3. Let $\mu \in \mathbb{R}$ and $M_t = e^{\mu B_t - \frac{\mu^2}{2} t}$. Show that the stopped martingale $M_{\sigma_a \wedge t}$ is closed if and only if $\mu \leq 1$.

Proof.

1. Since $\lim \inf_{t \to \infty} B_t = -\infty$ a.s., we see that $\lim \inf_{t \to \infty} (B_t - t) = -\infty$ a.s. and $\sigma_a < \infty$ a.s.

2. Given $\lambda \geq 0$. Set $\mu = 1 - \sqrt{1 + 2\lambda}$. Then $-\frac{\mu^2}{2} + \mu = -\lambda$ and $(M_t)_{t \geq 0}$ is a local martingale. Moreover, since

$$-a \leq B^{\sigma_a} - (\sigma_a \wedge t) < \infty$$

and

$$0 \leq e^{\mu(B^a - (\sigma_a \wedge t))} \leq e^{-\mu a}$$

for all $t \geq 0$, we see that $|M_t| \equiv |e^{\mu B^a - \frac{\mu^2}{2} \sigma_a \wedge t}| = |e^{\mu B^a - \mu(\sigma_a \wedge t)} e^{\mu(\sigma_a \wedge t) - \frac{\mu^2}{2} \sigma_a \wedge t}| \leq e^{-\mu a}$

for all $t \geq 0$ and therefore $M$ is an uniformly integrable martingale. By optional stopping theorem, we have

$$E[e^{\mu\sigma_a - \frac{\mu^2}{2} \sigma_a}] = E[e^{\mu B_a - \frac{\mu^2}{2} \sigma_a}] = 1.$$ 

Since

$$\mu = 1 - \sqrt{1 + 2\lambda}$$

and

$$\frac{\mu^2}{2} + \mu = -\lambda,$$

we get

$$E[e^{-\lambda \sigma_a}] = e^{\mu a} = e^{a \sqrt{1 + 2\lambda - 1}}.$$ 

Next, we show that the statement is true when $\lambda \in [-\frac{1}{2}, 0]$. Set $\Omega = \{ z \in \mathbb{C} \mid Re(z) > \frac{1}{2} \}$. Define $f : \Omega \mapsto \mathbb{Z}$ by

$$f(z) = E[e^{-z \sigma_a}].$$

Note that

$$\int_0^\infty \frac{1}{s^2} e^{-As^2 - \frac{B}{s} ds} = \frac{\sqrt{\pi} e^{-2AB}}{B}$$

for $A, B \geq 0$ and

$$P(\sigma_a \leq t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds.$$ 

For $z = c + id \in \Omega$, we have

$$|E[e^{-z \sigma_a}]| = \int_0^\infty e^{-zs} \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds \leq \int_0^\infty e^{-cs} \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds = \frac{ae^a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{s^2} e^{-\frac{1}{2} \frac{(a-c)^2}{s}} ds = \frac{ae^a}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2} e^{-\frac{a^2}{2}}} < \infty$$

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and, hence, \( f(z) \) is well-defined. Let \( \Gamma \) be a triangle in \( \Omega \). By using Fubini’s theorem, we have

\[
\int_{\Gamma} f(z) dz = \int_{\Omega} \int_{\Gamma} e^{-z\sigma_a} dz dP(dw) = 0.
\]

Thus, \( f(z) \) is holomorphic in \( \Omega \). Set \( g(z) = e^{-a(\sqrt{2z+1})} \). Then \( g(z) \) is holomorphic in \( \Omega \). Since \( f(z) = g(z) \) on the positive real line, we get \( g = f \) in \( \Omega \) and, hence,

\[
E[e^{-\lambda\sigma_a}] = e^{\mu a} = e^{-a(\sqrt{1+2\lambda}-1)}
\]

for \( \lambda \in (-\frac{1}{2}, 0] \). By monotone convergence theorem, we have

\[
E\left[e^{\frac{1}{2}\sigma_a}\right] = \lim_{\lambda \downarrow -\frac{1}{2}} E[e^{-\lambda\sigma_a}] = \lim_{\lambda \downarrow -\frac{1}{2}} e^{-a(\sqrt{1+2\lambda}-1)} = e^a
\]

and, hence,

\[
E[e^{-\lambda\sigma_a}] = e^{\mu a} = e^{-a(\sqrt{1+2\lambda}-1)}
\]

for \( \lambda \in [-\frac{1}{2}, 0] \).

3. Note that

\[
1 = E[M_{\sigma_a}] = E[e^{\mu(\sigma_a-a) - \frac{\mu^2}{2}\sigma_a}] = E[e^{-(\frac{\mu^2}{2} - \mu)a - \mu a}]
\]

if and only if

\[
E[e^{-(\frac{\mu^2}{2} - \mu)a}a] = e^{\mu a}
\]

Since \( \frac{\mu^2}{2} - \mu \geq -\frac{1}{2} \) for \( \mu \in \mathbb{R} \), we get, by the result in problem 2,

\[
E[e^{-(\frac{\mu^2}{2} - \mu)a}] = e^{-a(\sqrt{1+2\mu} - 1)} = \begin{cases} e^{-a(\mu-2)}, & \text{if } \mu > 1 \\ e^{a\mu}, & \text{if } \mu \leq 1 \end{cases}
\]

and, hence,

\[
1 = E[M_{\sigma_a}] \text{ if and only if } \mu \leq 1.
\]

Now, we show that \( M_{\sigma_a \wedge t} \) is closed if and only if \( \mu \leq 1 \).

It’s clear that

\[
1 = E[M_{0 \wedge \sigma_a}] = E[M_{\infty \wedge \sigma_a}] = E[M_{\sigma_a}]
\]

whenever \( M_{\sigma_a \wedge t} \) is closed. It remains to show that \( M_{\sigma_a \wedge t} \) is closed when \( 1 = E[M_{\sigma_a}] \).

Let \( t \geq 0 \). By using optional stopping theorem for supermartinale(Theorem 3.25), we have

\[
M_{t \wedge \sigma_a} \geq E[M_{\sigma_a} | \mathcal{F}_{t \wedge \sigma_a}], \text{ a.s.}
\]

If

\[
P(M_{t \wedge \sigma_a} > E[M_{\sigma_a} | \mathcal{F}_{t \wedge \sigma_a}] > 0,
\]

then we have

\[
1 = E[M_{0 \wedge \sigma_a}] = E[M_{t \wedge \sigma_a}] > E[E[M_{\sigma_a} | \mathcal{F}_{t \wedge \sigma_a}] = E[M_{\sigma_a}] = 1
\]

which is a contradiction. Thus, we have

\[
M_{t \wedge \sigma_a} = E[M_{\sigma_a} | \mathcal{F}_{t \wedge \sigma_a}], \text{ a.s.}
\]

This shows that \( M_{t \wedge \sigma_a} \) is closed.
3.4 Exercise 3.29

Let \((Y_t)_{t \geq 0}\) be a uniformly integrable martingale with continuous sample paths, such that \(Y_0 = 0\). We set \(Y_\infty = \lim_{t \to \infty} Y_t\). Let \(p \geq 1\) be a fixed real number. We say that Property (P) holds for the martingale \(Y\) if there exists a constant \(C\) such that, for every stopping time \(T\), we have

\[
E[|Y_\infty - Y_T|^p |\mathcal{F}_T] \leq C
\]

1. Show that Property (P) holds for \(Y\) if \(Y_\infty\) is bounded.

2. Let \(B\) be an \(\{\mathcal{F}_t\}\)-Brownian motion started from 0. Show that Property (P) holds for the martingale \(Y_t = B_{t\wedge 1}\).

3. Show that Property (P) holds for \(Y\), with the constant \(C\), if and only if, for any stopping time \(T\),

\[
E[|Y_T - Y_\infty|^p] \leq C P(T < \infty).
\]

4. We assume that Property (P) holds for \(Y\) with the constant \(C\). Let \(S\) be a stopping time and let \(Y^S\) be the stopped martingale defined by \(Y^S_t = Y_{S\wedge t}\). Show that Property (P) holds for \(Y^S\) with the same constant \(C\).

5. We assume in this question and the next one that Property (P) holds for \(Y\) with the constant \(C = 1\). Let \(a > 0\), and let \((R_n)_{n \geq 0}\) be the sequence of stopping times defined by induction by

\[
R_0 = 0 \quad \text{and} \quad R_{n+1} = \inf\{t \geq R_n | |Y_t - Y_{R_n}| \geq a\} \quad (\inf \emptyset = \infty).
\]

Show that, for every integer \(n \geq 0\),

\[
a^p P(R_{n+1} < \infty) \leq P(R_n < \infty).
\]

6. Infer that, for every \(x > 0\),

\[
P(\sup_{t \geq 0} Y_t > x) \leq 2^p 2^{-\frac{x}{T}}.
\]

Proof.

1. Since \((Y_t)_{t \geq 0}\) is an uniformly integrable martingale,

\[
Y_t = E[y_\infty |\mathcal{F}_t]
\]

for each \(0 \leq t \leq \infty\). Because \(Y_\infty\) is bounded, there exists \(C > 0\) such that a.s. \(|Y_t| \leq C\). Since the sample path is continuous, we have a.s. \(\sup_{t \geq 0} |Y_t| \leq C\) and therefore a.s. \(|Y_T| \leq C\). Thus, if \(p \geq 1\), then

\[
E[|Y_\infty - Y_T|^p |\mathcal{F}_T] \leq E[(|Y_\infty| + |Y_T|^p) |\mathcal{F}_T] \leq (2C)^p
\]

and therefore Property (P) holds for \(Y\).

2. First, note that \(Y_t\) is a uniformly integrable martingale, since \(Y_t = E[Y_1 |\mathcal{F}_t]\) for \(t \geq 1\).

Now, we show that Property (P) holds for the martingale \(Y_t = B_{t\wedge 1}\). First, we consider the case \(p = 1\). Let \(F \in \mathcal{F}_T\). Then

\[
E[E[|Y_T - Y_\infty||\mathcal{F}_T]|1_F] = E[|Y_T - Y_\infty|1_F] \leq E[|Y_\infty|1_F] + E[|Y_T|1_F].
\]

Since \(Y_t\) is a uniformly integrable martingale, \(Y_T = E[Y_\infty |\mathcal{F}_T]\) and, hence,

\[
E[|Y_T|1_F] = E[E[Y_\infty |\mathcal{F}_T]|1_F] \leq E[E[|Y_\infty| |\mathcal{F}_T]|1_F] = E[|Y_\infty|].
\]

Thus,

\[
E[E[|Y_T - Y_\infty||\mathcal{F}_T]|1_F] \leq 2E[|Y_\infty|]
\]

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for each \( F \in \mathcal{F}_T \). Since \( E[|Y_T - Y_\infty|] \) is \( \mathcal{F}_T \)-measurable, we get
\[
E[|Y_T - Y_\infty|] \leq 2E[|Y_\infty|]
\]
and therefore property (P) holds for the martingale \( Y_t = B_{t\land T} \) when \( p = 1 \).

Next, we suppose \( p > 1 \). By Doob’s inequality in \( L^p \), we get
\[
E[\sup_{t\geq 0} |Y_t|^p] \leq E[\sup_{0\leq t \leq 1} |B_t|^p] \leq \left( \frac{p}{p-1} \right)^p E[|B_1|^p]
\]
and therefore \( \sup_{t\geq 0} |Y_t|^p \) is in \( L^p \). Then, for each \( F \in \mathcal{F}_T \),
\[
\]
\[
\leq E([|Y_\infty| + |Y_T|]^p|F]
\]
\[
= E((2\sup_{t\geq 0} |Y_t|)^p|F]
\]
\[
= 2^p E[\sup_{t\geq 0} |Y_t|^p|F]
\]
\[
\leq 2^p E[|Y_\infty|^p]
\]
\[
\leq 2^p \left( \frac{p}{p-1} \right)^p E[|B_1|^p] < \infty
\]

Since \( E[|Y_\infty - Y_T|^p|\mathcal{F}_T] \) is \( \mathcal{F}_T \)-measurable, we get
\[
E[|Y_\infty - Y_T|^p|\mathcal{F}_T] \leq 2^p \left( \frac{p}{p-1} \right)^p E[|B_1|^p]
\]
and therefore property (P) holds for the martingale \( Y_t = B_{t\land T} \) when \( p > 1 \).

3. Suppose property (P) holds for the uniformly integrable martingale \((Y_t)_{t \geq 0}\). Since \( \{T < \infty\} \in \mathcal{F}_T \), we get
\[
E[|Y_\infty - Y_T|^p] = E[|Y_\infty - Y_T|^p 1_{T < \infty}] = E[E[|Y_\infty - Y_T|^p|\mathcal{F}_T]|1_{T < \infty}] \leq CP(T < \infty).
\]

Conversely, suppose that
\[
E[|Y_\infty - Y_T|^p] \leq CP(T < \infty)
\]
for each stopping time \( T \). Let \( T \) be any stopping time and \( F \in \mathcal{F}_T \). Then
\[
\]

Since \( E[|Y_\infty - Y_T|^p|\mathcal{F}_T] \) is \( \mathcal{F}_T \)-measurable, we get
\[
E[|Y_\infty - Y_T|^p|\mathcal{F}_T] \leq C
\]
and therefore property (P) holds for the martingale \((Y_t)_{t \geq 0}\).

4. Let \( S \) and \( T \) be stopping times. Since \((Y_t^S)_{t \geq 0}\) is an uniformly integrable martingale, \((Y_t^S)_{t \geq 0}\) and \((Y_t^T)_{t \geq 0}\) are also uniformly integrable martingales. Thus, we have
\[
Y_S^T = E[Y_{\infty}^T|\mathcal{F}_S] = E[Y_T|\mathcal{F}_S]
\]
and therefore
\[
Y_T^S = Y_{S\land T} = Y_S^T = E[Y_T|\mathcal{F}_S].
\]

Hence we get
\[
\]
\[
= E[E[Y_T|\mathcal{F}_S] - E[Y_\infty|\mathcal{F}_S]^p]
\]
\[
\leq E[|Y_T - Y_\infty|^p]
\]
\[
\leq CP(T < \infty).
\]

and therefore property (P) holds for \((Y_t^S)_{t \geq 0}\) with the same constant \( C \).
5. Given $a > 0$. By the definition of $\{R_n\}_{n \geq 0}$, we have $R_{n+1} \geq R_n$ for all $n \geq 0$. By considering uniformly integrable martingale $(Y_{t \geq 0}^{R_{n+1}})$ and using the result in problem 4, we get

$$E[|Y_{R_n+1} - Y_{R_n}|^p] = E[|Y_{R_n}^{R_{n+1}} - Y_{R_n}^{R_{n+1}}|^p] \leq P(R_n < \infty).$$

Since $|Y_{R_{n+1}} - Y_{R_n}| \geq a$ on $\{R_{n+1} < \infty\}$, we have

$$E[|Y_{R_n+1} - Y_{R_n}|^p] \geq a^p P(R_{n+1} < \infty)$$

and, hence,

$$a^p P(R_{n+1} < \infty) \leq P(R_n < \infty).$$

6. Observe that if $0 < x \leq 2$, then $2^{1-\frac{x}{2}} \geq 1$ and, hence, the inequality is true. Now, we suppose $x > 2$. Set

$$R_0 = 0 \quad \text{and} \quad R_{n+1} = \inf\{t \geq R_n | |Y_t - Y_{R_n}| \geq 2\}$$

for each $n \geq 0$. According the conclusion in problem 5, we get

$$P(R_n < \infty) \leq 2^{-np}$$

for all $n \geq 1$. Let $m$ be the smallest integer such that $2^m \geq x$. Then

$$P(\sup_{t \geq 0} Y_t > x) \leq P(R_{m-1} < \infty) \leq 2^{-(m-1)p} \leq 2^{(-\frac{x}{2}+1)p} = 2^p 2^{-\frac{x}{2}}.$$
Chapter 4
Continuous Semimartingales

4.1 Exercise 4.22
Let Z be a $\mathcal{F}_0$-measurable real random variable, and let M be a continuous local martingale. Show that the process $N_t = ZM_t$ is a continuous local martingale.

Proof.
Without loss of generality, we may assume $M_0 = 0$. Set
$$T_n = \inf \{ t \geq 0 | |N_t| \geq n \}$$
for each $n \geq 1$. Then $T_n$ is a stopping time for each $n \geq 1$. Clearly, $T_n \uparrow \infty$, $(T_n)$ reduce M, and $|ZM_{T_n}| \leq n$ for all $n \geq 1$. Thus, $ZM_{T_n}$ is bounded in $L^1$ for each $n \geq 1$. Now, we show that $ZM_{T_n}$ is a martingale for each $n \geq 1$. Fix $n \geq 1$. Choose a sequence of bounded simple function $\{Z_k\}$ such that $Z_k \rightarrow Z$ and $|Z_k| \leq |Z|$ for each $k \geq 1$ and for all $w \in \Omega$. Note that, $|Z_kM_{T_n}| \leq |ZM_{T_n}| \leq n$.

Fix $0 \leq s < t$. Let $\Gamma \in \mathcal{F}_s$. By Lebesgue’s dominated convergence theorem, we get
$$E[ZM_{T_n}^1|\Gamma] = \lim_{k \rightarrow \infty} E[Z_kM_{T_n}^1|\Gamma] = \lim_{k \rightarrow \infty} E[Z_kM_{T_n}^1] = E[ZM_{T_n}]$$
Thus,
$$ZM_{T_n} = E[ZM_{T_n}|\mathcal{F}_s]$$
for all $0 \leq s < t$ and, hence, $ZM_{T_n}$ is a martingale. Therefore $ZM$ is a continuous local martingale. 

4.2 Exercise 4.23
1. Let M be a martingale with continuous sample paths, such that $M_0 = 0$. We assume that $(M_t)_{t \geq 0}$ is also a Gaussian process. Show that, for every $t > 0$ and every $s > 0$, the random variable $M_{t+s} - M_t$ is independent of $\sigma(M_r, 0 \leq r \leq t)$.

2. Under the assumptions of question 1., show that there exists a continuous monotone nondecreasing function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\langle M, M \rangle_t = f(t)$ for all $t \geq 0$.

Proof.
1. Observe that
$$E[M_{s+t}M_t] = E[M_t^2]$$
for all $s > 0$ and $t > 0$. Since
$$E[(M_{t+s} - M_t)M_t] = E[M_t^2] - E[M_t^2] = 0$$
for all $0 \leq r \leq t$, we get span$\{M_{t+s} - M_t\}$ and span$\{M_r|0 \leq r \leq t\}$ are orthogonal. It followings form Theorem 1.9 that $M_{t+s} - M_t$ is independent of $\sigma(M_r, 0 \leq r \leq t)$.

2. Observe that if B is Brownian motion, $B$ is both continuous martingale and a Gaussian process. Moreover, we have
$$\langle B, B \rangle_t = t = E[B_t^2]$$
Therefore we consider the function
$$f(t) = E[M_t^2].$$
Now, we set \( \mathcal{F}_t = \sigma(M_r|0 \leq r \leq t) \) for all \( t \geq 0 \). First, we show that \( f(t) \) is a continuous monotone nondecreasing function. Let \( 0 \leq s < t \). Since

\[
M_s^2 = E[M_t|\mathcal{F}_s]^2 \leq E[M_t^2|\mathcal{F}_s],
\]

we have

\[
f(s) = E[M_s^2] \leq E[M_t^2] = f(t)
\]

and, hence, \( f(t) \) is monotone nondecreasing function. Let \( T > 0 \) and \( \{t_n\} \cup \{t\} \subseteq [0, T] \) such that \( t_n \to t \). By using Doob’s maximal inequality in \( L^2 \), we have

\[
E\left[ \sup_{0 \leq s \leq T} |M_s|^2 \right] \leq 4E[|M_T|^2] < \infty.
\]

By using dominated convergence theorem, we get

\[
\lim_{n \to \infty} f(t_n) = \lim_{n \to \infty} E[M_{t_n}^2] = E[M_t^2] = f(t)
\]

and, hence, \( f(t) \) is continuous.

Next, we show that \( \langle M, M \rangle_t = f(t) \) for all \( t \geq 0 \). Set \( \mathcal{N} \) to be the class of all \( (\sigma(M_t|t \geq 0), P) \)-negligible sets. That is,

\[
\mathcal{N} := \{ A : \exists A' \in \sigma(M_t|t \geq 0) \quad A \subseteq A' \text{ and } P(A') = 0 \}.
\]

Define

\[
\mathcal{G}_t := \sigma(M_s|s \leq t) \vee \sigma(\mathcal{N}) \quad t \geq 0
\]

and

\[
\mathcal{G}_\infty := \sigma(M_t|t \geq 0) \vee \sigma(\mathcal{N}) \quad t \geq 0.
\]

Then \( (\mathcal{G}_t)_{t \in [0, \infty]} \) is a complete filtration, \( \mathcal{G}_t \subseteq \mathcal{F}_t \) for every \( 0 \leq t \leq \infty \), \( M_t+s - M_t \perp \mathcal{G}_t \) for every \( t, s > 0 \), and \( (M_t)_{t \geq 0} \) is a \( (\mathcal{G}_t)_{t \in [0, \infty]} \)-martingale.

To show that \( \langle M, M \rangle_t = f(t) \) for every \( t \geq 0 \), it suffices to show that \( M_t^2 - f(t) \) is a \( (\mathcal{G}_t)_{t \in [0, \infty]} \)-continuous local martingale. Indeed, since

\[
\sum_{i=1}^{n} (M_{t_{i+1}} - M_{t_i})^2 \overset{P}{\to} \langle M, M \rangle_t,
\]

we see that finite variation process \( \langle (M, M)_t \rangle_{t \geq 0} \) does not depend on the filtration of \( (M_t)_{t \geq 0} \).

Now, we show that \( M_t^2 - f(t) \) is a \( (\mathcal{G}_t)_{t \in [0, \infty]} \)-martingale. Let \( 0 \leq s < t \). Observe that

\[
E[(M_t - M_s)^2|\mathcal{G}_s] = E[M_t^2 - M_s^2|\mathcal{G}_s]
\]

Since \( M_t - M_s \) is independent of \( \mathcal{G}_s \), we have

\[
E[(M_t - M_s)^2|\mathcal{G}_s] = E[(M_t - M_s)^2] = E[M_t^2 - M_s^2].
\]

Thus, if \( 0 \leq s < t \), we get

\[
\]

and therefore \( M_t^2 - f(t) \) is a \( (\mathcal{G}_t)_{t \in [0, \infty]} \)-martingale.

\[]
4.3 Exercise 4.24

Let $M$ be a continuous local martingale with $M_0 = 0$.

1. For every integer $n \geq 1$, we set $T_n = \inf\{t \geq 0 | |M_t| = n\}$. Show that, a.s.
   \[
   \{ \lim_{t \to \infty} M_t \text{ exists and finite } \} = \bigcup_{n \geq 1} \{ T_n = \infty \} \subseteq \{ \langle M, M \rangle_\infty < \infty \}.
   \]

2. We set $S_n = \inf\{t \geq 0 | \langle M, M \rangle_t = n\}$ for each $n \geq 1$. Show that, a.s.,
   \[
   \{ \langle M, M \rangle_\infty < \infty \} = \bigcup_{n \geq 1} \{ S_n = \infty \} \subseteq \{ \lim_{t \to \infty} M_t \text{ exists and finite } \}
   \]
   and conclude that
   \[
   \{ \lim_{t \to \infty} M_t \text{ exists and is finite } \} = \{ \langle M, M \rangle_\infty < \infty \}, \text{ a.s.}
   \]

Proof.

1. Since $M$ has continuous sample paths, we see that $T_n = \inf\{t \geq 0 | |M_t| \geq n\}$ and $(T_n)_{n \geq 1}$ reduces $M$ and, hence, $M_{T_n}^T$ is a uniformly integrable martingale for each $n \geq 1$. Thus, for each $n \geq 1$, $M_{T_n}^T$ exists a.s.
   Since $|M_{T_n}^T| \leq n$ for each $n \geq 1$, $M_{T_n}^T$ is bounded in $L^2$ and, hence, $E[\langle M_{T_n}^T, M_{T_n}^T \rangle_\infty] < \infty$. Thus, for each $n \geq 1$, $\langle M, M \rangle_{T_n} < \infty$ a.s.
   Set
   \[
   E = \bigcup_{n \geq 1} \{ M_{T_n}^T \text{ exists and } \langle M, M \rangle_{T_n} < \infty \}.
   \]
   Then $P(E) = 1$. To complete the proof, it suffices to show that the statement is true for each $w \in E$. Let
   \[
   w \in \{ \lim_{t \to \infty} M_t \text{ exists and finite } \} \bigcap E.
   \]
   Since $M(w)$ has continuous sample path and $M_\infty(w) < \infty$, there exists $K > 0$ such that $|M_t(w)| \leq K$ for all $t \geq 0$ and, hence, $T_m(w) = \infty$ for each $m > K$. Thus, $w \in E \bigcap (\bigcup_{n \geq 1} \{ T_n = \infty \})$. Conversely, let $w \in E$ and $T_m(w) = \infty$ for some $m \geq 1$. Then
   \[
   M_\infty(w) = M_{T_m}^T(w) \text{ exists}
   \]
   and
   \[
   |M_t(w)| = |M_{T_m}^T(w)| < m \text{ for all } 0 \leq t \leq \infty.
   \]
   Thus, $w \in \{ M_\infty \text{ exists and } M_\infty < \infty \} \bigcap E$. Moreover, since $w \in E$, we have
   \[
   \langle M, M \rangle_\infty(w) = \langle M, M \rangle_{T_m}(w) < \infty
   \]
   Thus, we get
   \[
   E \bigcap \{ \lim_{t \to \infty} M_t \text{ exists and finite } \} = E \bigcap (\bigcup_{n \geq 1} \{ T_n = \infty \}) \subseteq E \bigcap \{ \langle M, M \rangle_\infty < \infty \}
   \]
   and therefore a.s.
   \[
   \{ \lim_{t \to \infty} M_t \text{ exists and finite } \} = \bigcup_{n \geq 1} \{ T_n = \infty \} \subseteq \{ \langle M, M \rangle_\infty < \infty \}.
   \]
2. Since $\langle M, M \rangle$ is an increasing process, it’s clear that

$$\{ \langle M, M \rangle_\infty < \infty \} = \bigcup_{n \geq 1} \{ S_n = \infty \}.$$ 

Let $n \geq 1$. Then

$$\langle M^{S_n}, M^{S_n} \rangle_t = \langle M, M \rangle_{S_n \wedge t} \leq n$$

for all $t \geq 0$ and, hence, $E[\langle M^{S_n}, M^{S_n} \rangle_\infty] \leq n$. Thus, we see that $M^{S_n}$ is a $L^2$ bounded martingale and, hence, $\lim_{t \to \infty} M^{S_n}_t$ exists and finite (a.s.). Set

$$F = \bigcup_{n \geq 1} \{ \lim_{t \to \infty} M^{S_n}_t \text{ exists and is finite} \}.$$ 

Then $P(F) = 1$. Fix $w \in F \cap (\bigcup_{n \geq 1} \{ S_n = \infty \})$. Then $S_m(w) = \infty$ for some $m \geq 1$ and, hence,

$$\lim_{t \to \infty} M_t(w) = \lim_{t \to \infty} M^{S_m}_t(w)$$

exists and is finite. Thus, a.s.,

$$\{ \langle M, M \rangle_\infty < \infty \} = \bigcup_{n \geq 1} \{ S_n = \infty \} \subseteq \{ \lim_{t \to \infty} M_t \text{ exists and is finite} \}.$$ 

Combining the result with the above, we get

$$\{ \lim_{t \to \infty} M_t \text{ exists and finite} \} = \{ \langle M, M \rangle_\infty < \infty \}, \text{ a.s.}$$

\[\square\]

### 4.4 Exercise 4.25

For every integer $n \geq 1$, let $M^n = (M^n_t)_{t \geq 0}$ be a continuous local martingale with $M^n_0 = 0$. We assume that

$$\lim_{n \to \infty} \langle M^n, M^n \rangle_\infty = 0$$

in probability.

1. Let $\epsilon > 0$, and, for every $n \geq 1$, let

$$T^n_\epsilon = \inf\{ t \geq 0 | \langle M^n, M^n \rangle_t \geq \epsilon \}.$$ 

Justify the fact that $T^n_\epsilon$ is a stopping time, then prove that the stopped continuous local martingale

$$M^n_{t \wedge T^n_\epsilon}, \forall t \geq 0$$

is a true martingale bounded in $L^2$.

2. Show that

$$E[\sup_{0 \leq t} |M^n_{t \wedge T^n_\epsilon}|^2] \leq 4\epsilon.$$  

3. Writing, for every $a > 0$,

$$P(\sup_{t \geq 0} |M^n_t| \geq a) \leq P(\sup_{t \geq 0} |M^n_{t \wedge T^n_\epsilon}| \geq a) + P(T^n_\epsilon < \infty),$$

show that

$$\lim_{n \to \infty} (\sup_{t \geq 0} |M^n_t|) = 0$$

in probability.
Proof.

1. Since \( \langle M^n, M^n \rangle \) has continuous sample paths, it follows from proposition 3.9 (iii) that

\[
T^n_\epsilon = \inf\{t \geq 0|\langle M^n, M^n \rangle_t \in [\epsilon, \infty)\}
\]

is a stopping time. Hence \( M^{n,\epsilon} = (M^n)^{T^n_\epsilon} \) is a continuous local martingale with

\[
\langle M^{n,\epsilon}, M^{n,\epsilon} \rangle_\infty \leq \epsilon.
\]

Thus, \( M^{n,\epsilon} \) is a \( L^2 \) bounded martingale.

2. Since \( (M^{n,\epsilon}_t)_{t \geq 0} \) is a martingale bounded in \( L^2 \), we see that

\[
E[(M^{n,\epsilon}_\infty)^2] = E[\langle M^{n,\epsilon}, M^{n,\epsilon} \rangle_\infty] \leq \epsilon.
\]

By Doob's maximal inequality, we get

\[
E[\sup_{0 \leq s \leq t} |M^{n,\epsilon}_s|^2] \leq 4E[|M^{n,\epsilon}_t|^2]
\]

for each \( t > 0 \). Since \( M^{n,\epsilon} \) is a martingale, we see that

\[
E[(M^{n,\epsilon}_s)^2] \leq E[(M^{n,\epsilon}_t)^2]
\]

for each \( s \leq t \). Thus,

\[
E[\sup_{0 \leq s \leq t} |M^{n,\epsilon}_s|^2] \leq 4E[|M^{n,\epsilon}_t|^2] \leq 4E[|M^{n,\epsilon}_\infty|^2] \leq 4\epsilon.
\]

By the Monotone convergence theorem, we have

\[
E[\sup_{s \geq 0} |M^{n,\epsilon}_s|^2] \leq 4\epsilon.
\]

3. Given \( a > 0 \) and \( \epsilon > 0 \). It’s clear that

\[
P(\sup_{t \geq 0} |M^n_t| \geq a) \leq P(\sup_{t \geq 0} |M^n_t| \geq a, T^n_\epsilon = \infty) + P(T^n_\epsilon < \infty)
\]

\[
= P(\sup_{t \geq 0} |M^{n,\epsilon}_t| \geq a, T^n_\epsilon = \infty) + P(T^n_\epsilon < \infty)
\]

\[
\leq P(\sup_{t \geq 0} |M^{n,\epsilon}_t| \geq a) + P(T^n_\epsilon < \infty).
\]

Note that

\[
P(\sup_{t \geq 0} |M^{n,\epsilon}_t| \geq a) \leq \frac{1}{a^2} E[\sup_{0 \leq t} |M^{n,\epsilon}_t|^2] \leq \frac{4\epsilon}{a^2}
\]

and

\[
P(T^n_\epsilon < \infty) = P((M^n, M^n)_\infty \geq \epsilon).
\]

Thus,

\[
P(\sup_{t \geq 0} |M^n_t| \geq a) \leq \frac{4\epsilon}{a^2} + P((M^n, M^n)_\infty \geq \epsilon).
\]

By letting \( n \to \infty \) and then \( \epsilon \downarrow 0 \), we get

\[
\lim_{n \to \infty} P(\sup_{t \geq 0} |M^n_t| \geq a) = 0.
\]

Since \( a \) is arbitrary, we have

\[
\lim_{n \to \infty} \sup_{t \geq 0} |M^n_t| = 0 \text{ in probability.}
\]
4.5 Exercise 4.26

1. Let $A$ be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$) such that $A_\infty < \infty$ a.s., and let $Z$ be an integrable random variable. We assume that, for every stopping time $T$,

$$E[A_\infty - AT] \leq E[Z 1_{\{T < \infty\}}].$$

Show, by introducing an appropriate stopping time, that, for every $\lambda > 0$,

$$E[(A_\infty - \lambda) 1_{\{A_\infty > \lambda\}}] \leq E[Z 1_{\{A_\infty > \lambda\}}].$$

2. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuously differentiable monotone increasing function such that $f(0) = 0$ and set $F(x) = \int_0^x f(t)dt$ for each $x \geq 0$. Show that, under the assumptions of question 1., one has

$$E[F(A_\infty)] \leq E[Zf(A_\infty)].$$

3. Let $M$ be a (true) martingale with continuous sample paths and bounded in $L^2$ such that $M_0 = 0$, and let $M_\infty$ be the almost sure limit of $M_t$ as $t \to \infty$. Show that the assumptions of question 1 hold when $A_t = (M, M)_t$ and $Z = M_\infty^2$. Infer that, for every real $q \geq 1$,

$$E[(\langle M, M \rangle_\infty)^{q+1}] \leq (q + 1)E[(\langle M, M \rangle_\infty)^q M^2_\infty].$$

4. Let $p \geq 2$ be a real number such that $E[(\langle M, M \rangle_\infty)^p] < \infty$. Show that

$$E[(\langle M, M \rangle_\infty)^p] \leq p^p E[|M_\infty|^{2p}].$$

5. Let $N$ be a continuous local martingale such that $N_0 = 0$, and let $T$ be a stopping time such that the stopped martingale $N^T$ is uniformly integrable. Show that, for every real $p \geq 2$,

$$E[(\langle N, N \rangle_T)^p] \leq p^p E[|N_T|^{2p}].$$

6. Give an example showing that this result may fail if $N^T$ is not uniformly integrable.

Proof.

1. Set $T = \inf\{t \geq 0|A_t > \lambda\}$. Then $\{T < \infty\} = \{A_\infty > \lambda\}$ and therefore

$$E[Z 1_{\{A_\infty > \lambda\}}] = E[Z 1_{\{T < \infty\}}] \geq E[A_\infty - AT] = E[(A_\infty - AT) 1_{\{T < \infty\}}] = E[(A_\infty - \lambda) 1_{\{T < \infty\}}] = E[(A_\infty - \lambda) 1_{\{A_\infty > \lambda\}}].$$

2. Note that

$$F(x) = xf(x) - \int_0^x \lambda f'(\lambda)d\lambda$$

and $f'(\lambda) \geq 0$ for all $x, \lambda \geq 0$. Since

$$1_{\{A_\infty > \lambda\}} = \{(w, \lambda) \in \Omega \times \mathbb{R}_+|A_\infty > \lambda\} = \bigcup_{q \in \mathbb{Q}_+} (\{A_\infty > q\} \cap [0, q]) \in \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}_+}$$

for all $\lambda \in \mathbb{R}_+$, we see that $1_{\{A_\infty > \lambda\}}(w, \lambda)f'(\lambda)$ is $\mathcal{F} \otimes \mathcal{B}_{\mathbb{R}_+}$-measurable and, hence,

$$E[\int_0^\infty 1_{\{A_\infty > \lambda\}}f'(\lambda)d\lambda] = E[\int_0^{A_\infty} f'(\lambda)d\lambda].$$
is well-defined. Then

\[ E[F(A_\infty)] \]

\[ = E[A_\infty f(A_\infty)] - E[\int_0^{A_\infty} \lambda f'(\lambda) d\lambda] \]

\[ = E[A_\infty \int_0^\infty 1_{\{A_\infty > \lambda\}} f'(\lambda) d\lambda] - E[\int_0^\infty 1_{\{A_\infty > \lambda\}} \lambda f'(\lambda) d\lambda] \]

\[ = \int_0^\infty E[A_\infty 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda - \int_0^\infty E[A 1_{\{A > \lambda\}}] f'(\lambda) d\lambda \]

\[ \leq \int_0^\infty E[Z 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda \]

By using Fubini’s theorem, we get

\[ \int_0^\infty E[Z 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda = E[Z \int_0^\infty 1_{\{A_\infty > \lambda\}} f'(\lambda) d\lambda] = E[Z f(A_\infty)] \]

and, hence,

\[ E[F(A_\infty)] \leq E[Z f(A_\infty)]. \]

3. First, we show that the assumptions of question 1. hold when \( A_t = \langle M, M \rangle_t \) and \( Z = M_{2\infty}. \) Let \( T \) be any stopping time. Since \( M \) is \( L^2 \)-bounded martingale, we see that \( M^2 - \langle M, M \rangle \) is an uniformly integrable martingale and, hence,

\[ E[M_{2\infty}^2 - \langle M, M \rangle_T] = E[M_{2\infty}^2 - \langle M, M \rangle_{\infty}]. \]

Thus,

\[ E[\langle M, M \rangle_{\infty} - \langle M, M \rangle_T] = E[M_{2\infty}^2 - M_T^2] \]

\[ = E[(M_{2\infty}^2 - M_T^2) 1_{\{T<\infty\}}] \]

\[ \leq E[M_{2\infty}^2 1_{\{T<\infty\}}] \]

and therefore

\[ E[A_{\infty} - AT] \leq E[Z 1_{\{T<\infty\}}]. \]

Next, by taking \( F(x) = x^{q+1} \) in problem 2, we have

\[ E[(\langle M, M \rangle_{\infty})^{q+1}] \leq (q + 1) E[(\langle M, M \rangle_{\infty})^q M_{2\infty}^2]. \]

4. Given \( p \geq 2. \) Set \( q = \frac{p}{p-1}. \) Then \( \frac{1}{p} + \frac{1}{q} = 1. \) By Holder’s inequality, we get

\[ E[(\langle M, M \rangle_{\infty})^p] \leq p E[(\langle M, M \rangle_{\infty})^{p-1} M_{2\infty}^2] \]

\[ \leq p E[(\langle M, M \rangle_{\infty})^{q(p-1)}]^{\frac{1}{2}} E[|M_{\infty}|^{2p}]^{\frac{1}{2}} \]

\[ = p E[(\langle M, M \rangle_{\infty})^p]^{\frac{1}{2}} E[|M_{\infty}|^{2p}]^{\frac{1}{2}}. \]

By assumption, we have \( E[(\langle M, M \rangle_{\infty})^p] < \infty \) and, hence,

\[ E[(\langle M, M \rangle_{\infty})^p]^{q-1} \leq p^q E[|M_{\infty}|^{2p}]^{\frac{q}{2}}. \]

That is,

\[ E[(\langle M, M \rangle_{\infty})^p] \leq p^\frac{q-1}{p} E[|M_{\infty}|^{2p}]^{\frac{q-1}{p}} = p^p E[|M_{\infty}|^{2p}]. \]

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5. Given $p \geq 2$. If $E[|N_T|^{2p}] = \infty$, then there is nothing to prove. Now, we suppose $E[|N_T|^{2p}] < \infty$. Observe that $N^T$ is a $L^{2p}$-bounded martingale. Indeed, since $N^T$ is uniformly integrable martingale, one has

$$N_{T \wedge t} = E[N_T | \mathcal{F}_t]$$

for all $t \geq 0$ and, hence,

$$E[|N_{T \wedge t}|^{2p}] \leq E[|N_T|^{2p}] < \infty$$

for all $t \geq 0$. Thus we see that $N^T$ is a $L^{2p}$-bounded martingale, which implies that $N^T$ is a $L^2$-bounded martingale. Set

$$\tau_n = \{ t \geq 0 | \langle N^T, N^T \rangle_t \geq n \}$$

for each $n \geq 1$. Since $N^T$ is uniformly integrable martingale, we have

$$N_{T \wedge \tau_n} = E[N_T | \mathcal{F}_{T \wedge \tau_n}]$$

for each $n \geq 1$ and, hence,

$$E[|N_{T \wedge \tau_n}|^{2p}] \leq E[|N_T|^{2p}]$$

for each $n \geq 1$. Note that $N^{T \wedge \tau_n} = (N^T)^{\tau_n}$ is a $L^2$-martingale with continuous sample paths and

$$E[\langle N^{T \wedge \tau_n}, N^{T \wedge \tau_n} \rangle_\infty] \leq n^p.$$ 

By using the result in problem 2, we get

$$E[\langle N, N \rangle_{T \wedge \tau_n}^p] = E[\langle N^{T \wedge \tau_n}, N^{T \wedge \tau_n} \rangle_\infty^p] \leq p^p E[|N_{T \wedge \tau_n}|^{2p}]$$

for each $n \geq 1$. By using monotone convergence theorem, we have

$$E[\langle N, N \rangle_T^p] = \lim_{n \to \infty} E[\langle N, N \rangle_{T \wedge \tau_n}^p] \leq \lim_{n \to \infty} \sup_{n \to \infty} p^p E[|N_{T \wedge \tau_n}|^{2p}] \leq p^p E[|N_T|^{2p}].$$

6. Let $a \neq 0$, $p \geq 1$, and $B$ is a Brownian motion starting from 0. Then $B$ is a martingale and $\langle B, B \rangle_t = t$. Set $T = \inf \{ t \geq 0 | B_t = a \}$. Note that $T < \infty$ (a.s.) and

$$E[|B_T|^{2p}] = |a|^{2p} < \infty.$$ 

By using the result in Chapter 2(Corollary 2.22), we see that $E[T] = \infty$ and, hence, $E[T^p] = \infty$. Thus,

$$\infty = E[T^p] = E[\langle B, B \rangle_T^p] > p^p |a|^{2p} = p^p E[|B_T|^{2p}]$$

and, hence, the inequality fails. Finally, $B^T$ isn’t uniformly integrable. Indeed, if $B^T$ is uniformly integrable, then

$$0 = E[B^T_T] = E[B^T_{\infty}] = E[B_T] = a \neq 0$$

which is a contradiction.

\[\Box\]

### 4.6 Exercise 4.27

Let $(X_t)_{t \geq 0}$ be an adapted process with continuous sample paths and taking nonnegative values. Let $(A_t)_{t \geq 0}$ be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$). We consider the following condition:

(D) For every bounded stopping time $T$, we have $E[X_T] \leq E[A_T]$.

1. Show that, if $M$ is a square integrable martingale with continuous sample paths and $M_0 = 0$, the condition (D) holds for $X_t = M_t^2$ and $A_t = \langle M, M \rangle_t$. 

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2. Show that the conclusion of the previous question still holds if one only assumes that \( M \) is a continuous local martingale with \( M_0 = 0 \).

3. We set \( X_t^* = \sup_{s \leq t} X_s \). Show that, under the condition \( (D) \), we have, for every bounded stopping time \( S \) and every \( c > 0 \),

\[
P(X_S^* \geq c) \leq \frac{1}{c} E[A_S].
\]

4. Infer that, still under the condition \( (D) \), one has, for every (finite or not) stopping time \( S \),

\[
P(X_S^* > c) \leq \frac{1}{c} E[A_S].
\]

(when \( S \) takes the value \( \infty \), we of course define \( X_S^* = \sup_{s \geq 0} X_s \))

5. Let \( c > 0 \) and \( d > 0 \), and \( S = \inf\{t \geq 0 | A_t \geq d\} \). Let \( T \) be a stopping time. Noting that

\[
\{X_T^* > c\} \subseteq \{X_{T \wedge S}^* > c\} \cup \{A_T \geq d\}.
\]

Show that, under the condition \( (D) \), one has

\[
P(X_T^* > c) \leq \frac{1}{c} E[A_T \wedge d] + P(A_T \geq d).
\]

6. Use questions (2) and (5) to verify that, if \( M^{(n)} \) is a sequence of continuous local martingales and \( T \) is a stopping time such that \( \langle M^{(n)}, M^{(n)} \rangle_T \) converges in probability to 0 as \( n \to \infty \), then,

\[
\lim_{n \to \infty} (\sup_{s \leq T} |M_s^{(n)}|) = 0, \text{ in probability.}
\]

**Proof.**

1. Let \( T \) be a bounded stopping time. Since \( M \) is a \( L^2 \)-bounded martingale, we see that \( M^2 - \langle M, M \rangle \) is uniformly integrable and, hence,

\[
E[M_T^2 - \langle M, M \rangle_T] = E[M_0^2 - \langle M, M \rangle_0] = 0.
\]

Thus,

\[
\]

2. Let \( T \) be a bounded stopping time. Set

\[
\tau_n = \inf\{t \geq 0 | |M_t| \geq n\}
\]

for each \( n \geq 1 \). Then \( \tau_n \to \infty \) as \( n \to \infty \), \( (\tau_n) \) reduce \( M \), and \( M^{\tau_n} \) is a bounded martingale for each \( n \geq 1 \). By (1), we have

\[
E[M_{T \wedge \tau_n}^2] \leq E[\langle M, M \rangle_{T \wedge \tau_n}]
\]

for each \( n \geq 1 \). By Fatou’s lemma and monotone convergence theorem, we get

\[
E[(M_T)^2] \leq \liminf_{n \to \infty} E[(M_{T \wedge \tau_n})^2] = \lim_{n \to \infty} E[\langle M, M \rangle_{T \wedge \tau_n}] = E[\langle M, M \rangle_T].
\]

3. Given a bounded stopping time \( S \) and \( c > 0 \). Set \( R = \inf\{t \geq 0 | X_t \geq c\} \) and \( T = S \wedge R \). According to the assumption, we have

\[
E[X_T] \leq E[A_T] \leq E[A_S].
\]

Note that

\[
\{T = R\} = \{R \leq S\} = \{X_S^* \geq c\}.
\]
Since $X$ is continuous and $S$ is bounded, we see that

\[ X_R = c \text{ on } \{T = R\} \]

and, hence,

\[ E[X_T 1_{\{T=R\}}] = cP(T = R) = cP(X^*_S \geq c). \]

Therefore

\[ P(X^*_S \geq c) = \frac{1}{c} E[X_T 1_{\{T=R\}}] \leq \frac{1}{c} E[X_T] \leq \frac{1}{c} E[A_S]. \]

4. Given a stopping time $S$ (finite or not) and $c > 0$. Set $S_n = S \land n$. Then $S_n \uparrow S$ and $S_n$ is a bounded stopping time for each $n \geq 1$. By using the result in problem 3, we get

\[ P(X^*_S > c) \leq \frac{1}{c} E[A_{S_n}], \]

By using monotone convergence theorem, we get

\[ E[A_S] = \lim_{n \to \infty} E[A_{S_n}]. \]

Note that

\[ \{X^*_{S_n} > c\} \subseteq \{X^*_{S_{n+1}} > c\} \]

for each $n \geq 1$ and

\[ \bigcup_{n \geq 1} \{X^*_{S_n} > c\} = \{X^*_S > c\}. \]

Thus

\[ P(X^*_S > c) = \lim_{n \to \infty} P(X^*_{S_n} > c) \leq \frac{1}{c} \lim_{n \to \infty} E[A_{S_n}] = \frac{1}{c} E[A_S]. \]

5. Note that

\[ \{X^*_T > c\} \subseteq \{AT < d, X^*_T > c\} \bigcup \{AT \geq d\} \]

\[ \subseteq \{T \leq S, X^*_T \land S > c\} \bigcup \{AT \geq d\} \]

\[ \subseteq \{X^*_T \land S > c\} \bigcup \{AT \geq d\}. \]

and, hence,

\[ P(X^*_T > c) \leq P(X^*_{T \land S} > c) + P(AT \geq d). \]

Since $A_{S \land T} = AT \land d$, by using the result in problem 4, we get

\[ P(X^*_{S \land T} > c) \leq \frac{1}{c} E[A_{T \land S}] = \frac{1}{c} E[AT \land d]. \]

and, so,

\[ P(X^*_T > c) \leq \frac{1}{c} E[AT \land d] + P(AT \geq d). \]

6. Given $\epsilon > 0$. Let $d > 0$. Set $X^{(n)} = (M^{(n)})^2$ and $A^{(n)} = \langle M^{(n)}, M^{(n)} \rangle$. Then $A^{(n)}_T \to 0$ in probability. By using the result in problem 5, we get

\[ P(\sup_{0 \leq s \leq T} |M^{(n)}_s|^2 > \epsilon) \leq \frac{1}{\epsilon} E[A^{(n)}_T \land d] + P(A^{(n)}_T \geq d) \leq \frac{d}{\epsilon} + P(A^{(n)}_T \geq d). \]

By letting $n \to \infty$ and $d \downarrow 0$, we have

\[ \lim_{n \to \infty} P(\sup_{0 \leq s \leq T} |M^{(n)}_s| > \sqrt{\epsilon}) = \lim_{n \to \infty} P(\sup_{0 \leq s \leq T} |M^{(n)}_s|^2 > \epsilon) = 0 \]

and therefore

\[ \lim_{n \to \infty} (\sup_{s \leq T} |M^{(n)}_s|) = 0, \text{ in probability.} \]

\[ \square \]
Chapter 5

Stochastic Integration

5.1 Exercise 5.25

Let $B$ be an $(\mathcal{F}_t)$-Brownian motion with $B_0 = 0$, and let $H$ be an adapted process with continuous sample paths. Show that $\frac{1}{B_t} \int_0^t H_s dB_s$ converges in probability when $t \to 0$ and determine the limit.

Proof.

To determine the limit of $\frac{1}{B_t} \int_0^t H_s dB_s$, consider the special case

$$H_s(w) = \sum_{i=0}^{p-1} H(i)(w)1_{(t_i,t_{i+1})}(s),$$

where $H(i)$ be $\mathcal{F}_{t_i}$-measurable and $0 < t < t_1$. We see that

$$\frac{1}{B_t} \int_0^t H_s dB_s = \frac{1}{B_t} \left( \sum_{i=0}^{p-1} H(i)(B_{t_{i+1} \wedge t} - B_{t_i \wedge t}) \right) = \frac{1}{B_t} H(0)B_t = H(0).$$

From the above observation, we will show that $\frac{1}{B_t} \int_0^t H_s dB_s \overset{p}{\to} H_0$ and we may suppose that $H_0 = 0$.

First, we consider the case that $H$ is bounded. By Cauchy–Schwarz’s inequality and Jensen’s inequality, we get

$$E[|\frac{1}{B_t} \int_0^t H_s dB_s|^\frac{1}{2}] \leq E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} E[(\int_0^t H_s dB_s)^2]^\frac{1}{2} \leq E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} E[(\int_0^t H_s dB_s)^2]^\frac{1}{2} = E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} E[\int_0^t H_s^2 ds]^\frac{1}{2} \leq E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} E[\sup_{0 \leq s \leq t} H_s^2 \times t]^\frac{1}{2} \leq E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} E[\sup_{0 \leq s \leq t} H_s^2]^\frac{1}{2} t^\frac{1}{2}. $$

Note that

$$E[|B_t|^{-\frac{1}{2}}] \frac{1}{2} = (2 \int_0^\infty \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx)^\frac{1}{2} = (2 \int_0^\infty \frac{1}{\sqrt{y}} (2t)^\frac{1}{2} \frac{1}{\sqrt{\pi}} e^{-y^2 / 2t} dy)^\frac{1}{2} = c \times t^{-\frac{1}{2}},$$

where $0 < c = (\frac{2}{2^\frac{1}{4} \sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{y}} e^{-y^2} dy)^\frac{1}{2} < \infty$. By Lebesgue dominated convergence theorem, we shows that

$$E[\sup_{0 \leq s \leq t} H_s^2]^\frac{1}{2} \rightarrow 0 \text{ as } t \rightarrow 0^+$$

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and therefore

\[
P(\frac{1}{B_t} \int_0^t H_s dB_s \geq \epsilon) \leq \frac{1}{\epsilon^\frac{1}{4}} E[\frac{1}{B_t} \int_0^t H_s dB_s \frac{1}{\epsilon}] \\
\leq \frac{1}{\epsilon^\frac{1}{4}} E[|B_t|^{-\frac{1}{2}} \frac{1}{\epsilon} E[\sup_{0 \leq s \leq t} H_s^2] \frac{1}{\epsilon} t^{\frac{1}{2}}] \\
\leq \frac{1}{\epsilon^\frac{1}{4}} c \times t^{\frac{1}{2}} E[\sup_{0 \leq s \leq t} H_s^2] \frac{1}{\epsilon} t^{\frac{1}{2}} \\
= \frac{1}{\epsilon^\frac{1}{4}} c E[\sup_{0 \leq s \leq t} H_s^2] \frac{1}{\epsilon} \to 0 \text{ as } t \to 0^+.
\]

Next, we prove the statement for unbounded case. Set

\[
H^{(R)}_s(w) = \begin{cases} 
H_s(w) & \text{if } |H_s(w)| < R \\
R, & \text{if } H_s(w) \geq R \\
-R, & \text{if } H_s(w) \leq -R.
\end{cases}
\]

Then \(H^{(R)}_s(w)\) is an adapted process with continuous sample paths. Now, we show that, for \(0 < a < 1\), a.s.

\[
\int_0^a H_s dB_s = \int_0^a H^{(R)}_s dB_s \text{ in } \{ \sup_{0 \leq s \leq 1} |H_s| < R \}.
\]

That is,

\[
P(\int_0^a H_s dB_s = \int_0^a H^{(R)}_s dB_s, \sup_{0 \leq s \leq 1} |H_s| < R) = 1.
\]

Given \(0 < a < 1\). Note that, if \(0 = t_0 < \ldots < t_p\) and \(w \in \{ \sup_{0 \leq s \leq 1} |H_s| < R \}, \) then

\[
\sum_{i=0}^{p-1} H^{(R)}(w)(B_{t_{i+1} \wedge a}(w) - B_{t_i \wedge a}(w)) = \sum_{i=0}^{p-1} H^{(R)}(w)(B_{t_{i+1} \wedge a}(w) - B_{t_i \wedge a}(w)).
\]

Choose \(0 = t_0^a < \ldots < t_p^a = a\) of subdivisions of \([0, a]\) whose mesh tends to 0. By using Proposition 5.9, we have

\[
A_n \equiv \sum_{i=0}^{p_n-1} H^{(R)}_t(B_{t_{i+1}^a \wedge a} - B_{t_i^a \wedge a}) \to \int_0^a H_s dB_s \text{ in probability}
\]

and

\[
B_n \equiv \sum_{i=0}^{p_n-1} H^{(R)}_t(B_{t_{i+1}^a \wedge a} - B_{t_i^a \wedge a}) \to \int_0^a H^{(R)}_s dB_s \text{ in probability}.
\]

Choose some subsequences \(A_{n_k}\) and \(B_{n_k}\) such that a.s.

\[
A_{n_k} \to \int_0^a H_s dB_s
\]

and

\[
B_{n_k} \to \int_0^a H^{(R)}_s dB_s.
\]

Since \(A_{n_k} = B_{n_k}\) in \(\{ \sup_{0 \leq s \leq 1} |H_s| < R \}\), we see that a.s.

\[
\int_0^a H_s dB_s = \int_0^a H^{(R)}_s dB_s \text{ in } \{ \sup_{0 \leq s \leq 1} |H_s| < R \}.
\]
Given $\epsilon > 0$. Let $R > 0$ and $0 < t < 1$. Then

$$P\left( \left| \frac{1}{B_t} \int_0^t H_s dB_s \right| \geq \epsilon \right) \leq P\left( \sup_{0 \leq s \leq 1} |H_s| < R, \left| \frac{1}{B_t} \int_0^t H_s dB_s \right| \geq \epsilon \right) + P\left( \sup_{0 \leq s \leq 1} |H_s| \geq R \right)$$

$$= P\left( \sup_{0 \leq s \leq 1} |H_s| < R, \left| \frac{1}{B_t} \int_0^t H_s dB_s \right| \geq \epsilon \right) + P\left( \sup_{0 \leq s \leq 1} |H_s| \geq R \right)$$

$$\leq P\left( \frac{1}{B_t} \int_0^t H_s(R) dB_s \geq \epsilon \right) + P\left( \sup_{0 \leq s \leq 1} |H_s| \geq R \right).$$

By using the result in first case, we get

$$\lim_{t \to 0^+} P\left( \frac{1}{B_t} \int_0^t H_s(R) dB_s \geq \epsilon \right) = 0.$$

Because $H$ is continuous and $H_0 = 0$, we see that

$$P\left( \sup_{0 \leq s \leq 1} |H_s| \geq R \right) \to 0 \text{ as } R \to \infty.$$

By letting $t \to 0^+$ and then $R \to \infty$, we get

$$P\left( \frac{1}{B_t} \int_0^t H_s dB_s \geq \epsilon \right) \to 0 \text{ as } t \to 0^+.$$

### 5.2 Exercise 5.26

1. Let $B$ be a one-dimensional $(\mathcal{F}_t)$-Brownian motion with $B_0 = 0$. Let $f$ be a twice continuously differentiable function on $\mathbb{R}$, and let $g$ be a continuous function on $\mathbb{R}$. Verify that the process

$$X_t = f(B_t) e^{-\int_0^t g(B_s) ds}$$

is a semimartingale, and give its decomposition as the sum of a continuous local martingale and a finite variation process.

2. Prove that $X$ is a continuous local martingale if and only if the function $f$ satisfies the differential equation

$$f'' = 2fg.$$

3. From now on, we suppose in addition that $g$ is nonnegative and vanishes outside a compact subinterval of $(0, \infty)$. Justify the existence and uniqueness of a solution $f_1$ of the equation $f'' = 2fg$ such that $f_1(0) = 1$ and $f_1'(0) = 0$. Let $a > 0$ and $T_a = \inf\{t \geq 0 \mid B_t = a\}$. Prove that

$$E[e^{-\int_0^{T_a} g(B_s) ds}] = \frac{1}{f_1(a)}.$$

**Proof.**

1. Set $F(x, y) = f(x)e^{-y}$. Then $F \in C^2(\mathbb{R}^2)$. Note that $(\int_0^t g(B_s) ds)_{t \geq 0}$ is a finite variation process. By using Itô’s formula, we get

$$X_t = F(B_t, \int_0^t g(B_s) ds)$$

$$= f(0) + \int_0^t f'(B_s) e^{-\int_0^s g(B_r) dr} dB_s + \int_0^t -f(B_s) e^{-\int_0^s g(B_r) dr} g(B_s) ds + \frac{1}{2} \int_0^t f''(B_s) e^{-\int_0^s g(B_r) dr} ds.$$
Since
\[ f(0) + \int_0^t f'(B_s)e^{-\int_0^s g(B_r)dr} dB_s \]
is a continuous local martingale and
\[ \int_0^t -f(B_s)e^{-\int_0^s g(B_r)dr} g(B_s)ds + \frac{1}{2} \int_0^t f''(B_s)e^{-\int_0^s g(B_r)dr} ds \]
is a finite variation process, we see that
\[ X_t = f(B_t)e^{-\int_0^t g(B_r)dr}d_s \]
is a simimartingale.

2. Note that \( X \) is a continuous local martingale if and only if
\[ \int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s))ds = 0, \forall t \geq 0 \text{ a.s.} \]
It’s clear that \( X \) is a continuous local martingale whenever \( f'' = 2fg \). Now, we show that \( f'' = 2fg \) when
\[ \int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s))ds = 0, \forall t \geq 0 \text{ a.s.} \]
We prove it by contradiction. Without loss of generality, we assume that there exists \( a \in \mathbb{R} \) and \( \delta > 0 \) such that
\[ f''(x) - 2f(x)g(x) > 0 \text{ on } B(a, \delta). \]
Choose \( t_a > a + \delta \). Set \( T = \inf \{ t \geq 0 : B_t = a \} \). Then
\[ P(\int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s))ds \neq 0 \text{ for some } t \in (0, t_a) \geq P(T < t_a) > 0 \]
which is a contradiction.

3. We show that existence and uniqueness of the problem:
\[
\begin{align*}
\begin{cases}
  f''(x) = 2g(x)f(x), & \forall x \in \mathbb{R} \\
  f \in C^2(\mathbb{R}) \\
  f(0) = 1 \text{ and } f'(0) = 0.
\end{cases}
\end{align*}
\]
(a) Choose \([\alpha, \beta] \subseteq (0, \infty)\) such that \( g(x) = 0 \) for every \( x \not\in [\alpha, \beta] \). Observe that if \( f \) is a solution of the problem, then \( f''(x) = 0 \) for every \( x \leq \alpha \) and so
\[ f(x) = 1 \quad \forall x \leq \alpha. \]
(b) Let \( f(x) \) be a solution of the problem. By continuity, we see that \( f(\alpha) = 1 \) and \( f'(\alpha) = 0 \). By \([2], \text{Theorem 4.1.1}\), there exists a unique solution \( F \in C^2([\alpha, \beta]) \) such that
\[
\begin{align*}
\begin{cases}
  F''(x) = 2g(x)F(x), & \forall x \in [\alpha, \beta] \\
  F(\alpha) = 1 \text{ and } F'(\alpha) = 0.
\end{cases}
\end{align*}
\]
(c) Since \( g(x) = 0 \) for every \( x \geq \beta \), we see that \( f''(x) = 0 \) for every \( x \geq \beta \) and so
\[ f(x) = F'(\beta)x + F(\beta) - F'(\beta)\beta \quad \forall x \geq \beta. \]
Thus, we define
\[ f_1(x) = \begin{cases} 1, & \text{if } -\infty < x \leq \alpha \\ F(x), & \text{if } \alpha \leq x \leq \beta \\ F'(\beta)x + F(\beta) - F'(\beta)\beta, & \text{if } \beta \leq x < \infty. \end{cases} \]
and so \( f_1 \) is a solution of the problem. Moreover, by the construction as mentioned above, \( f_1 \) is the unique solution of the problem.

4. Now, we show that
\[ E[\exp(-\int_0^{T_a} g(B_s)ds)] = \frac{1}{f_1(a)}. \]
Fix \( a > 0 \). Define \( T_a := \inf\{t \geq 0 : B_t = a\} \). Let \( c > 0 \). Then
\[ M^c_t := X_{t \wedge T_a \wedge c}, \quad \forall t \geq 0 \]
is a continuous local martingale. It’s clear that \( \sup_{x \leq a} |f'_1(x)| \leq M < \infty \) for some \( M > 0 \). Thus,
\[ E[(M^c, M^c)_\infty] = E[\int_0^{c \wedge T_a} f'_1(B_s)^2 \exp(-2 \int_0^s g(B_u)du)ds] \leq M^2c < \infty \]
and so \( M^c \) is a \( L^2 \)-bounded martingale. Therefore, we have
\[ E[f_1(B_{c \wedge T_a}) \exp(-\int_0^{c \wedge T_a} g(B_s)ds)] = E[M^c_\infty] = E[M^c_0] = f_1(0) = 1. \]
Note that \( \sup_{x \leq a} |f(x)| < \infty \) and \( P(T_a < \infty) = 1 \). By dominated convergence theorem, we get
\[ E[f_1(a) \exp(-\int_0^{T_a} g(B_s)ds)] = \lim_{c \to \infty} E[f_1(B_{c \wedge T_a}) \exp(-\int_0^{c \wedge T_a} g(B_s)ds)] = 1 \]
and so
\[ E[\exp(-\int_0^{T_a} g(B_s)ds)] = \frac{1}{f_1(a)}. \]

5.3 Exercise 5.27 (Stochastic calculus with the supremum)
1. Let \( m : \mathbb{R}_+ \to \mathbb{R} \) be a continuous function such that \( m(0) = 0 \), and let \( s : \mathbb{R}_+ \to \mathbb{R} \) be the monotone increasing function defined by
\[ s(t) = \sup_{0 \leq r \leq t} m(r). \]
Show that, for every bounded Borel function \( h \) on \( \mathbb{R} \) and every \( t > 0 \),
\[ \int_0^t (s(r) - m(r))h(r)ds(r) = 0. \]

2. Let \( M \) be a continuous local martingale such that \( M_0 = 0 \), and for every \( t \geq 0 \), let
\[ S_t = \sup_{0 \leq r \leq t} M_r. \]
Let \( \varphi : \mathbb{R}_+ \to \mathbb{R} \) be a twice continuously differentiable function. Justify the equality
\[ \varphi(S_t) = \varphi(0) + \int_0^t \varphi'(S_s)dS_s. \]
3. Show that
\[ (S_t - M_t)\varphi(S_t) = \Phi(S_t) - \int_0^t \varphi(S_s)dM_s \]
where \( \Phi(x) = \int_0^x \varphi(y)dy \) for each \( x \in \mathbb{R} \).

4. Infer that, for every \( \lambda > 0 \),
\[ e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t} \]
is a continuous local martingale.

5. Let \( a > 0 \) and \( T = \inf\{t \geq 0 \mid S_t - M_t = a\} \). We assume that a.s. \( \langle M, M \rangle_\infty = \infty \). Show that \( T < \infty \) a.s. and \( S_T \) is exponentially distributed with parameter \( \frac{1}{a} \).

Proof.

1. Given \( t > 0 \) and a bounded Borel function \( h \) on \( \mathbb{R} \). Observe that \( s(r) \) is a nonnegative continuous function. Then
\[ E = \{r \in [0, t] \mid s(r) - m(r) > 0\} \]
is an open subset in \([0, t]\) and, hence, there exists a sequence of disjoint intervals \( \{I_n\}_{n \geq 1} \) in \([0, t]\) (these intervals may be open or half open) such that
\[ E = \bigcup_{n \geq 1} I_n. \]
Moreover, \( s \) is a constant in \( I_n \) for each \( n \geq 1 \). Indeed, if \( r_0 \in I_n = (a_n, b_n) \) (\( I_n \) may be half open interval, but the argument remain the same) for some \( n \geq 1 \), there exists \( \delta > 0 \) such that
\[ m(r) < s(r_0) \text{ in } B(r_0, \delta) \]
and, hence, \( s \) is a constant in \( B(r_0, \delta) \). By using the connectedness of \( I_n \), we see that \( s \) is a constant in \( I_n \). Thus
\[ \int_{I_n} (s(r) - m(r))h(r)ds(r) = 0 \]
for each \( n \geq 1 \) and, hence,
\[ \int_0^t (s(r) - m(r))h(r)ds(r) = \int_E (s(r) - m(r))h(r)ds(r) + \int_{[0,t]\setminus E} (s(r) - m(r))h(r)ds(r) = \sum_{n=1}^\infty \int_{I_n} (s(r) - m(r))h(r)ds(r) + 0 = 0 \]

2. Since \( S \) is an increasing process, we see that \( S \) is a finite variation process and, hence, \( \langle S, S \rangle = 0 \). By Itô’s formula, we get
\[ \varphi(S_t) = \varphi(0) + \int_0^t \varphi'(S_s)dS_s + \frac{1}{2} \int_0^t \varphi''(S_s)d\langle S, S \rangle_s = \varphi(0) + \int_0^t \varphi'(S_s)dS_s. \]

3. Set
\[ F(x, y) = (y - x)\varphi(y) - \Phi(y). \]
Then \( F \in C^2(\mathbb{R}^2), \frac{\partial F}{\partial y}(x, y) = (y - x)\varphi'(y) \), and \( \frac{\partial^2 F}{\partial y^2}(x, y) = 0 \). By Itô’s formula, we get
\[ (S_t - M_t)\varphi(S_t) - \Phi(S_t) = F(M_t, S_t) \]
\[ = F(0, 0) + \int_0^t \frac{\partial F}{\partial x}(M_s, S_s)dM_s + \int_0^t \frac{\partial F}{\partial y}(M_s, S_s)dS_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(M_s, S_s)d\langle M, M \rangle_s \]
\[ = -\int_0^t \varphi(S_s)dM_s + \int_0^t (S_s - M_s)\varphi'(S_s)dS_s. \]
Fix \( w \in \Omega \). Note that \( s \in [0,t] \mapsto \varphi'(S_s(w)) \) is continuous and, hence \( \varphi'(S_s(w)) \) is bounded in \([0,t]\). It follows for, problem 1 that

\[
\left( \int_0^t (S_s - M_s) \varphi'(S_s)dS_s \right)(w) = 0
\]

and therefore

\[
(S_t - M_t)\varphi(S_t) = \Phi(S_t) - \int_0^t \varphi(S_s)dM_s.
\]

4. Given \( \lambda > 0 \). Set \( \varphi(x) = \lambda e^{-\lambda x} \). Then \( \Phi(x) = 1 - e^{-\lambda x} \). By using the result in problem 4, we get

\[
e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t} = 1 - \int_0^t e^{-\lambda S_s}dM_s.
\]

Because \( \int_0^t \lambda e^{-\lambda S_s}dM_s \) is a continuous local martingale, so is

\[
e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}.
\]

5. Fix \( a > 0 \). By Theorem 5.13, we see that there exists a Brownian motion \( (\beta_s)_{s \geq 0} \) such that

\[
M_t = \beta_{(M,M)_t}, \forall t \geq 0, \text{ a.s.}
\]

By Proposition 2.14, we have a.s. \( \lim_{t \to \infty} \beta_t = -\infty \). Because \( (M,M)_\infty = \infty \) a.s., we have a.s.

\[
\lim_{t \to \infty} M_t = -\infty.
\]

Since \( S \) is nonnegative, we have a.s. \( T = \inf\{t \geq 0 \mid S_t - M_t = a\} < \infty \). Now, we show that \( S_T \) is exponentially distributed with parameter \( \frac{1}{a} \). For this, it suffices to show that

\[
E[e^{-\lambda S_T}] = \frac{1}{1 + \lambda \times a}
\]

for each \( \lambda \geq 0 \). Let \( \lambda > 0 \). By using the result in problem 4, we see that

\[
e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}
\]

is a continuous local martingale and, hence, there exists a sequence of stopping times \( \{\sigma_n\}_{n \geq 1} \) such that \( \sigma_n \uparrow \infty \) and

\[
e^{-\lambda S_{\sigma_n \wedge T_n}} + \lambda(S_{\sigma_n \wedge T_n} - M_{\sigma_n \wedge T_n})e^{-\lambda S_{\sigma_n \wedge T_n}}
\]

is an uniformly integrable martingale where \( T_n \equiv \sigma_n \wedge T \) and \( n \geq 1 \). Then \( T_n \uparrow T \) and

\[
E[e^{-\lambda S_T}] + \lambda E[(S_{T_n} - M_{T_n})e^{-\lambda S_{T_n}}] = E[e^{-\lambda S_{T_n}}] + \lambda E[(S_{0 \wedge T_n} - M_{0 \wedge T_n})e^{-\lambda S_{0 \wedge T_n}}] = 1
\]

for each \( n \geq 1 \). Note that

\[
0 \leq S_{T_n} - M_{T_n} \leq a
\]

for all \( n \geq 1 \). By using Lebesgue dominated convergence theorem, we see that

\[
1 = \lim_{n \to \infty} E[e^{-\lambda S_{T_n}}] + \lim_{n \to \infty} \lambda E[(S_{T_n} - M_{T_n})e^{-\lambda S_{T_n}}]
\]

\[
= E[e^{-\lambda S_T}] + \lambda E[(S_T - M_T)e^{-\lambda S_T}]
\]

\[
= E[e^{-\lambda S_T}](1 + \lambda \times a).
\]

and, hence,

\[
E[e^{-\lambda S_T}] = \frac{1}{1 + \lambda \times a}.
\]
5.4 Exercise 5.28

Let $B$ be an $(\mathcal{F}_t)$-Brownian motion started from 1. We fix $\epsilon \in (0, 1)$ and set $T_\epsilon = \{t \geq 0 \mid B_t = \epsilon\}$. We also let $\lambda > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

1. Show that $Z_t = (B_{\lambda T_t})^\alpha$ is a semimartingale and give its canonical decomposition as the sum of a continuous local martingale and a finite variation process.

2. Show that the process

$$Z_t = (B_{\lambda T_t})^\alpha e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds}$$

is a continuous local martingale if $\alpha$ and $\lambda$ satisfy a polynomial equation to be determined.

3. Compute

$$E[e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds}].$$

Proof.

1. Observe that

$$T_\epsilon < \infty \text{ a.s.}$$

and

$$B_{\lambda T_t} \geq \epsilon \forall t \geq 0 \text{ a.s.}$$

Define $F : \mathbb{R}^+ \mapsto \mathbb{R}$ by $F(x) = x^\alpha$. By Itô’s formula, we have

$$(B_{\lambda T_t})^\alpha = 1 + \alpha \int_0^t (B_{\lambda T_s})^{\alpha-1} dB_s + \frac{\alpha(\alpha-1)}{2} \int_0^t (B_{\lambda T_s})^{\alpha-2} ds \text{ a.s.}$$

for all $t \geq 0$.

2. Define $F : \mathbb{R}^+ \mapsto \mathbb{R}$ by $F(x) = \ln(x)$. By Itô’s formula, we have

$$\ln(B_{\lambda T_t})^\alpha = \alpha \ln(B_{\lambda T_t}) = \alpha \int_0^{T_t} \frac{1}{B_s} dB_s - \frac{\alpha}{2} \int_0^{T_t} \frac{1}{B_s^2} ds,$$

and, hence,

$$Z_t = (B_{\lambda T_t})^\alpha e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds} = e^{\ln(B_{\lambda T_t})^\alpha} e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds} = e^{\alpha \int_0^{T_t} \frac{1}{\beta_s} dB_s - \frac{\alpha}{2} \int_0^{T_t} \frac{1}{\beta_s^2} ds} = e^{\alpha \int_0^{T_t} \frac{1}{\beta_s} dB_s - \frac{\alpha}{2} \int_0^{T_t} \frac{1}{\beta_s^2} ds - \lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds}$$

is a continuous local martingale whenever $\frac{\alpha^2}{2} = \frac{\alpha}{2} + \lambda$ (i.e. $\alpha = \frac{1+\sqrt{1+8\lambda}}{2}$).

3. Let $\lambda > 0$. Set $\alpha = \frac{1-\sqrt{1+8\lambda}}{2}$ be a negative real number. Choose stopping times $(T_n)_{n \geq 1}$ such that $T_n \to \infty$ and $Z_{T_n}$ is an uniformly integrable martingale for $n \geq 1$. Then

$$1 = E[Z_0^{T_n}] = E[Z_{T_n}] = E[(B_{T_n \wedge T_t})^\alpha e^{-\lambda \int_0^{T_n \wedge T_t} \frac{1}{\beta_s^2} ds}]$$

for all $n \geq 1$. Observe that

$$0 \leq (B_{T_n \wedge T_t})^\alpha e^{-\lambda \int_0^{T_n \wedge T_t} \frac{1}{\beta_s^2} ds} \leq (B_{T_n \wedge T_t})^\alpha \leq \epsilon^\alpha \text{ a.s.}$$

for all $n \geq 1$. By using the Lebesgue dominated convergence theorem, we have

$$1 = \lim_{n \to \infty} E[(B_{T_n \wedge T_t})^\alpha e^{-\lambda \int_0^{T_n \wedge T_t} \frac{1}{\beta_s^2} ds}] = E[e^{\alpha} e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds}]$$

and therefore

$$E[e^{-\lambda \int_0^{T_t} \frac{1}{\beta_s^2} ds}] = \frac{1}{e^\alpha}. \square$$
5.5 Exercise 5.29

Let \((X_t)_{t \geq 0}\) be a semimartingale. We assume that there exists an \((\mathcal{F}_t)\)-Brownian motion \((B_t)_{t \geq 0}\) started from 0 and a continuous function \(b : \mathbb{R} \to \mathbb{R}\), such that

\[
X_t = B_t + \int_0^t b(X_s)ds.
\]  

(7)

1. Let \(F : \mathbb{R} \to \mathbb{R}\) be a twice continuously differentiable function on \(\mathbb{R}\). Show that, for \(F(X_t)\) to be a continuous local martingale, it suffices that \(F\) satisfies a second-order differential equation to be determined.

2. Give the solution of this differential equation which is such that \(F(0) = 0\) and \(F'(0) = 1\). In what follows, \(F\) stands for this particular solution, which can be written in the form

\[
F(x) = \int_0^x e^{-2\beta(y)}dy,
\]

with a function \(\beta\) that will be determined in terms of \(b\).

3. In this question only, we assume that \(b\) is integrable, i.e. \(\int_{\mathbb{R}} |b(x)|dx < \infty\).

(a) Show that the continuous local martingale \(M_t = F(X_t)\) is a martingale.

(b) Show that \(\langle M, M \rangle_{\infty} = \infty\) a.s.

(c) Infer that

\[
\limsup_{t \to \infty} X_t = +\infty, \liminf_{t \to \infty} X_t = -\infty, \text{ a.s.}
\]

4. We come back to the general case. Let \(c < 0\) and \(d > 0\), and

\[
T_c = \inf\{t \geq 0 \mid X_t \leq c\}, \quad T_d = \inf\{t \geq 0 \mid X_t \geq d\}.
\]

Show that, on the event \(\{T_c \land T_d\}\), the random variables \(|B_{n+1} - B_n|\) for \(n \geq 0\), are bounded above by a (deterministic) constant which does not depend on \(n\). Infer that

\[
P(T_c \land T_d = \infty) = 0.
\]

5. Compute \(P(T_c < T_d)\) in terms of \(F(c)\) and \(F(d)\).

6. We assume that \(b\) vanishes on \((-\infty, 0]\) and that there exists a constant \(\alpha > \frac{1}{2}\) such that \(b(x) \geq \frac{2}{\alpha} x\) for all \(x \geq 1\).

Show that, for every \(\epsilon > 0\), one can choose \(c < 0\) such that

\[
P(T_n < T_c, \ \forall n \geq 1) \geq 1 - \epsilon.
\]

Infer that \(X_t \to \infty\) as \(t \to \infty\) a.s.

7. Suppose now \(b(x) = \frac{1}{2x^2}\) for all \(x \geq 1\). Show that

\[
\liminf_{t \to \infty} X_t = -\infty, \text{ a.s.}
\]

Proof.

1. By Itô’s formula, we get

\[
F(X_t) = \int_0^t F'(X_s)dB_s + \int_0^t F'(X_s)b(X_s)ds + \frac{1}{2} \int_0^t F''(X_s)ds.
\]
Thus,
\[ F(X_t) = \int_0^t F'(X_s) dB_s \quad \forall t \geq 0 \text{ a.s.} \quad (8) \]
is a continuous local martingale whenever
\[ \frac{1}{2} F''(x) + F'(x)b(x) = 0 \text{ for all } x \in \mathbb{R}. \]

2. By integrating both sides of the equation, we get
\[ F'(x) = e^{\int_0^x -2b(t)dt} \quad (9) \]
and, hence,
\[ F(x) = \int_0^x e^{\int_0^s -2b(t)dt} dy \quad (10) \]

3. (a) Since \( b \in L^1(\mathbb{R}) \), there exists \( 0 < l < L < \infty \) such that
\[ l \leq e^{\int_0^x -2b(t)dt} \leq L \quad (11) \]
for all \( x \in \mathbb{R} \). By the formula (1), we get
\[ l \leq F'(X_s)(w) \leq L \quad (12) \]
for all \( s \geq 0 \) and \( w \in \Omega \) and, hence, \( (F'(X_t))_{t \geq 0} \in L^2(B^a) \) for all \( a > 0 \). Thus \( (\int_0^{t \wedge a} F'(X_s) dB_s)_{t \geq 0} \) is a \( L^2 \)-bounded martingale for \( a > 0 \) and therefore \( (\int_0^t F'(X_s) dB_s)_{t \geq 0} \) is a martingale. By (32), we see that \( M_t = F(X_t) \) is a martingale.

(b) By (32) and (12)
\[ \langle M, M \rangle_t = \int_0^t F'(X_s)^2 ds \geq l^2 \times t \quad \forall t \geq 0 \text{ a.s.} \]
and, hence, \( \langle M, M \rangle_{\infty} = \infty \) a.s.

(c) Since
\[ M_t = \beta_{(M,M)}t \quad \forall t \geq 0 \text{ a.s.} \]
for some Brownian motion \( \beta \) and \( \langle M, M \rangle_{\infty} = \infty \) a.s., we see that
\[ \limsup_{t \to \infty} M_t = +\infty, \liminf_{t \to \infty} M_t = -\infty, \text{ a.s.} \]

By (9), (10), and (11), we see that \( F \) is nondecreasing and
\[ F(\pm \infty) \equiv \lim_{x \to \pm \infty} F(x) = \pm \infty. \]

Since \( M_t = F(X_t) \), we have
\[ \limsup_{t \to \infty} X_t = +\infty, \liminf_{t \to \infty} X_t = -\infty, \text{ a.s.} \]

4. Given \( c < 0 \) and \( d > 0 \). Let \( w \in \{ T_c \wedge T_d = \infty \} \). Then \( c < X_t(w) < d \) for all \( t \geq 0 \). By (7), we get
\[
|B_n - B_{n-1}| = |X_n - X_{n-1} - \int_{n-1}^n b(X_s) ds| \leq |X_n| + |X_{n-1}| + \int_{n-1}^n |b(X_s)|ds \\
\leq 2 \times (d \vee (-c)) + \sup_{t \in [c,d]} |b(t)| \equiv R < \infty.
\]
for all \( n \geq 1 \). Thus, we see that
\[
\{T_c \land T_d = \infty\} \subseteq \{|B_n - B_{n-1}| \leq R, \forall n \geq 1\}.
\]
Because \( \{B_n - B_{n-1} \mid n \geq 1\} \) are independent and
\[
0 < P(|B_n - B_{n-1}| \leq R) \equiv c < 1
\]
for all \( n \geq 1 \), we see that
\[
P(|B_n - B_{n-1}| \leq R, \forall n \geq 1) = \lim_{m \to \infty} P(|B_n - B_{n-1}| \leq R, \forall 1 \leq n \leq m) = \lim_{m \to \infty} c^m = 0
\]
and, hence,
\[
P(T_c \land T_d = \infty) = 0. \quad (13)
\]
5. Set \( T = T_c \land T_d \). Because \( P(T < \infty) = 1 \) and \( M \) is a continuous local martingale, we get
\[
|M_t^T| = |F(X_t^T)| \leq \sup_{x \in [c,d]} |F(x)| < \infty, \forall t \geq 0, a.s.
\]
and, hence, \( M^T \) is an uniformly integrable martingale. Thus,
\[
0 = E[M_T^T] = E[M_T^T] = E[E[1_{T < T_d} M_{T_d}]] + E[1_{T_d \leq T_c} M_{T_d}] = F(c)P(T_c < T_d) + F(d)P(T_d \leq T_c)
\]
and, hence,
\[
P(T_c < T_d) = \frac{F(d)}{F(d) - F(c)}, \quad P(T_d \leq T_c) = \frac{-F(c)}{F(d) - F(c)}. \quad (14)
\]
6. Observe that, for each \( x \geq 1 \) and \( z < 0 \),
\[
F(x) = \int_0^x e^{-2 \int_0^y b(t)dt} dy
\]
\[
= \int_0^1 e^{-2 \int_0^y b(t)dt} dy + e^{-2 \int_0^1 b(t)dt} \int_1^x e^{-2 \int_0^y b(t)dt} dy
\]
\[
\leq \int_0^1 e^{-2 \int_0^y b(t)dt} dy + e^{-2 \int_0^1 b(t)dt} \int_1^x e^{-2 \int_0^y \frac{2}{y^{2\alpha}} dt} dy
\]
\[
= \int_0^1 e^{-2 \int_0^y b(t)dt} dy + e^{-2 \int_0^1 b(t)dt} \int_1^x \frac{1}{y^{2\alpha}} dy
\]
and
\[
F(z) = -\int_z^0 e^{\int_0^y 2b(t)dt} dy = -\int_z^0 1 dy = z.
\]
This implies that
\[
0 < F(\infty) < \infty \quad \text{and} \quad F(-\infty) = -\infty. \quad (15)
\]
Given \( \epsilon > 0 \). By (15), there exists \( c < 0 \) such that \( \frac{F(\infty)}{F(\infty) - F(c)} < \epsilon \). Since \( T_n \geq T_{n-1} \), we see that
\[
P(T_n < T_c, \forall n \geq 1) = \lim_{n \to \infty} P(T_n < T_c) = 1 - \frac{F(\infty)}{F(\infty) - F(c)} \geq 1 - \epsilon.
\]
For \( k \geq 1 \), there exists \( c_k < 0 \) such that
\[
P(T_n \geq T_{c_k} \text{ for some } n \geq 1) \leq 2^{-k}.
\]
By Borel Cantelli’s lemma, we see that \( P(E^c) = 0 \), where

\[ E^c = \{ \{T_n \geq T_{c_k} \text{ for some } n \geq 1 \} \text{ i.o k} \}. \]

For \( k \geq 1 \), since \( F(c_k) \leq M_{t \wedge T_{c_k}} = F(X_{t \wedge T_{c_k}}) \leq F(\infty) < \infty \), we see that \( M_{T_{c_k}}^{c_k} \) is an uniformly integrable martingale and, hence, \( \lim_{t \to \infty} M_{t \wedge T_{c_k}}^{c_k} \) exists (a.s.). Set

\[ G = \bigcap_{k \geq 1} \{ \lim_{t \to \infty} M_{t \wedge T_{c_k}}^{c_k} \text{ exists } \}. \]

Then \( P(G \cap E) = 1 \). Let \( w \in E \cap G \). Then \( T_n(w) < T_{c_k}(w) \) for some \( k \geq 1 \) and all \( n \geq 1 \). Since \( T_n(w) \uparrow \infty \), we see that \( T(c_k)(w) = \infty \), and, hence, \( \lim_{t \to \infty} M_t(w) = \lim_{n \to \infty} M_{T_{n}}(w) = \lim_{n \to \infty} F(n) = F(\infty) \),

we get \( \lim_{t \to \infty} X_t(w) = \infty \). Therefore \( \lim_{t \to \infty} X_t = \infty \) (a.s.).

7. Let \( x > 1 \). We see that

\[ F(x) = \int_0^1 e^{-2 \int_0^t b(t)dt} dy + e^{-2 \int_0^1 b(t)dt} \int_1^x \frac{1}{y} dy \]

and, hence, \( F(\infty) = \infty \). Choose \( \{ c_k \} \subseteq \mathbb{R} \) such that \( c_k \to -\infty \). For \( k \geq 1 \), by (14), there exists \( d_k > 0 \) such that

\[ \lim_{t \to \infty} M_t(w) = \lim_{n \to \infty} M_{T_n}(w) = \lim_{n \to \infty} F(n) = F(\infty), \]

we get \( \lim_{t \to \infty} X_t(w) = \infty \). Therefore \( \lim \inf_{t \to \infty} X_t = \infty \) (a.s.).

\[ \square \]

### 5.6 Exercise 5.30 (Lévy Area)

Let \((X_t, Y_t)_{t \geq 0}\) be a two-dimensional \((\mathcal{F}_t)\)-Brownian motion started from 0. We set, for every \( t \geq 0 \):

\[ \mathcal{A}_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s \text{ (Lévy area)} \]

1. Compute \( \langle \mathcal{A}, \mathcal{A} \rangle_t \) and infer that \( \langle \mathcal{A} \rangle_{t \geq 0} \) is a square-integrable (true) martingale.

2. Let \( \lambda > 0 \). Justify the equality

\[ E[e^{i \lambda \mathcal{A}_t}] = E[\cos(\lambda \mathcal{A}_t)]. \]

3. Let \( f \in C^3(\mathbb{R}_+) \). Give the canonical decomposition of the semimartingales

\[ Z_t = \cos(\lambda \mathcal{A}_t), W_t = -\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t). \]

Verify that \( \langle Z, W \rangle_t = 0 \).
4. Show that, for the process $Z_t e^{W_t}$ to be a continuous local martingale, it suffices that $f$ solves the differential equation

$$f''(t) = f'(t)^2 - \lambda^2.$$ 

5. Let $r > 0$. Verify that the function

$$f(t) = -\ln(\cosh(\lambda(r - t)))$$

solves the differential equation of question 4. and derive the formula

$$E[e^{i\lambda \xi r}] = \frac{1}{\cosh(\lambda r)}.$$ 

Proof.

1. By Fubini’s theorem, we get

$$E[\langle \mathcal{A}, \mathcal{A} \rangle_t] = E\left[ \int_0^t X_s^2 ds \right] + E\left[ \int_0^t Y_s^2 ds \right]$$

$$= \int_0^t E[X_s^2] ds + \int_0^t E[Y_s^2] ds$$

$$= \int_0^t ds + \int_0^t ds = t^2$$

for all $t \geq 0$. By Theorem 4.13, we see that $\mathcal{A}$ is a true martingale and $\mathcal{A}_t \in L^2$ for all $t \geq 0$.

2. Fix $\lambda > 0$ and $t > 0$. Let $0 = t_0 < t_1 < ... < t_n = t$ be a sequence of subdivisions of $[0, t]$ whose mesh tends to 0. By Proposition 5.9, we have

$$\sum_{i=0}^{p_n-1} X_{t_{i+1}}(Y_{t_{i+1}} - Y_{t_{i-1}}) - \sum_{i=0}^{p_n-1} Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_{i-1}}) \to \int_0^t X_s dY_s - \int_0^t Y_s dX_s = \mathcal{A}_t$$

and

$$\sum_{i=0}^{p_n-1} Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_{i-1}}) - \sum_{i=0}^{p_n-1} X_{t_{i+1}}(Y_{t_{i+1}} - Y_{t_{i-1}}) \to \int_0^t Y_s dX_s - \int_0^t X_s dY_s = -\mathcal{A}_t.$$ 

Let

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{t_2 - t_1}...(t_p - t_{p-1})} e^{-\sum_{k=0}^{p_n-1} (x_{t_{k+1}} - x_k)^2 \frac{2}{(t_{k+1} - t_k)^2}}.$$ 

Since $(X_t, Y_t)_{t \geq 0}$ is two-dimensional Brownian motion, we get

$$E[e^{i\xi(\sum_{i=0}^{p_n-1} X_{t_{i+1}}(Y_{t_{i+1}} - Y_{t_{i-1}}) - \sum_{i=0}^{p_n-1} Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_{i-1}}))}]$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{i\xi(\sum_{k=0}^{p_n-1} x_{t_{k+1}} - y_{t_{k+1}} - \sum_{k=0}^{p_n-1} y_{t_{k+1}} - x_{t_{k+1}}))} p(x)p(y)dx dy$$

$$= E[e^{i\xi(\sum_{i=0}^{p_n-1} Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_{i-1}}) - \sum_{i=0}^{p_n-1} X_{t_{i+1}}(Y_{t_{i+1}} - Y_{t_{i-1}}))}]$$

for all $n \geq 1$ and $\xi \in \mathbb{R}$. By Lévy’s continuity theorem, we see that

$$E[e^{i\xi \mathcal{A}_t}] = E[e^{i\xi (-\mathcal{A}_t)}]$$

for all $\xi \in \mathbb{R}$ and, hence $\mathcal{A}_t \overset{D}{=} -\mathcal{A}_t$. Therefore

$$E[\cos(\lambda \mathcal{A}_t)] + iE[\sin(\lambda \mathcal{A}_t)] = E[\cos(\lambda \mathcal{A}_t)] - iE[\sin(\lambda \mathcal{A}_t)]$$

and, hence $E[\sin(\lambda \mathcal{A}_t)] = 0$. 

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3. By Itô’s formula, we get

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda \alpha_s) d\alpha_s - \frac{1}{2} \lambda^2 \int_0^t \cos(\lambda \alpha_s) d(\alpha_s, \alpha_s)_s
\]

\[
= 1 - \lambda \int_0^t \sin(\lambda \alpha_s) d\alpha_s - \frac{1}{2} \lambda^2 \int_0^t \cos(\lambda \alpha_s) (X^2_s + Y^2_s) ds
\]

\[
= 1 - \lambda \int_0^t \sin(\lambda \alpha_s) d\alpha_s - \frac{1}{2} \lambda^2 \int_0^t Z_s (X^2_s + Y^2_s) ds.
\]

Also we have

\[
f'(t)(X^2_t + Y^2_t)
\]

\[
= \int_0^t f''(s)(X^2_s + Y^2_s) ds + \int_0^t f'(s)2X_s dX_s + \int_0^t f'(s)2Y_s dY_s + \frac{1}{2} \int_0^t f'(s) \times 2 ds + \frac{1}{2} \int_0^t f'(s) \times 2 ds
\]

\[
= \int_0^t f''(s)(X^2_s + Y^2_s) ds + \int_0^t f'(s)2X_s dX_s + \int_0^t f'(s)2Y_s dY_s + 2(f(t) - f(0))
\]

and, hence,

\[
W_t = -\frac{1}{2} f'(t)(X^2_t + Y^2_t) + f(t) = f(0) - \int_0^t f'(s)X_s dX_s - \int_0^t f'(s)Y_s dY_s - \frac{1}{2} \int_0^t f''(s)(X^2_s + Y^2_s) ds.
\]

Therefore

\[
\langle W, Z \rangle_t = X_t f'(t) \lambda \cos(\alpha_t) + Y_t f'(t) \lambda \cos(\alpha_t)
\]

\[
= X_t f'(t) \lambda \cos(\alpha_t) \times (-W_t) + Y_t f'(t) \lambda \cos(\alpha_t)(X_t t) = 0
\]

4. By Itô’s formula, we get

\[
Z_t e^{W_t} = \int_0^t e^{W_s} dZ_s + \int_0^t Z_s e^{W_s} dW_s + \frac{1}{2} \int_0^t Z_s e^{W_s} d\langle W, W \rangle_s.
\]

Note that

\[
dZ_s = -\lambda \sin(\lambda \alpha_s) d\alpha_s - \frac{1}{2} \lambda^2 Z_s (X^2_s + Y^2_s) ds,
\]

\[
dW_s = f'(s)X_s dX_s - f'(s)Y_s dY_s - \frac{1}{2} f''(s)(X^2_s + Y^2_s) ds,
\]

and

\[
d\langle W, W \rangle_s = (X^2_s f'(s)^2 + Y^2_s f'(s)^2) ds.
\]

Thus, \(Z_t e^{W_t}\) is a continuous local martingale when

\[
f''(t) = f'(t)^2 - \lambda^2.
\]

5. Fix \(r > 0\) and \(\lambda > 0\). It’s clear that \(f(t) = -\ln(\cosh(\lambda(r - t))) \in C^3(\mathbb{R}_+\cup 0)\) and satisfy

\[
f''(t) = f'(t)^2 - \lambda^2.
\]

Thus \((Z_t e^{W_t})_{t \geq 0}\) is a continuous local martingale. Choose \((T_n)_{n \geq 1}\) such that \((Z_t^{T_n} e^{W_{T_n}})_{t \geq 0}\) is an uniformly integrable martingale for \(n \geq 1\) and \(T_n \uparrow \infty\). Then

\[
E[\cos(\lambda \alpha_{T_n \land r}) e^{-\frac{1}{2} f'(T_n \land r)(X^2_{T_n \land r} + Y^2_{T_n \land r}) + f(T_n \land r)}] = E[Z_{T_n} e^{W_{T_n}}] = E[Z_0 e^{W_0}] = \frac{1}{\cosh(\lambda r)}.
\]
Because $r - T_n \wedge r \geq 0$ for all $n \geq 1$, we see that
\[
f'(T_n \wedge r) = \frac{\sinh(\lambda(r - T_n \wedge r))}{\cosh(\lambda(r - T_n \wedge r))} \lambda \geq 0
\]
and, hence,
\[
0 \leq e^{-\frac{1}{2}f'(T_n \wedge r)(X_n^2 + Y_n^2)} \leq 1
\]
for all $n \geq 1$. Since $\cosh(\lambda(r - T_n \wedge r)) \geq 1$ for all $n \geq 1$, we get
\[
f(T_n \wedge r) = -\ln(\cosh(\lambda(r - T_n \wedge r))) \leq 0
\]
and, hence
\[
0 \leq e^{f(T_n \wedge r)} \leq 1.
\]
By Lebesgue dominated convergence theorem, we see that
\[
\lim_{n \to \infty} E\left[\cos(\lambda A_T \wedge r)e^{-\frac{1}{2}f'(T_n \wedge r)(X_n^2 + Y_n^2) + f(T_n \wedge r)}\right] = E[\cos(\lambda A_T)r] = E[\cos(\lambda A_T)] = \frac{1}{\cosh(\lambda r)}.
\]

5.7 Exercise 5.31 (Squared Bessel processes)

Let $B$ be an $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion started from 0, and let $X$ be a continuous semimartingale. We assume that $X$ takes values in $\mathbb{R}_+$, and is such that, for every $t \geq 0$,
\[
X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \alpha t
\]
where $x$ and $\alpha$ are nonnegative real numbers.

1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, and let $\varphi$ be a twice continuously differentiable function on $\mathbb{R}_+$, taking strictly positive values, which solves the differential equation
\[
\varphi'' = 2f\varphi
\]
and satisfies $\varphi(0) = 1$ and $\varphi'(1) = 0$. Observe that the function $\varphi$ must then be decreasing over the interval $[0, 1]$. We set
\[
u(t) = \frac{\varphi'(t)}{2\varphi(t)}
\]
for every $t \geq 0$. Verify that we have, for every $t \geq 0$,
\[
u'(t) + 2u(t)^2 = f(t),
\]
then show that, for every $t \geq 0$,
\[
u(t)X_t - \int_0^t f(s)X_s ds = u(0)x + \int_0^t u(s)dX_s - 2 \int_0^t u(s)^2 X_s ds.
\]
We set
\[
Y_t = u(t)X_t - \int_0^t f(s)X_s ds.
\]
2. Show that, for every \( t \geq 0 \),
\[
\varphi(t) - \frac{\beta}{2} e^{\frac{\lambda}{2}} = E(N)_t
\]
where \( E(N)_t = \exp(N_t - \frac{1}{2} \langle N, N \rangle_t) \) denotes the exponential martingale associated with the continuous local martingale
\[
N_t = u(0)x + 2 \int_0^t u(s) \sqrt{X_s} dB_s.
\]

3. Infer from the previous question that
\[
E[\exp(- \int_0^t f(s) X_s ds)] = \varphi(1) \exp(\frac{x}{2} \varphi'(0)).
\]

4. Let \( \lambda > 0 \). Show that
\[
E[\exp(-\lambda \int_0^1 X_s ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{\varphi}{2}} \exp(-\frac{x^2}{2} \sqrt{2\lambda} \tanh(\sqrt{2\lambda})).
\]

5. Show that, if \( \beta = (\beta_t)_{t \geq 0} \) is a real Brownian motion started from \( y \), one has, for every \( \lambda > 0 \),
\[
E[\exp(-\lambda \int_0^1 \beta^2_s ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{\varphi}{2}} \exp(-\frac{y^2}{2} \sqrt{2\lambda} \tanh(\sqrt{2\lambda})).
\]

Proof.

1. Since \( f \geq 0 \) and \( \varphi \geq 0 \), we see that \( \varphi'' = 2f \varphi \geq 0 \). Because \( \varphi'(1) = 0 \) and \( \varphi' \) is nondecreasing, one has \( \varphi' \leq 0 \) in \([0, 1]\) and, hence, \( \varphi \) is decreasing over the interval \([0, 1]\). Note that
\[
\frac{\varphi''(t)}{2\varphi(t)} + 2 \frac{\varphi'(t)}{\varphi(t)} = f(t).
\]

By Itô's formula, we get
\[
u(t)X_t = u(0)x + \int_0^t u(s) X_s ds + \int_0^t u(s) dX_s
\]
\[
= u(0)x + \int_0^t f(s) X_s ds - 2 \int_0^t u(s)^2 X_s ds + \int_0^t u(s) dX_s,
\]
and, hence,
\[
u(t)X_t - \int_0^t f(s) X_s ds = u(0)x + \int_0^t u(s)dX_s - 2 \int_0^t u(s)^2 X_s ds.
\]

2. Note that
\[
Y_t = u(0)x + \int_0^t u(s) dB_s - 2 \int_0^t u(s)^2 dB_s
\]
\[
= u(0)x + \int_0^t u(s) \sqrt{X_s} dB_s + \alpha \int_0^t u(s) ds - 2 \int_0^t u(s)^2 X_s ds
\]
\[
= u(0)x + \int_0^t u(s) \sqrt{X_s} dB_s - 2 \int_0^t u(s)^2 X_s ds + \alpha \int_0^t \frac{\varphi(s)}{2\varphi(t)} ds
\]
\[
= u(0)x + \int_0^t u(s) \sqrt{X_s} dB_s - 2 \int_0^t u(s)^2 X_s ds + \frac{\alpha \ln(\varphi(t))}{2}.
\]

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Then we have
\[ \mathcal{E}(N)_t = \exp(N_t - \langle N, N \rangle_t) \]
\[ = \exp(u(0)x + 2 \int_0^1 u(s)\sqrt{X_s}dB_s - 2 \int_0^1 u(s)^2X_sds) \]
\[ = \exp(u(0)x + 2 \int_0^1 u(s)\sqrt{X_s}dB_s - 2 \int_0^1 u(s)^2X_sds + \alpha \ln(\varphi(t)))\varphi(t)^{-\frac{\alpha}{2}} \]
\[ = \exp(Y_t)\varphi(t)^{-\frac{\alpha}{2}}. \]

3. Choose \( m \) such that \( \ln(\varphi(t)) \geq m \) for all \( t \in [0, 1] \). Fix \( t \in [0, 1] \). Because \( \varphi' \leq 0 \) in \([0, 1]\) (problem 1), we see that \( u \leq 0 \) in \([0, 1]\). Because \( f \geq 0 \) in \([0, 1]\) and \( X_t, \alpha \geq 0 \), we see that
\[ \mathcal{E}(N)_t = \exp(Y_t)\varphi(t)^{-\frac{\alpha}{2}} = \exp(u(t)X_t - \int_0^t f(s)X_sds - \frac{\alpha}{2} \ln(\varphi(t))) \leq \exp(-\frac{\alpha}{2} m) < \infty. \]

and, hence, \( \mathcal{E}(N)_{t \wedge 1} \) is a uniformly integrable martingale. Because \( u(1) = \varphi'(1) = 0 \) and \( \varphi(0) = 1 \), we have
\[ \varphi(1)^{-\frac{\alpha}{2}} E[\exp(-\int_0^1 f(s)X_sds)] = \varphi(1)^{-\frac{\alpha}{2}} E[\exp(u(1)X_1 - \int_0^1 f(s)X_sds)] = E[\varphi(1)^{-\frac{\alpha}{2}} \exp Y_1] \]
\[ = E[\mathcal{E}(N)_1] = E[\mathcal{E}(N)_0] = E[\exp(N_0)] = \exp(u(0)x) \]
\[ = \exp(x\varphi'(0)) = \exp(\frac{x\varphi'(0)}{2}) \]

and, so
\[ E[\exp(-\int_0^1 f(s)X_sds)] = \varphi(1)^{\frac{\alpha}{2}} \exp(\frac{x\varphi'(0)}{2}). \]

4. Set \( f = \lambda \). Then we have \( \varphi''(t) - 2\lambda \varphi(t) = 0 \) and, hence, \( \varphi(t) = c_1 \exp(\sqrt{2\lambda}t) + c_2 \exp(-\sqrt{2\lambda}t) \). Combining with initial conditions, we get
\[ \varphi(t) = \frac{\exp(-\sqrt{2\lambda}t)}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} \exp(\sqrt{2\lambda}t) + \frac{\exp(\sqrt{2\lambda}t)}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} \exp(-\sqrt{2\lambda}t). \]

Thus,
\[ \varphi(1) = \frac{2}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} = \frac{1}{\cosh(\sqrt{2\lambda})} \]

and
\[ \varphi'(0) = \sqrt{2\lambda} \frac{-\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} = -\sqrt{2\lambda} \tanh(\sqrt{2\lambda}). \]

By problem 3, we get
\[ E[\exp(-\lambda \int_0^1 X_sds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}} \exp(-\frac{x}{2} \sqrt{2\lambda} \tanh(\sqrt{2\lambda})). \]

5. Suppose \( \beta \) is a \((\mathcal{F}_t)_{t \geq 0}\)-real Brownian motion. By Itô’s formula, we get
\[ \beta_t^2 = y^2 + 2 \int_0^t \beta_s dB_s + t \]

Set \( B_t = \int_0^t \text{sgn}(\beta_s) dB_s \). Then \((B_t)_{t \geq 0}\) is a process \((B, B)_t = t\), we see that \( B \) is a \((\mathcal{F}_t)_{t \geq 0}\)-real Brownian motion and
\[ \beta_t^2 = y^2 + 2 \int_0^t |\beta_s|dB_s + t. \]
Thus, by problem 4, we get
\[ E[\exp(-\lambda \int_0^1 \beta_s^2 ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} \exp(-\frac{y^2}{2}\sqrt{2\lambda}\tanh(\sqrt{2\lambda})). \]

5.8 Exercise 5.32 (Tanaka’s formula and local time)

Let \( B \) be an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion started from 0. For every \( \epsilon > 0 \), we define a function \( g_\epsilon : \mathbb{R} \to \mathbb{R} \) by setting \( g_\epsilon(x) = \sqrt{\epsilon^2 + x^2} \).

1. Show that
\[ g_\epsilon(B_t) = g_\epsilon(0) + M^\epsilon_t + A_t^\epsilon \]
where \( M^\epsilon \) is a square integrable continuous martingale that will be identified in the form of a stochastic integral, and \( A^\epsilon \) is an increasing process.

2. We set \( \text{sgn}(x) = 1_{\{x > 0\}} - 1_{\{x < 0\}} \) for all \( x \in \mathbb{R} \). Show that, for every \( t \geq 0 \),
\[ M^\epsilon_t \to \int_0^t \text{sgn}(B_s)dB_s \text{ in } L^2 \text{ as } \epsilon \to 0. \]
Infer that there exists an increasing process \( L \) such that, for every \( t \geq 0 \),
\[ |B_t| = \int_0^t \text{sgn}(B_s)dB_s + L_t. \]

3. Observing that \( A^\epsilon_t \to L_t \) as \( \epsilon \to 0 \) (It seems that the author want us to prove
\[ A^\epsilon_t \to L_t \text{ as } \epsilon \to 0 \forall t \geq 0 \text{ (a.s.)}, \]
but this statement is to strong to prove. You can prove the following problems without this statement). Show that, for every \( \delta > 0 \), for every choice of \( 0 < u < v \), the condition \( (|B_t| \geq \delta \text{ for every } t \in [u,v]) \text{ a.s.} \) implies that \( L_u = L_v \). Infer that the function \( t \mapsto L_t \) is a.s. constant on every connected component of the open set \( \{t \geq 0 | B_t \neq 0\} \).

4. We set \( \beta_t = \int_0^t \text{sgn}(B_s)dB_s \) for all \( t \geq 0 \). Show that \( (\beta_t)_{t \geq 0} \) is a \((\mathcal{F}_t)_{t \geq 0}\) Brownian motion started from 0.

5. Show that \( L_t = \sup_{s \leq t}(-\beta_s) \) (a.s.). (In order to derive the bound \( L_t \leq \sup_{s \leq t}(-\beta_s) \), one may consider the last zero of \( B \) before time \( t \), and use question 3.) Give the law of \( L_t \).

6. For every \( \epsilon > 0 \), we define two sequences of stopping times \((S^\epsilon_n)_{n \geq 1}\) and \((T^\epsilon_n)_{n \geq 1}\), by setting
\[ S^\epsilon_1 = 0, T^\epsilon_1 = \inf\{t \geq S^\epsilon_1 | |B_t| = \epsilon\} \]
and then, by induction,
\[ S^\epsilon_{n+1} = \inf\{t \geq T^\epsilon_n | |B_t| = 0\}, T^\epsilon_{n+1} = \inf\{t \geq S^\epsilon_{n+1} | |B_t| = \epsilon\}. \]
For every \( t \geq 0 \), we set
\[ N^\epsilon_t = \sup\{n \geq 1 | T^\epsilon_n \leq t\}, \]
where \( \sup\emptyset = 0 \). Show that
\[ \epsilon N^\epsilon_t \xrightarrow{L^2} L_t \text{ as } \epsilon \to 0. \]
(One may observe that
\[ L_t + \int_0^t \sum_{n=1}^\infty 1_{[S^\epsilon_n,T^\epsilon_n]}(s)\text{sgn}(B_s)dB_s = \epsilon N^\epsilon_t + r^\epsilon_t \text{ (a.s.)}, \]
where the “remainder” \( r^\epsilon_t \) satisfies \( |r^\epsilon_t| \leq \epsilon \).)
7. Show that $N_{t\sqrt{t}}$ converges in law as $t \to \infty$ to $|U|$, where $U$ is $N(0, 1)$-distributed.

Proof.

1. By Itô’s formula, we get

$$g_{\epsilon}(B_t) = g_{\epsilon}(0) + \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s + \frac{1}{2} \int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{1/2}} ds.\tag{16}$$

It’s clear that

$$A_{\epsilon}^t = \frac{1}{2} \int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{1/2}} ds$$

is an increasing process. For $t \geq 0$,

$$E[\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s, \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s] = E[\int_0^t \frac{B_s^2}{\epsilon^2 + B_s^2} ds] \leq t.$$

By theorem 4.13, we see that

$$M_{\epsilon}^t = \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s$$

is a square integrable continuous martingale.

2. Fix $t > 0$. Then

$$\frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} \rightarrow \frac{B_s}{|B_s|} = \text{sgn}(B_s) \text{ as } \epsilon \rightarrow 0 \forall s \in [0, t] \text{ (a.s.),}$$

where $\frac{B_s}{|B_s|} = 0$ when $B_s = 0$.

By Proposition 5.8, we see that

$$\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \overset{P}{\rightarrow} \int_0^t \text{sgn}(B_s) dB_s \text{ as } \epsilon \rightarrow 0.$$

Recall that

**Lieb’s theorem [1, Theorem 6.2.3].**

Let $(E, \mathcal{B}, \mu)$ be a measure space, $p \in [1, \infty)$, and \{fn\} \cup \{f\} \subset L^p(\mu; \mathbb{R})$. If sup$_{n \geq 1} ||f_n||_{L^p(\mu; \mathbb{R})} < \infty$ and $f_n \rightarrow f$ in $\mu$-measure, then

$$||f_n - f||_{L^p(\mu; \mathbb{R})} \rightarrow 0 \text{ whenever } ||f_n||_{L^p(\mu; \mathbb{R})} \rightarrow ||f||_{L^p(\mu; \mathbb{R})}.\tag{17}$$

Since

$$||\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s||_{L^2}^2 = E[(\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s)^2] = E[\int_0^t \frac{B_s^2}{\epsilon^2 + B_s^2} ds] \leq t$$

for all $\epsilon > 0$ and

$$\lim_{\epsilon \rightarrow 0} ||\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s||_{L^2}^2 = t = E[(\int_0^t \text{sgn}(B_s) dB_s)^2] = ||\int_0^t \text{sgn}(B_s) dB_s||_{L^2}^2,$$

we get

$$M_{\epsilon}^t = \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \rightarrow \int_0^t \text{sgn}(B_s) dB_s \text{ in } L^2 \text{ as } \epsilon \rightarrow 0.$$
Let us now construct the corresponding increasing process \((L_t)_{t \geq 0}\). We just define

\[
L_t = |B_s| - \int_0^t sgn(B_s)dB_s. \tag{18}
\]

It remains to show that \((L_t)_{t \geq 0}\) is an increasing process. Fix \(t > 0\). By Lieb’s theorem, we see that

\[
g_\epsilon(B_t) = \sqrt{\epsilon^2 + |B_t|^2} \xrightarrow{\epsilon \to 0} |B_t|
\]

and therefore

\[
A_t^\epsilon = g_\epsilon(B_t) - g_\epsilon(0) - M_t^{|B_t|} \xrightarrow{\epsilon \to 0} L_t - \int_0^t sgn(B_s)dB_s = L_t.
\]

Since \((A_t^\epsilon)_{t \geq 0}\) is an increasing process for all \(\epsilon > 0\), we see that \((L_t)_{t \geq 0}\) is an increasing process.

3. First we show that the condition \(|B_t| \geq \delta\) for every \(t \in [u, v]\) a.s. implies that \(L_u = L_v\). Fix \(\delta > 0\) and \(0 < u < v\). Since \(A_t^{|B_t|} \xrightarrow{\epsilon \to 0} L_t\) for \(i = u, v\), there exists \(\{\epsilon_k\}\) such that \(\epsilon_k \downarrow 0\) and \(A_t^{|B_t|} \xrightarrow{\epsilon \to 0} L_t\) for \(i = u, v\). Let

\[
w \in \big\{ \lim_{k \to \infty} A_u^\epsilon_k = L_u \big\} \bigcap \big\{ \lim_{k \to \infty} A_v^\epsilon_k = L_v \big\} \bigcap \{|B_t| \geq \delta\} \text{ for all } t \in [u, v].
\]

Then

\[
\frac{\epsilon_k^2}{(\epsilon_k^2 + B_s^2(w))^2} \leq \frac{1}{\delta^2}
\]

for \(s \in [u, v]\) and \(k \geq 1\). By Lebesgue’s dominated convergence theorem, we get

\[
L_v(w) - L_u(w) = \lim_{k \to \infty} \frac{1}{2} \int_u^v \frac{\epsilon_k^2}{(\epsilon_k^2 + B_s^2(w))^2} ds = 0.
\]

Thus, the condition \(|B_t| \geq \delta\) for every \(t \in [u, v]\) a.s. implies that \(L_u = L_v\).

Next, we show that the function \(t \mapsto L_t\) is a.s. constant on every connected component of the open set \(\{ t \geq 0 \mid B_t \neq 0 \}\). Set

\[
Z_{\delta, u, v} = \{|B_t| \geq \delta \text{ for every } t \in [u, v]\} \implies L_u = L_v
\]

for all positive rational numbers \(\delta\) and \(u < v\). Then

\[
Z = \bigcup_{\delta, u, v} Z_{\delta, u, v} \tag{19}
\]

is a zero set. Let \(w \in Z^c\). Let \((a, b)\) be a connected component of \(\{ t \geq 0 \mid B_t(w) \neq 0 \}\). For any two rational numbers \(u\) and \(v\) such that \(a < u < v < b\), there exists positive rational number \(\delta\) such that \(|B_t(w)| \geq \delta\) for all \(t \in [u, v]\) and therefore \(L_u(w) = L_v(w)\). Since \(t \in (a, b) \mapsto L_t(w)\) is increasing, we see that \(t \in (a, b) \mapsto L_t(w)\) is a constant. Hence \(t \mapsto L_t\) is a.s. constant on every connected component of the open set \(\{ t \geq 0 \mid B_t \neq 0 \}\).

4. It’s clear that \((\beta_t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-continuous local martingale with \(\langle \beta, \beta \rangle_t = t\) for all \(t \geq 0\). Thus, \((\beta_t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\) Brownian motion started from \(0\).

5. Fix \(t_0 > 0\). Since \(|B_t| = \beta_t + L_t \forall t \geq 0\) (a.s.), we have \(\sup_{s \leq t_0} (-\beta_s) \leq \sup_{s \leq t_0} L_s = L_{t_0}\) (a.s.). We show that

\[
\sup_{s \leq t_0} (-\beta_s) \geq L_{t_0}\text{ (a.s.)}
\]

Let \(w \in Z^c \cap \{|B_t| = \beta_t + L_t \forall t \geq 0\}\), where \(Z\) is defined in (19). Set \(r = \sup\{0 \leq s \leq t_0 \mid B_s(w) = 0\}\). Then \(B_{r}(w) = 0\) and

\[
L_{t_0}(w) = -\beta_{t_0}(w) \leq \sup_{s \leq t_0} (-\beta_s)(w) \text{ whenever } B_{t_0}(w) = 0.
\]
Since \( t \in \mathbb{R}_+ \mapsto L_t(w) \in C(\mathbb{R}_+) \) is constant on every connected component of \( \{ t \geq 0 \mid B_t(w) \neq 0 \} \), we have
\[
L_t(w) = L_t(w) = -\beta_t(w) \leq \sup_{s \leq t}(-\beta_s)(w) \text{ whenever } B_t(w) \neq 0.
\]
Thus
\[
\sup_{s \leq t}(-\beta_s) \geq L_{t_0} \text{ (a.s.)}
\]
and therefore
\[
\sup_{s \leq t}(-\beta_s) = L_{t_0} \text{ (a.s.)}.
\]
To find the law of \( L_t \), we define stopping times
\[
\Gamma_a = \inf\{ t \geq 0 \mid -\beta_t = a \}
\]
for \( a \in \mathbb{R} \). By the result of problem 4 and Corollary 2.22, we get
\[
P(L_t \leq a) = P(\sup_{s \leq t}(-\beta_s) \leq a) = P(\Gamma_a \geq t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s})ds.
\]

6. Fix \( t > 0 \) and \( \epsilon > 0 \). Note that \( N^c_t \) is the number of upcrossing from 0 to \( \pm\epsilon \) by \( (B_s)_{s \in [0,t]} \). First, we show that
\[
L_t + \int_0^t \sum_{n=1}^{\infty} 1_{[S_n^*,T_n^*]}(s)\text{sgn}(B_s)dB_s = \epsilon N_t^c + r_t^c \text{ (a.s.)},
\]
where \( |r_t^c| \leq \epsilon \). By (18) and Proposition 5.8, we get
\[
L_t + \int_0^t \sum_{n=1}^{\infty} 1_{[S_n^*,T_n^*]}(s)\text{sgn}(B_s)dB_s = |B_t| - \int_0^t \sum_{n=1}^{\infty} 1_{(T_n^*,S_{n+1}^*)}(s)\text{sgn}(B_s)dB_s
\]
outside a zero set \( N \). Let \( w \in N^c \). We consider the following cases:

(a) Suppose that \( 0 = S_1^*(w) < T_1^*(w) < S_2^*(w) \ldots < T_{m-1}^*(w) < S_m^*(w) < t < T_m^*(w) \) for some \( m \geq 1 \). Then \( |B_t(w)| \leq \epsilon \), \( N_t^c = m-1 \), and \( \text{sgn}(B_s)(w) = \text{sgn}(B_{T_k^*})(w) \) for \( s \in [T_k^*(w),S_{k+1}^*(w)] \) for each \( k = 1,\ldots, m-1 \). If we set \( r_t^c(w) = |B_t(w)| \), then we have
\[
|B_t(w)| = \left( \sum_{k=1}^{m-1} \int_0^t 1_{(T_k^*,S_{k+1}^*)}(s)\text{sgn}(B_s)dB_s \right)(w)
\]
\[
= r_t^c(w) - \left( \sum_{k=1}^{m-1} \text{sgn}(B_{T_k^*}) \int_0^t 1_{(T_k^*,S_{k+1}^*)}(s)dB_s \right)(w)
\]
\[
= r_t^c(w) - \sum_{k=1}^{m-1} \text{sgn}(B_{T_k^*})(w)(B_{S_{k+1}^*}(w) - B_{T_k^*}(w))
\]
\[
= r_t^c(w) - \sum_{k=1}^{m-1} \text{sgn}(B_{T_k^*})(w)(0 - \text{sgn}(B_{T_k^*})(w) \times \epsilon)
\]
\[
= r_t^c(w) + (m-1)\epsilon
\]
\[
= r_t^c(w) + N_t^c(w)\epsilon.
\]

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(b) Suppose that \( 0 = S'_1(w) < T'_1(w) < S'_2(w) \ldots < T'_m(w) < S'_m(w) < T'_m(w) \leq t < S'_{m+1}(w) \) for some \( m \geq 1 \). Similar, we get \( N'_1 = m \), and \( \text{sgn}(B_s)(w) = \text{sgn}(B_T(w)) \) for \( s \in [T'_k(w), S'_{k+1}(w)] \) for each \( k = 1, \ldots, m + 1 \). If we set \( r'_T(w) = \epsilon \), then we have

\[
|B_t(w)| - (\sum_{k=1}^{\infty} \int_{0}^{t} 1_{(T'_k, S'_{k+1})}(s)\text{sgn}(B_s)dB_s(w))
\]

\[
= |B_t(w)| - (\sum_{k=1}^{m} \text{sgn}(B_{T'_k})(\int_{0}^{t} 1_{(T'_k, S'_{k+1})}(s)dB_s(w)) - \text{sgn}(B_t)(\int_{0}^{t} 1_{(T'_m, T'_1)}(s)dB_s(w))
\]

\[
= |B_t(w)| - (\sum_{k=1}^{m} \text{sgn}(B_{T'_k})(B_{S'_{k+1}}(w) - B_{T'_k}(w)) - \text{sgn}(B_t)(B_t(w) - B_{T'_m}(w))
\]

\[
= |B_t(w)| - (\sum_{k=1}^{m} \text{sgn}(B_{T'_k})(w)(0 - \text{sgn}(B_{T'_k})(w) \times \epsilon) - \text{sgn}(B_t)(w)(B_t(w) - \text{sgn}(B_t)(w) \times \epsilon)
\]

\[
= \epsilon + m\epsilon
\]

\[
r'_T(w) + N'_T(w)\epsilon.
\]

Thus we have, a.s.,

\[
L_t + \int_{0}^{t} \sum_{n=1}^{\infty} 1_{[S'_n, T'_n]}(s)\text{sgn}(B_s)dB_s = \epsilon N'_T + r'_T.
\]

where \(|r'_T| \leq \epsilon\).

Next, we show that

\[
\epsilon N'_T \stackrel{L^2}{\rightharpoonup} L_t \text{ as } \epsilon \to 0.
\]

Fix \( t \geq 0 \). Note that

\[
\sum_{k=1}^{\infty} 1_{[S'_n, T'_n]}(s) \leq 1_{|B_s| \leq \epsilon}(w) \text{ for all } 0 \leq s \leq t \text{ and } w \in \Omega.
\]

and so

\[
||\epsilon N'_T - L_t||_{L^2} \leq || \int_{0}^{t} \sum_{n=1}^{\infty} 1_{[S'_n, T'_n]}(s)\text{sgn}(B_s)dB_s||_{L^2} + ||r'_T||_{L^2}
\]

\[
= E[\int_{0}^{t} \sum_{n=1}^{\infty} 1_{[S'_n, T'_n]}(s)ds] + ||r'_T||_{L^2}
\]

\[
= \int_{0}^{t} E[\sum_{n=1}^{\infty} 1_{[S'_n, T'_n]}(s)]ds + ||r'_T||_{L^2}
\]

\[
\leq \int_{0}^{t} E[1_{|B_s| \leq \epsilon}(w)]ds + ||r'_T||_{L^2}
\]

\[
= \int_{0}^{t} P(|B_s| \leq \epsilon)ds + ||r'_T||_{L^2} \overset{c\to 0}{\longrightarrow} \int_{0}^{t} P(|B_s| = 0)ds = 0.
\]

7. First we show that \( \frac{L_t}{\sqrt{t}} \overset{d}{=} |U| \) for all \( t > 0 \). Define stopping times \( \Gamma_a \) as (33). Fix \( t_0 > 0 \). By (20) and Corollary 2.22, we get

\[
P\left( \frac{L_{t_0}}{\sqrt{t_0}} \leq a \right) = P\left( \sup_{s \leq t_0} (-\beta_s) \leq a \times \sqrt{t_0} \right) = P\left( \Gamma_a \sqrt{t_0} \geq t_0 \right) = \int_{t_0}^{\infty} \frac{\sqrt{t_0}}{\sqrt{2\pi t^3}} \exp\left( -\frac{t_0a^2}{2t} \right)dt.
\]
Set \( x = \frac{\sqrt{a_0}}{\sqrt{t}} \). Then \( dx = \frac{1}{2} \frac{\sqrt{a_0}}{t^{3/2}} dt \) and

\[
\int_0^\infty \frac{\sqrt{a_0}}{\sqrt{2\pi t^3}} \exp(-\frac{t_0a_0^2}{2t}) dt = \int_0^a \frac{2}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx = P(|U| \leq a).
\]

Recall that if \( X_n \overset{d}{\to} X \) and \( Y_n \overset{d}{\to} 0 \), then \( X_n + Y_n \overset{d}{\to} X \). To show that \( \frac{N^1_t}{\sqrt{t}} \overset{d}{\to} |U| \), it suffices to show that, as \( t \to \infty \),

\[
\frac{1}{\sqrt{t}} (N^1_t - L_t) = \frac{1}{\sqrt{t}} \left( \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) dB_s - r^1_t \right) \overset{L^2}{\to} 0.
\]

Note that

\[
|| \frac{1}{\sqrt{t}} \left( \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) dB_s - r^1_t \right) ||_{L^2} \leq || \frac{1}{\sqrt{t}} \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) dB_s ||_{L^2} + || \frac{1}{\sqrt{t}} r^1_t ||_{L^2}
\]

and

\[
|| \frac{1}{\sqrt{t}} r^1_t ||_{L^2} \leq \frac{1}{\sqrt{t}}.
\]

It suffices to show that

\[
\frac{1}{\sqrt{t}} \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) dB_s \overset{L^2}{\to} 0 \text{ as } t \to \infty.
\]

By (32), we get

\[
|| \frac{1}{\sqrt{t}} \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) dB_s ||_{L^2}^2
\]

\[
= E \left[ \frac{1}{t} \int_0^t \sum_{n=1}^\infty 1_{[s_n^1, T_n^1]}(s) \text{sgn}(B_s) ds \right] \leq E \left[ \frac{1}{t} \int_0^t 1_{|B_s| \leq 1} ds \right]
\]

\[
= \frac{1}{t} \int_0^t P(|B_s| \leq 1) ds = \frac{1}{t} \int_0^t P(|B_1| \leq 1) ds
\]

\[
= \frac{2}{t} \int_0^t \int_0^{\frac{\sqrt{t}}{2}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx ds
\]

\[
= \frac{2}{t} \left( \int_0^{\frac{\sqrt{t}}{2}} \int_0^t \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx ds + \int_0^\infty \int_0^{\frac{\sqrt{t}}{2}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx ds \right)
\]

\[
= \frac{2}{t} \int_0^{\frac{\sqrt{t}}{2}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \int_0^\infty \frac{1}{x^2} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx
\]

\[
\leq \frac{2}{t} \int_0^{\frac{\sqrt{t}}{2}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \int_0^\infty \frac{1}{x^2} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx
\]

\[
= \frac{2}{t} \int_0^{\frac{\sqrt{t}}{2}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \frac{1}{\sqrt{2\pi}} \sqrt{t}
\]

\[
= \int_0^{\frac{\sqrt{t}}{2}} \frac{2}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \frac{2}{\sqrt{2\pi}} \sqrt{t} \to 0.
\]

\[
\square
\]

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5.9 Exercise 5.33 (Study of multidimensional Brownian motion)

Let $B_t = (B_t^1, ..., B_t^N)$ be an $N$-dimensional $(\mathcal{F}_t)$-Brownian motion started from $x = (x_1, ..., x_N)$. We suppose that $N \geq 2$.

1. Verify that $|B_t|^2$ is a continuous semimartingale, and that the martingale part of $|B_t|^2$ is a true martingale.

2. We set

$$
\beta_t = \sum_{i=1}^{N} \int_{0}^{t} \frac{B_s^i}{|B_s|} dB_s^i
$$

with the convention that $\frac{B_s^i}{|B_s|} = 0$ if $|B_s| = 0$. Justify the definition of the stochastic integrals appearing in the definition of $\beta_t$, then show that the process $(\beta_t)_{t \geq 0}$ is an $(\mathcal{F}_t)$-Brownian motion started from 0.

3. Show that

$$
|B_t|^2 = |x|^2 + 2 \int_{0}^{t} |B_s| dB_s + Nt.
$$

4. From now on, we assume that $x \neq 0$. Let $\epsilon \in (0, |x|)$ and $T_\epsilon = \inf\{t \geq 0 \mid |B_t| \leq \epsilon\}$. Define $f : (0, \infty) \to \mathbb{R}$ by

$$
f(a) = \begin{cases} 
\log(a), & \text{if } N = 2 \\
2^{-N}, & \text{if } N \geq 3
\end{cases}
$$

Verify that $f(|B_{t\vee T_\epsilon}|)$ is a continuous local martingale.

5. Let $R > |x|$ and set $S_R = \inf\{t \geq 0 \mid |B_t| \geq R\}$. Show that

$$
P(T_\epsilon < S_R) = \frac{f(R) - f(|x|)}{f(R) - f(\epsilon)}.
$$

Observing that $P(T_\epsilon < S_R) \to 0$ as $\epsilon \to 0$, show that $B_t \neq 0$ for all $t \geq 0$, a.s.

6. Show that, a.s., for every $t \geq 0$,

$$
|B_t| = |x| + \beta_t + \frac{N-1}{2} \int_{0}^{t} ds.
$$

7. We assume that $N \geq 3$. Show that $\lim_{t \to \infty} |B_t| = \infty$ (a.s.) (Hint: Observe that $|B_t|^{2-N}$ is a nonnegative supermartingale.)

8. We assume $N = 3$. Using the form of the Gaussian density, verify that the collection of random variables $\{B_t^{i\vee T}\}_{t \geq 0}$ is bounded in $L^2$. Show that $\{B_t^{i\vee T}\}_{t \geq 0}$ is a continuous local martingale but is not a (true) martingale.

**Proof.**

1. By Itô’s formula and Doob’s inequality in $L^2$, we get

$$
|B_t|^2 = |x|^2 + \sum_{i=1}^{N} \int_{0}^{t} 2B_s^i dB_s^i + Nt
$$

and

$$
E[\int_{0}^{t} 2B_s^i dB_s^i, \int_{0}^{t} 2B_s^j dB_s^j] = 4E[\int_{0}^{t} (B_s^i)^2 ds] \leq 4tE[\sup_{0 \leq s \leq t} (B_s^i)^2] \leq 4t^2E[(B_t^i)^2] \leq 16t(t + x_i^2)
$$

for $1 \leq i \leq N$. Thus, $\{\int_{0}^{t} 2B_s^i dB_s^i\}_{t \geq 0}$ is a true $(\mathcal{F}_t)$-martingale for $1 \leq i \leq N$. 

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2. Since \((B_t^N)^2 \leq 1\), we see that \(\frac{B_t^N}{|B_t^N|} \in L^2_{loc}(B^t)\) and, hence, \(\int_0^t \frac{B_s^N}{|B_s^N|} dB_s^i\) is well-defined continuous local martingale. Thus, \((\beta_t)_{t \geq 0}\) is a \((\mathcal{F}_t)\)-continuous local martingale. Because

\[
\langle \beta, \beta \rangle_t = \sum_{i=1}^N \int_0^t \frac{(B_s^i)^2}{|B_s^i|^2} ds = t,
\]

we see that \((\beta_t)_{t \geq 0}\) is an \((\mathcal{F}_t)\)-Brownian motion started from 0.

3. Note that

\[
B_t^i = \frac{B_t^i}{|B_t^i|}|B_t^i|,
\]

where \(\frac{B_t^i}{|B_t^i|}\) is defined in problem 2, and

\[
d\beta_t = \sum_{i=1}^N \frac{B_t^i}{|B_t^i|} dB_t^i.
\]

Then

\[
|B_t|^2 = |x|^2 + \sum_{i=1}^N \int_0^t 2B_t^i dB_t^i + Nt = |x|^2 + 2 \int_0^t |B_s| dB_s + Nt.
\]

4. Define \(F : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}\) by \(F(x) = f(|x|)\). Then we have

\[
\frac{\partial F}{\partial x_i}(x) = \begin{cases} \frac{(2-N)x_i}{|x|^2}, & \text{if } N \geq 3 \\ \frac{2}{|x|^2}, & \text{if } N = 2 \end{cases}
\]

and

\[
\frac{\partial^2 F}{\partial x_i^2}(x) = \begin{cases} \frac{N-2}{|x|^2} (1 - \frac{x_i^2}{|x|^2}), & \text{if } N \geq 3 \\ 1 - \frac{2x_i^2}{|x|^2}, & \text{if } N = 2. \end{cases}
\]

Note that \(|B_{t \wedge T}(w)| \geq \epsilon\) for all \(t \geq 0\) and \(w \in \Omega\). By Itô’s formula, we get

\[
f(|B_{t \wedge T}|) = F(B_{t \wedge T})
\]

\[
= f(|x|) + \sum_{i=1}^N \int_0^t \frac{\partial F}{\partial x_i}(B_{s \wedge T}) dB_s^i + \frac{N}{2} \sum_{i=1}^N \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_{s \wedge T}) ds
\]

\[
= \begin{cases} f(|x|) + \sum_{i=1}^N \int_0^t (2-N)\frac{B_{s \wedge T}^i}{|B_{s \wedge T}^i|^2} dB_s^i + \frac{1}{2} \sum_{i=1}^N \int_0^t \frac{N-2}{|B_{s \wedge T}^i|^2} (1 - \frac{N(B_{s \wedge T}^i)^2}{|B_{s \wedge T}^i|^2}) ds, & \text{if } N \geq 3 \\ f(|x|) + \sum_{i=1}^N \int_0^t \frac{B_{s \wedge T}^i}{|B_{s \wedge T}^i|^2} dB_s^i + \frac{1}{2} \sum_{i=1}^N \int_0^t (1 - \frac{B_{s \wedge T}^i}{|B_{s \wedge T}^i|^2}) ds, & \text{if } N = 2 \end{cases}
\]

and, hence, \(f(|B_{t \wedge T}|)\) is a continuous local martingale.

5. Set \(T = T_{\epsilon} \wedge S_R\). Then \(|f(|B_T^T|)| \leq M\) for some \(M > 0\) and all \(t \geq 0\) (a.s.). Since \(f(|B_{t \wedge T}|)\) is a continuous local martingale, we see that \(f(|B_T^T|)\) is a bounded continuous local martingale and, hence, \(f(|B_T^T|)\) is an uniformly bounded martingale. Then we have

\[
f(|x|) = E[f(|B_T^T|)] = E[f(|B_T|)] = f(\epsilon)P(T_\epsilon < S_R) + f(R)P(T_\epsilon \geq S_R).
\]

Since \(P(T_\epsilon < S_R) + P(T_\epsilon \geq S_R) = 1\), we get

\[
P(T_\epsilon < S_R) = \frac{f(R) - f(|x|)}{f(R) - f(\epsilon)}.
\]
Because \( f(\epsilon) \to \pm \infty \) (depending on \( N \)) as \( \epsilon \to 0 \), we see that \( P(T_\epsilon < S_R) \to 0 \) as \( \epsilon \to 0 \). Next we show that \( B_t \neq 0 \) for all \( t \geq 0 \) (a.s.). Choose a sequence of positive real number \( \{\epsilon_n\} \) such that \( \epsilon_n \downarrow 0 \) and

\[
\sum_{n=1}^{\infty} P(T_{\epsilon_n} < S_n) < \infty.
\]

By Borel Cantelli’s lemma, we get \( P(Z) = 0 \), where \( Z = \limsup_{n \to \infty} \{T_{\epsilon_n} < S_n\} \). Then \( B_t \neq 0 \) for all \( t \geq 0 \) in \( Z^c \). Indeed, if \( w \in Z^c \) and \( B_t(w) = 0 \) for some \( t > 0 \), then \( T_{\epsilon_n}(w) < t \) for all \( n \geq 1 \) and, hence, \( S_n(w) < t \) for some \( m \geq 1 \) and all \( n \geq m \). Since \( \{S_n(w)\} \) is nondecreasing, we see that \( \lim_{n \to \infty} S_n(w) \) exists, \( s = \lim_{n \to \infty} S_n(w) \leq t \) and, hence, \( B_s(w) = \infty \) which is a contradiction. Thus, \( B_t \neq 0 \) for all \( t \geq 0 \), a.s.

6. Define \( F : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+ \) by \( F(x) = |x| \). Then \( F \in C^\infty(\mathbb{R}^N \setminus \{0\}) \), \( \frac{\partial F}{\partial x_i}(x) = \frac{x_i}{|x|} \), and \( \frac{\partial^2 F}{\partial x_i^2}(x) = \frac{|x|^2 - x_i^2}{|x|^3} \). Since \( B_t \in \mathbb{R}^N \setminus \{0\} \) for all \( t \geq 0 \) (a.s.), we get (see the proof of problem 4)

\[
|B_t|^{2-N} = |x|^{2-N} + \sum_{i=1}^{N} \int_0^t \frac{(2-N)B_i^2}{|B_s|^N} \, dB_s.
\]

Then \( |B_t|^{2-N} \) is a non-negative continuous local martingale and, hence, \( |B_t|^{2-N} \) is a non-negative supermartingale. Thus,

\[
E[|B_t|^{2-N}] \leq E[|B_0|^{2-N}] = |x|^{2-N}
\]

for all \( t \geq 0 \). By Theorem 3.19, \( |B_\infty|^{2-N} \) exists (a.s.) and, hence, \( \lim_{t \to \infty} |B_t| \) exists (a.s.). Since \( \limsup_{t \to \infty} B_t = \infty \) (a.s.), we see that \( \lim_{t \to \infty} |B_t| = \infty \) (a.s.).

8. First, we show that \( (|B_t|^{-1})_{t \geq 0} \) is bounded in \( L^2 \). Set \( \delta = \frac{|x|}{2} > 0 \). Then

\[
E[|B_t|^{-2}] = \int_{\mathbb{R}^3} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|y-x|^2}{2t}\right) dy = \int_{|y|<\delta} + \int_{|y|\geq\delta}.
\]

Since

\[
\int_{|y|\geq\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \leq \frac{1}{\delta^2} \int_{\mathbb{R}^3} \frac{1}{(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \leq \frac{1}{\delta^2}
\]

for all \( t > 0 \), it suffices to show that

\[
\int_{|y|<\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|y-x|^2}{2t}\right) dy
\]

is bounded in \( t > 0 \). Note that, if \( |y| < \delta = \frac{|x|}{2} \), then \( |y-x| \geq |x| - |y| \geq \frac{|x|}{2} \). Then we see that

\[
\int_{|y|<\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \leq \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{|y|<\delta} \frac{1}{|y|^2} dy = \frac{1}{(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|x|^2}{8t}\right) w_3,
\]

where \( w_3 \) is the area of unit sphere in \( \mathbb{R}^3 \). Define \( \varphi : (0, \infty) \to \mathbb{R}_+ \) by

\[
\varphi(t) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|x|^2}{8t}\right).
\]
Then $\varphi \in C_0((0, \infty))$ and $\lim_{t \downarrow 0} \varphi(t) = 0$. There exists $M > 0$ such that $\sup_{t > 0} |\varphi(t)| \leq M < \infty$. Thus,

$$\sup_{t > 0} \int_{|y| < \delta} \frac{1}{|y|^2 (2\pi t)^{3/2}} \exp\left(-\frac{|y - x|^2}{2t}\right) dy \leq M w_3$$

and therefore $(|B_t|^{-1})_{t \geq 0}$ is bounded in $L^2$. Now we show that $(|B_t|^{-1})_{t \geq 0}$ is a continuous local martingale but is not a true martingale. Assume that $(|B_t|^{-1})_{t \geq 0}$ is a true martingale. Then $(|B_t|^{-1})_{t \geq 0}$ is a $L^2$-bounded martingale. Recall that $\lim_{t \to \infty} |B_t| = \infty$ (a.s.). Together with Theorem 4.13, we get

$$0 = E[|B_{\infty}|^{-2}] = E[|B_0|^{-2}] + E[(|B|^{-1}, |B|^{-1})_{\infty}]$$

which is a contradiction. Thus $(|B_t|^{-1})_{t \geq 0}$ is a continuous local martingale (see the proof of problem 7) but is not a true martingale.

\[ \square \]
Chapter 6
General Theory of Markov Processes

6.1 Exercise 6.23 (Reflected Brownian motion)

We consider a probability space equipped with a filtration \((\mathcal{F}_t)_{t \in [0, \infty]}\). Let \(a \geq 0\) and let \(B = (B_t)_{t \geq 0}\) be an \((\mathcal{F}_t)\)-Brownian motion such that \(B_0 = a\). For every \(t > 0\) and every \(z \in \mathbb{R}\), we set

\[
p_t(z) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{z^2}{2t}).
\]

1. We set \(X_t = |B_t|\) for every \(t \geq 0\). Verify that, for every \(s \geq 0\) and \(t \geq 0\), for every bounded measurable function \(f : \mathbb{R}_+ \mapsto \mathbb{R}\),

\[
E[f(X_{s+t}) | \mathcal{F}_s] = Q_tf(X_s),
\]

where \(Q_0f = f\) and, for every \(t > 0\), for every \(x \geq 0\),

\[
Q_tf(x) = \int_0^\infty (p_t(y-x) + p_t(y+x))f(y)dy.
\]

2. Infer that \((Q_t)_{t \geq 0}\) is a transition semigroup, then that \((X_t)_{t \geq 0}\) is a Markov process with values in \(E = \mathbb{R}_+\), with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\), with semigroup \((Q_t)_{t \geq 0}\).

3. Verify that \((Q_t)_{t \geq 0}\) is a Feller semigroup. We denote its generator by \(L\).

4. Let \(f\) be a twice continuously differentiable function on \(\mathbb{R}_+\), such that \(f\) and \(f''\) belong to \(C_0(\mathbb{R}_+)\). Show that, if \(f'(0) = 0\), \(f\) belongs to the domain of \(L\), and \(Lf = \frac{1}{2}f''\). (Hint: One may observe that the function \(g : \mathbb{R} \mapsto \mathbb{R}\) defined by \(g(y) = f(|y|)\) is then twice continuously differentiable on \(\mathbb{R}\).) Show that, conversely, if \(f(0) \neq 0\), \(f\) does not belong to the domain of \(L\).

Proof.

1. Set \(Q^B_t\) to be the semigroup of real Brownian motion (i.e. \(Q^B_t(x, dy) = p_t(y-x)dy\)). Given a bounded measurable function \(f : \mathbb{R}_+ \mapsto \mathbb{R}\). Define \(g : \mathbb{R} \mapsto \mathbb{R}\) by \(g(y) = f(|y|)\). By definition of Markov process,

\[
E[f(X_{s+t}) | \mathcal{F}_s] = E[g(B_{s+t}) | \mathcal{F}_s] = Q^B_tg(B_s)
\]

\[
= \int_{-\infty}^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t})dy
\]

\[
= \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t})dy + \int_{-\infty}^0 f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t})dy
\]

\[
= \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t})dy + \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y+B_s)^2}{2t})dy
\]

\[
= \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t})dy + \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y+B_s)^2}{2t})dy
\]

\[
= Q_tf(X_s).
\]

2. It’s clear that

\[
(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto Q_t(x, A) = \int_0^\infty \left( \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right) \right) 1_A(y)dy
\]

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is a measurable function. Thus, it suffices to show that \((Q_t)_{t \geq 0}\) satisfy Chapman-Kolmogorov’s identity. Let \(f\) be a bounded measurable function on \(\mathbb{R}_+\). Define \(g : \mathbb{R} \mapsto \mathbb{R}\) by \(g(y) = f(|y|)\). By using similar argument as the proof of problem 1, we have
\[
Q_t f(|x|) = Q_t B g(x) \quad \forall x \in \mathbb{R}.
\] (23)
and therefore
\[
Q_{t+s} f(x) = Q_{t+s} B g(x) = Q_t B Q_s B g(x) = \int_{\mathbb{R}} Q_t B g(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy
\]
\[
= \int_{\mathbb{R}^+} Q_t B g(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy + \int_{\mathbb{R}^-} Q_t B g(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right) dy
\]
\[
= \int_{\mathbb{R}^+} Q_t B g(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy + \int_{\mathbb{R}^+} Q_t B g(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right) dy
\]
\[
= Q_t Q_s B x \quad \forall x \in \mathbb{R}_+.
\]

3. Given \(f \in C_0(\mathbb{R}_+)\). Then \(g(x) = f(|x|) \in C_0(\mathbb{R})\). Since \((Q_t B)_{t \geq 0}\) is Feller semigroup, we see that \(Q_t f(x) = Q_t B g(x) \in C_0(\mathbb{R}_+)\) and
\[
\sup_{x \in \mathbb{R}_+} |Q_t f(x) - f(x)| \leq \sup_{x \in \mathbb{R}} |Q_t B g(x) - g(x)| \xrightarrow{t \to 0} 0.
\]
Therefore \((Q_t)_{t \geq 0}\) is a Feller semigroup.

4. Let \(f\) be a twice continuously differentiable function on \(\mathbb{R}_+\), such that \(f\) and \(f''\) belong to \(C_0(\mathbb{R}_+)\). Define \(g : \mathbb{R} \mapsto \mathbb{R}\) by \(g(y) = f(|y|)\). Observe that
\[
\lim_{t \to 0^+} \frac{g(x) - g(0)}{x} = \lim_{t \to 0^+} \frac{f(x) - f(0)}{x} = f'(0).
\]
and
\[
\lim_{t \to 0^-} \frac{g(x) - g(0)}{x} = \lim_{t \to 0^-} \frac{f(-x) - f(0)}{-x} = -f'(0).
\]
Since \(f'(0) = 0\), \(g'(0)\) exists and therefore
\[
g'(y) = f'(|y|) \text{ sgn}(y)
\]
and
\[
g''(y) = f''(|y|),
\]
where \(\text{sgn}(y) = 1_{\{y > 0\}} - 1_{\{y < 0\}}\). Thus \(g\) is a twice continuously differentiable function on \(\mathbb{R}\), such that \(g\) and \(g''\) belong to \(C_0(\mathbb{R})\). Let \(L^B\) be the generator of \((Q_t B)_{t \geq 0}\). Then \(L^B h = \frac{1}{2} h''\) (see the example after Corollary 6.13). By (32), we have
\[
L f(x) = \lim_{t \to 0} \frac{Q_t f(x) - f(x)}{t} = \lim_{t \to 0} \frac{Q_t B g(x) - g(x)}{t} = \frac{1}{2} g''(x) = \frac{1}{2} f''(x) \quad \forall x \in \mathbb{R}_+
\]
and therefore \(L f = \frac{1}{2} f''\). Conversely, assume that there exists \(f \in C_0(\mathbb{R}_+) \cap D(L)\) such that \(f'(0) \neq 0\). Then \(g'(0)\) doesn’t exist and \(\lim_{t \to 0} \frac{Q_t f(x) - f(x)}{t}\) exists for all \(\forall x \in \mathbb{R}_+.\) By (32), we see that
\[
\lim_{t \to 0} \frac{Q_t B g(x) - g(x)}{t} = \lim_{t \to 0} \frac{Q_t f(x) - f(x)}{t} = L f(x) \quad \forall x \geq 0,
\]
\[
\lim_{t \to 0} \frac{Q_t B g(x) - g(x)}{t} = \lim_{t \to 0} \frac{Q_t f(-x) - f(-x)}{t} = L f(-x) \quad \forall x < 0,
\]
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and therefore $L^B g(x) = L_x f(|x|)$ for all $x \in \mathbb{R}$. Since $L_t f \in C_0(\mathbb{R}_+)$, we see that $L^B g \in C_0(\mathbb{R})$ and, hence, $g \in D(L^B) = \{ h \in C^2(\mathbb{R}) \mid h \text{ and } h'' \in C_0(\mathbb{R}) \}$ (see the example after Corollary 6.13) which is a contradiction. Thus, we see that

$$D(L) = \{ h \in C^2(\mathbb{R}_+) \mid h, h'' \in C_0(\mathbb{R}_+) \text{ and } h'(0) = 0 \}.$$

and $Lf = \frac{1}{2} f''$.

\[ \square \]

### 6.2 Exercise 6.24

Let $(Q_t)_{t \geq 0}$ be a transition semigroup on a measurable space $E$. Let $\pi$ be a measurable mapping from $E$ onto another measurable space $F$. We assume that, for any measurable subset $A$ of $F$, for every $x, y \in E$ such that $\pi(x) = \pi(y)$, we have

$$Q_t(x, \pi^{-1}(A)) = Q_t(y, \pi^{-1}(A)) \quad \forall t > 0. \quad (24)$$

We then set, for every $z \in F$ and every measurable subset $A$ of $F$, for every $t > 0$,

$$Q'_t(z, A) = Q_t(x, \pi^{-1}(A)) \quad (25)$$

where $x$ is an arbitrary point of $E$ such that $\pi(x) = z$. We also set $Q'_0(z, A) = 1_A(z)$. We assume that the mapping $(t, z) \mapsto Q'_t(z, A)$ is measurable on $\mathbb{R}_+ \times F$, for every fixed $A$.

1. Verify that $(Q'_t)_{t \geq 0}$ forms a transition semigroup on $F$.

2. Let $(X_t)_{t \geq 0}$ be a Markov process in $E$ with transition semigroup $(Q_t)_{t \geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Set $Y_t = \pi(X_t)$ for every $t \geq 0$. Verify that $(Y_t)_{t \geq 0}$ is a Markov process in $F$ with transition semigroup $(Q'_t)_{t \geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

3. Let $(B_t)_{t \geq 0}$ be a $d$-dimensional Brownian motion, and set $R_t = B_t$ for every $t \geq 0$. Verify that $(R_t)_{t \geq 0}$ is a Markov process and give a formula for its transition semigroup (the case $d = 1$ was treated via a different approach in Exercise 6.23).

**Proof.**

1. To show that $(Q'_t)_{t \geq 0}$ forms a transition semigroup on $F$, it remain to show that $(Q'_t)_{t \geq 0}$ satisfies Chapman–Kolmogorov identity. Since

$$\int_F 1_A(y)Q'_t(\pi(x), dy) = \int_E 1_A(\pi(y))Q_t(x, dy),$$

we get

$$(Q'_t f)(\pi(x)) = Q_t g(x), \quad (26)$$

where $f$ is a bounded measurable function on $F$, $g = f \circ \pi$, and $x \in E$. Given $z \in F$. Since $\pi$ is surjective, there exists $x \in E$ such that $z = \pi(x)$. By (26) and (25), we get

$$Q'_{t+s} f(z) = Q_{t+s} g(x) = Q_t Q_s g(x) = \int_E Q_s g(y)Q_t(x, dy)$$

$$= \int_E Q'_s f(\pi(y))Q_t(x, dy) = \int_E Q'_s f(w)Q_t(\pi(x), dw)$$

$$= Q'_s Q'_t f(\pi(x)) = Q'_t Q'_s f(z).$$

2. It’s clear that $(Y_t)_{t \geq 0}$ is an adapted process. It remain to show that has $(Y_t)_{t \geq 0}$ Markov property. Let $f$ be a bounded measurable function on $F$ and $g = f \circ \pi$. By (26), we get

$$E[f(Y_{t+s}) \mid \mathcal{F}_s] = E[g(X_{t+s}) \mid \mathcal{F}_s] = Q_t g(X_s) = Q'_t f(\pi(X_s)) = Q'_t f(Y_s).$$

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3. The case \( d = 1 \) was solved in Exercise 6.23. Now we assume that \( d \geq 2 \). Recall that
\[
Q_{t}f(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-|w-x|^2/2t) f(w) dw.
\]
for all bounded measurable function \( f \) on \( \mathbb{R}^d \). Define \( \pi(x) = |x| \) and \( Q_t'(z,A) \) as (25) for \( z \in \mathbb{R}_+ \) and \( A \in \mathcal{B}_{\mathbb{R}_+} \). First we show that \((Q_t)_{t \geq 0}\) satisfies condition (24). Let \( A \in \mathcal{B}_{\mathbb{R}_+} \) and \( B = \pi^{-1}(A) \). Then
\[
OB \equiv \{ Ox \mid x \in B \} = B
\]
for all orthogonal matrix \( O \). Given \( x, y \in \mathbb{R}^d \) such that \( \pi(x) = \pi(y) \). Choose an orthogonal matrix \( O \) such that \( x = Oy \). Then
\[
Q_t(x, \pi^{-1}(A)) = Q_t(x, B) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-|w-x|^2/2t) 1_B(w) dw
\]
\[
= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-|Ou-Oy|^2/2t) 1_B(Ou) du \quad (w = Ou)
\]
\[
= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-|u-y|^2/2t) 1_{O^{-1}B}(u) du
\]
\[
= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-|u-y|^2/2t) 1_B(u) du
\]
\[
= Q_t(y, B) = Q_t(y, \pi^{-1}(A))
\]
Next we show that the mapping \((t, z) \mapsto Q_t'(z,A)\) is measurable on \( \mathbb{R}_+ \times \mathbb{R}_+ \) for all \( A \in \mathcal{B}_{\mathbb{R}_+} \). Given a bounded measurable function \( f \) on \( \mathbb{R}_+ \) and \( z \in \mathbb{R}_+ \). Set \( x = (z, 0, ..., 0) \) and \( g = f \circ \pi \). By (26), we have
\[
Q_t f(z) = Q_t g(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-1/2t((w_1 - z)^2 + \sum_{k=2}^{d} w_k^2)) f(|w|) dw.
\]
(27)
This shows that the mapping \((t, z) \mapsto Q_t'(z,A)\) is measurable on \( \mathbb{R}_+ \times \mathbb{R}_+ \) for all \( A \in \mathcal{B}_{\mathbb{R}_+} \). By problem 2, we see that \((R_t)_{t \geq 0}\) is a Markov process with semigroup (27).

In the remaining exercises, we use the following notation. \((E, d)\) is a locally compact metric space, which is countable at infinity, and \((Q_t)_{t \geq 0}\) is a Feller semigroup on \( E \). We consider an \( E \)-valued process \((X_t)_{t \geq 0}\) with càdlàg sample paths, and a collection \((P_x)_{x \in E}\) of probability measures on \( E \), such that, under \((P_x, (X_t)_{t \geq 0})\) is a Markov process with semigroup \((Q_t)_{t \geq 0}\) with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) and \( P_x(X_0 = x) = 1 \). We write \( L \) for the generator of the semigroup \((Q_t)_{t \geq 0}\), \( D(L) \) for the domain of \( L \) and \( R_\lambda \) for the \( \lambda \)-resolvent, for every \( \lambda > 0 \).

6.3 Exercise 6.25 (Scale Function)
In this exercise, we assume that \( E = \mathbb{R}_+ \) and that the sample paths of \( X \) are continuous. For every \( x \in \mathbb{R}_+ \), we set
\[
T_x \equiv \inf \{ t \geq 0 \mid X_t = x \}
\]
and
\[
\varphi(x) \equiv P_x(T_0 < \infty).
\]
1. Show that, if \( 0 \leq x \leq y \),
\[
\varphi(y) = \varphi(x) P_y(T_x < \infty).
\]
2. We assume that \( \varphi(x) < 1 \) and \( P_x(\sup_{t \geq 0} X_t = \infty) = 1 \), for every \( x > 0 \). Show that, if \( 0 < x \leq y \),
\[
P_x(T_0 < T_y) = \frac{\varphi(x) - \varphi(y)}{1 - \varphi(y)}.
\]
Proof.

1. By strong Markov property, we have

\[ P_y(T_0 < \infty) = P_y(T_0 < \infty, T_x < \infty) = E_y[1_{\{T_x < \infty\}} 1_{\{T_0 < \infty\}}] = E_y[1_{\{T_x < \infty\}} E_{X_{T_x}}[1_{\{T_0 < \infty\}}]]. \]

Since \((X_t)_{t \geq 0}\) has continuous sample path, we get \(X_{T_x} = x\) on \(\{T_x < \infty\}\) and therefore

\[ P_y(T_0 < \infty) = E_y[1_{\{T_x < \infty\}} E_{X_{T_x}}[1_{\{T_0 < \infty\}}]] = P_y(T_x < \infty) P_x(T_0 < \infty) = \varphi(x) P_y(T_x < \infty). \]

2. Because \(P_x(T_y < \infty) = 1\), we get

\[ P_x(T_0 < \infty) = P_x(T_0 < T_y) + P_x(T_0 < \infty, T_y < T_0). \]

By strong Markov property, we have

\[ E_x[1_{\{T_y < T_0\}} 1_{\{T_0 < \infty\}}] = E_x[1_{\{T_y < T_0\}} E_{X_{T_y}}[1_{\{T_0 < \infty\}}]]. \]

Since \((X_t)_{t \geq 0}\) has continuous sample path, we get \(X_{T_y} = y\) (a.s.) and therefore

\[ E_x[1_{\{T_y < T_0\}} 1_{\{T_0 < \infty\}}] = P_x(T_y < T_0) P_y(T_0 < \infty). \]

Hence

\[ \varphi(x) = P_x(T_0 < \infty) = P_x(T_0 < T_y) + P_x(T_0 < T_y) P_y(T_0 < \infty) = P_x(T_0 < T_y) + P_x(T_y < T_0) \varphi(y). \]

Since

\[ 1 = P_x(T_0 < T_y) + P_x(T_y < T_0) \]

and

\[ \varphi(x) < 1 \quad \forall x > 0, \]

we have

\[ P_x(T_0 < T_y) = \frac{\varphi(x) - \varphi(y)}{1 - \varphi(y)}. \]

\[ \Box \]

6.4 Exercise 6.26 (Feynman–Kac Formula)

Let \(v\) be a nonnegative function in \(C_0(E)\). For every \(x \in E\) and every \(t \geq 0\), we set, for every \(\varphi \in B(E)\),

\[ Q_t^* \varphi(x) \equiv E_x[\varphi(X_t) \exp(-\int_0^t v(X_s) ds)]. \]

1. Show that, for every \(\varphi \in B(E)\), and \(s, t \geq 0\), \(Q_{s+t}^* \varphi = Q_t^*(Q_s^* \varphi)\).

2. After observing that

\[ 1 - \exp(-\int_0^t v(X_s) ds) = \int_0^t v(X_s) \exp(-\int_s^t v(X_u) du) ds, \]

show that, for every \(\varphi \in B(E)\),

\[ Q_t \varphi - Q_t^* \varphi = \int_0^t Q_s(v Q_{t-s}^* \varphi) ds. \quad (28) \]

3. Assume that \(\varphi \in D(L)\). Show that

\[ \frac{d}{dt} Q_t^* \varphi|_{t=0} = L \varphi - v \varphi. \]
Proof.

1. Fix $s, t ≥ 0$. Define $\Phi^s(f) = \varphi(f(s)) \exp(-\int_0^s v(f(u))du)$. By simple Markov property, we get

$$Q^*_t(Q^*_s \varphi)(x) = E_x[E_{X_t}[\varphi(X_s) \exp(-\int_0^s v(X_u)du)] \exp(-\int_0^t v(X_u)du)]$$

$$= E_x[E_{X_t}[\Phi^s] \exp(-\int_0^t v(X_u)du)]$$

$$= E_x[E_{X_t}[(X_{t+r})_{r≥0} : \mathcal{F}_t] \exp(-\int_0^t v(X_u)du)]$$

$$= E_x[\varphi(X_{s+t}) \exp(-\int_0^s v(X_{u+t})du) \exp(-\int_0^t v(X_u)du)]$$

$$= E_x[\varphi(X_{s+t}) \exp(-\int_0^t v(X_u)du) \exp(-\int_0^t v(X_u)du)] = Q^*_t \varphi(x)$$

2. Observe that

$$\frac{d}{ds} \exp(-\int_0^t v(X_u)du) = v(X_s) \exp(-\int_s^t v(X_u)du).$$

Then we have

$$1 - \exp(-\int_0^t v(X_u)du) = \int_0^t v(X_s) \exp(-\int_s^t v(X_u)du)ds.$$ 

By Fubini’s theorem and simple Markov property, we get

$$Q_t \varphi(x) - Q^*_t \varphi(x) = E_x[\varphi(X_t)] - E_x[\varphi(X_t) \exp(-\int_0^t v(X_u)du)]$$

$$= E_x[\varphi(X_t)](1 - \exp(-\int_0^t v(X_u)du))]$$

$$= E_x[\varphi(X_t) \times \int_0^t v(X_s) \exp(-\int_s^t v(X_u)du)ds]$$

$$= \int_0^t E_x[\varphi(X_t) \times v(X_s) \exp(-\int_s^t v(X_u)du)ds]$$

$$= \int_0^t E_x[\varphi(X_t) \times (X_{s+t})_{r≥0}]ds$$

$$= \int_0^t E_x[\varphi(X_s)E_x[\Phi^{t-s}((X_{s+r})_{r≥0} : \mathcal{F}_s)]ds$$

$$= \int_0^t E_x[\varphi(X_s)E_{X_s}[\Phi^{t-s}((X_{s+r})_{r≥0} : \mathcal{F}_s)]ds$$

$$= \int_0^t E_x[\varphi(X_s)Q^{t-s} \varphi(X_s)]ds$$

$$= \int_0^t Q_s(vQ^{t-s} \varphi)(x)ds$$

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3. Note that
\[ Q_t \varphi(x) = \varphi(x) + \int_0^t Q_s(L\varphi)(x) \, ds \]
and \( Q_0 \varphi(x) = \varphi(x) \). By differentiating (32), we have
\[ \frac{d}{dt} Q_t \varphi(x) \Big|_{t=0} = L\varphi(x) - v(x) \varphi(x). \]

\[ \square \]

6.5 Exercise 6.27 (Quasi left-continuity)

Throughout the exercise we fix the starting point \( x \in E \). For every \( t > 0 \), we write \( X_{t-}(w) \) for the left-limit of the sample path \( s \mapsto X_s(w) \) at \( t \).

Let \((T_n)_{n \geq 1}\) be a strictly increasing sequence of stopping times, and \( T = \lim_{n \to \infty} T_n \). We assume that there exists a constant \( C < \infty \) such that \( T \leq C \). The goal of the exercise is to verify that \( X_T = X_{T-} \), \( P_x\)-a.s.

1. Let \( f \in D(L) \) and \( h = Lf \). Show that, for every \( n \geq 1 \),
\[ E_x[f(X_T) \mid \mathcal{F}_{T_n}] = f(X_{T_n}) + E_x \left[ \int_{T_n}^T h(X_s) \, ds \mid \mathcal{F}_{T_n} \right]. \]

2. We recall from the theory of discrete time martingales that
\[ E_x[f(X_T) \mid \mathcal{F}_{T_n}] \overset{a.s.}{\to} E_x[f(X_T) \mid \mathcal{F}_T], \]
where
\[ \mathcal{F}_T = \bigcup_{n=1}^\infty \mathcal{F}_{T_n}. \]

Infer from question (1) that
\[ E[f(X_T) \mid \mathcal{F}_T] = f(X_{T-}). \]

3. Show that the conclusion of question (2) remains valid if we only assume that \( f \in C_0(E) \), and infer that, for every choice of \( f, g \in C_0(E) \),
\[ E_x[f(X_T)g(X_{T-})] = E_x[f(X_{T-})g(X_{T-})]. \]

Conclude that \( X_{T-} = X_T \), \( P_x\)-a.s.

Proof.

1. By Theorem 6.14, we see that \( (f(X_t) - \int_0^t h(X_s) \, ds)_{t \geq 0} \) is a martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\). By Corollary 3.23, we have
\[ E_x[f(X_T) - \int_0^T h(X_s) \, ds \mid \mathcal{F}_{T_n}] = f(X_{T_n}) - \int_0^{T_n} h(X_s) \, ds \]
and so
\[ E_x[f(X_T) \mid \mathcal{F}_{T_n}] = f(X_{T_n}) + E_x \left[ \int_{T_n}^T h(X_s) \, ds \mid \mathcal{F}_{T_n} \right]. \]

2. Note that
\[ E_x[f(X_T) \mid \mathcal{F}_T] \leq ||f||_u < \infty, \]
where \( ||f||_u = \sup_{x \in E} |f(x)| \). Then the discrete time martingale
\[ (E_x[f(X_T) \mid \mathcal{F}_{T_n}])_{n \geq 0} = (E_x[E_x[f(X_T) \mid \mathcal{F}_T] \mid \mathcal{F}_{T_n}])_{n \geq 0} \]
is closed and, hence,
\[ f(X_{T_n}) + E_x \left[ \int_{T_n}^T h(x_s) ds \mid \mathcal{F}_{T_n} \right] = E_x[f(X_T) \mid \mathcal{F}_{T_n}] \xrightarrow{a.s., L^1} E_x[f(X_T) \mid \mathcal{F}_T]. \]

Note that \( \lim_{n \to \infty} X_{T_n} = X_T \), \( P_x \)-a.s. and \( ||h||_u < \infty \). By Lebesgue’s dominated convergence theorem, we get
\[ \|f(X_T) - f(X_{T_n}) - E_x \left[ \int_{T_n}^T h(x_s) ds \mid \mathcal{F}_{T_n} \right] \|_{L^1} \]
\[ \leq \|f(X_T) - f(X_{T_n})\|_{L^1} + \|E_x \left[ \int_{T_n}^T h(x_s) ds \mid \mathcal{F}_{T_n} \right] \|_{L^1} \]
\[ \leq E_x[|f(X_T) - f(X_{T_n})|] + E_x[|h(x_s)| ds] \]
\[ \leq E_x[|f(X_T) - f(X_{T_n})|] + ||h||_u E_x[T - T_n] \xrightarrow{n \to \infty} 0 \]
and therefore \( E[f(X_T) \mid \mathcal{F}_T] = f(X_{T-}), \ P_x \)-a.s.

3. First, we show that
\[ E[f(X_T) \mid \mathcal{F}_T] = f(X_{T-}) \quad \forall f \in C_0(E). \]

By proposition 6.8 and proposition 6.12, we see that
\[ D(L) = \mathcal{R} = \{ R_\lambda f \mid f \in C_0(E) \} \]
is dense in \( C_0(E) \). Given \( f \in C_0(E) \) and \( \epsilon > 0 \). Choose \( g \in D(L) \) such that \( ||f - g||_u < \epsilon \). Then
\[ E[g(X_T) \mid \mathcal{F}_T] = g(X_{T-}) \]
and, hence,
\[ E_x[|E[f(X_T) \mid \mathcal{F}_T] - f(X_{T-})|] \]
\[ \leq E_x[|E[f(X_T) \mid \mathcal{F}_T] - E[g(X_T) \mid \mathcal{F}_T]|] + E_x[|g(X_{T-}) - f(X_{T-})|] \]
\[ \leq E_x[|g(X_T) - f(X_{T-})|] + E_x[|g(X_{T-}) - f(X_{T-})|] \]
\[ \leq 2||f - g||_u \leq 2\epsilon. \]

By letting \( \epsilon \to 0 \), we get
\[ E[f(X_T) \mid \mathcal{F}_T] = f(X_{T-}). \]

Next, we show that \( X_{T-} = X_T \). Let \( f, g \in C_0(E) \). Then \( g(X_{T-}) \) is \( \mathcal{F}_T \)-measurable and, hence,
\[ E_x[f(X_T)g(X_{T-})] = E_x[E_x[f(X_T) \mid \mathcal{F}_T]g(X_{T-})] = E_x[f(X_{T-})g(X_{T-})]. \]

Thus, we have
\[ E_x[f(X_T)g(X_{T-})] = E_x[f(X_{T-})g(X_{T-})] \quad \forall f, g \in C_0(E). \]

Hence
\[ E_x[f(X_T)g(X_{T-})] = E_x[f(X_{T-})g(X_{T-})] \quad \forall f, g \in B(E) \]
and therefore
\[ E_x[h(X_T, X_{T-})] = E_x[h(X_{T-}, X_{T-})] \quad \forall h \in B(E \times E). \]

For \( \epsilon > 0 \), if we set \( h(x, y) = 1_{d(x,y) \leq \epsilon}(x, y) \), then
\[ P_x(d(X_T, X_{T-}) > \epsilon) = E_x[h(X_T, X_{T-})] = E_x[h(X_{T-}, X_{T-})] = 0. \]

Therefore \( X_{T-} = X_T, \ P_x \)-a.s.
6.6 Exercise 6.28 (Killing operation)

In this exercise, we assume that $X$ has continuous sample paths. Let $A$ be a compact subset of $E$ and

$$T_A = \inf\{t \geq 0 \mid X_t \in A\}.$$

1. We set, for every $t \geq 0$ and every bounded measurable function $\varphi$ on $E$,

$$Q_t^* \varphi(x) = E_x[\varphi(X_t)1_{\{t < T_A\}}], \quad \forall x \in E.$$

Verify that $Q_t^* \varphi = Q_t^*(Q^\ast_t \varphi)$, for every $s, t > 0$.

2. We set $\overline{E} = (E \setminus A) \cup \{\Delta\}$, where $\Delta$ is a point added to $E \setminus A$ as an isolated point. For every bounded measurable function $\varphi$ on $E$ and every $t \geq 0$, we set

$$\overline{Q}_t^* \varphi(x) = \begin{cases} E_x[\varphi(X_t)1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta), & \text{if } x \in E \setminus A \\ \varphi(\Delta), & \text{if } x = \Delta. \end{cases}$$

Verify that $(\overline{Q}_t^*)_{t \geq 0}$ is a transition semigroup on $\overline{E}$. (The proof of the measurability of the mapping $(t, x) \mapsto \overline{Q}_t^* \varphi(x)$ will be omitted.)

3. Show that, under the probability measure $P_x$, the process $\overline{X}$ defined by

$$\overline{X}_t = \begin{cases} X_t, & \text{if } t < T_A \\ \Delta, & \text{if } t \geq T_A. \end{cases}$$

is a Markov process with semigroup $(\overline{Q}_t^*)_{t \geq 0}$, with respect to the canonical filtration of $X$.

4. We take it for granted that the semigroup $(\overline{Q}_t^*)_{t \geq 0}$ is Feller, and we denote its generator by $\overline{L}$. Let $f \in D(\overline{L})$ such that $f$ and $Lf$ vanish on an open set containing $A$. Write $\overline{f}$ for the restriction of $f$ to $E \setminus A$, and consider $\overline{f}$ as a function on $\overline{E}$ by setting $\overline{f}(\Delta) = 0$. Show that $\overline{f} \in D(\overline{L})$ and $\overline{L}\overline{f}(x) = Lf(x)$ for every $x \in E \setminus A$.

Proof.

1. By the simple Markov property, we have

$$Q_t^*(Q^\ast_t \varphi)(x) = E_x[Q^\ast_t \varphi(X_t)1_{\{t < T_A\}}]$$

$$= E_x[E_x[\varphi(X_s)1_{\{s < T_A\}}]1_{\{t < T_A\}}]$$

$$= E_x[E_x[\varphi(X_{s+t})1_{\{s < T_A\}}1_{\{r \geq 0 \mid X_{r+t} \in A\}}}| \mathcal{F}_t]1_{\{t < T_A\}}]$$

$$= E_x[\varphi(X_{s+t})1_{\{s < T_A\}}1_{\{r \geq 0 \mid X_{r+t} \in A\}}1_{\{t < T_A\}}]$$

$$= E_x[\varphi(X_{s+t})1_{\{t+s < T_A\}}] = Q_{t+s}^* \varphi(x)$$

2. First, we show that $x \in \overline{E} \mapsto \overline{Q}_t^* \varphi(x)$ is measurable for every bounded measurable function $\varphi$ on $\overline{E}$ and every $t \geq 0$. Observe that

$$\{x \in \overline{E} \mid \overline{Q}_t^* \varphi(x) \in \Gamma\} = \{(\overline{Q}_t \varphi \in \Gamma) \cap (E \setminus A)\} \cup \{\{\Delta\}, \text{ if } \varphi(\Delta) \in \Gamma \text{ otherwise.} \}

Define $\overline{\varphi} : E \mapsto \mathbb{R}$ by

$$\overline{\varphi}(x) = \begin{cases} \varphi(x), & \text{if } x \in E \setminus A \\ 0, & \text{if } x \in A. \end{cases}$$

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Then \( \tilde{\varphi} \) is a bounded measurable function on \( E \) and, hence,
\[
x \in E \mapsto E_x[\tilde{\varphi}(X_t)1_{\{t < T_A\}}]
\]
is measurable on \( E \). Note that
\[
\tilde{\varphi}(X_t) = \varphi(X_t) \text{ in } \{t < T_A\}.
\]
Then we see that
\[
x \in E \setminus A \mapsto E_x[\tilde{\varphi}(X_t)1_{\{t < T_A\}}] = E_x[\varphi(X_t)1_{\{t < T_A\}}]
\]
is measurable on \( E \setminus A \). Similarly, we see that
\[
x \in E \setminus A \mapsto P_x(T_A \leq t)
\]
is measurable on \( E \setminus A \). Thus,
\[
x \in E \setminus A \mapsto E_x[\varphi(X_t)1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) = \overline{Q}_t\varphi(x)
\]
is measurable on \( E \setminus A \) and, hence,
\[
\{x \in E \mid \overline{Q}_t\varphi(x) \in \Gamma\} = (\{\overline{Q}_t\varphi \in \Gamma\} \cap (E \setminus A)) \bigcup \left\{\{\Delta\}, \text{ if } \varphi(\Delta) \in \Gamma \right\} \bigcup \{0\}, \text{ otherwise.}
\]
is a measurable set on \( E \setminus A \).
Next, we show that \( \overline{Q}_t\overline{Q}_s\varphi = \overline{Q}_{t+s}\varphi \) for all bounded measurable function \( \varphi \) on \( E \). It’s clear that
\[
\overline{Q}_t\overline{Q}_s\varphi(\Delta) = \overline{Q}_s\varphi(\Delta) = \varphi(\Delta) = \overline{Q}_{t+s}\varphi(\Delta).
\]
Now, we suppose \( x \in E \setminus A \). By the simple Markov property, we get
\[
\overline{Q}_t\overline{Q}_s\varphi(x)
\]
\[
\begin{align*}
&= E_x[\overline{Q}_s\varphi(X_t)1_{\{t < T_A\}}] + P_x(T_A \leq t)\overline{Q}_s\varphi(\Delta) \\
&= E_x[\overline{Q}_s\varphi(X_t)1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) \\
&= E_x[E_x[\varphi(X_s)1_{\{s < T_A\}}] + P_x(T_A \leq s)\varphi(\Delta)1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) \\
&= E_x[E_x[\varphi(X_s)1_{\{s < T_A\}}]1_{\{t < T_A\}}] + E_x[P_x(T_A \leq s)\varphi(\Delta)1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) \\
&= E_x[\varphi(X_{s+t})1_{\{s + \inf\{r \geq 0 \mid X_{s+r} \in A\} \leq s \mid t \leq T_A\}}1_{\{t < T_A\}}] + E_x[1_{\{\inf\{r \geq 0 \mid X_{s+r} \in A\} \leq s \}}1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) \\
&= E_x[\varphi(X_{s+t})1_{\{t + \inf\{r \geq 0 \mid X_{s+r} \in A\} \leq s \mid t \leq T_A\}}] + \varphi(\Delta)E_x[1_{\{\inf\{r \geq 0 \mid X_{s+r} \in A\} \leq s \}}1_{\{t < T_A\}}] + P_x(T_A \leq t)\varphi(\Delta) \\
&= E_x[\varphi(X_{s+t})1_{\{t + \inf\{r \geq 0 \mid X_{s+r} \in A\} \leq s \mid t \leq T_A\}}] + P_x(T_A \leq s + t)\varphi(\Delta) = \overline{Q}_{s+t}(x).
\end{align*}
\]
3. For \( t \geq 0 \) and a measurable set \( \Gamma \) of \( E \) such that \( \Delta \notin \Gamma \),
\[
\{X_t \in \Gamma\} = \{X_t \in \Gamma\} \cap \{t < T_A\} \in \mathcal{F}_t
\]
and, hence, \((X_t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-adapted process. Now, we show that \((X_t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-Markov process on \( E \). Let \( \varphi \in B(E) \). Note that
\[
\varphi(X_t) = \begin{cases} 
\varphi(X_t), & \text{if } t < T_A \\
\varphi(\Delta), & \text{if } t \geq T_A.
\end{cases}
\]
By the simple Markov property, we get

\[ E_x[\varphi(X_{t+s}) | \mathcal{F}_s] = E_x[\varphi(X_{t+s})1_{\{t+s\leq T_A\}} | \mathcal{F}_s] + E_x[\varphi(X_{t+s})1_{\{t+s > T_A\}} | \mathcal{F}_s] \]

and

\[ = E_x[\varphi(X_{t+s})1_{\{t \leq \inf\{r \geq 0 | X_{s+r} \in A\}\}} | \mathcal{F}_s] + E_x[\varphi(\Delta)(1_{\{s < T_A\}}1_{\{t \geq T_A\}} + 1_{\{s \geq T_A\}}) | \mathcal{F}_s] \]

4. Let us show that

\[ T \mathcal{J}(x) = \begin{cases} Lf(x), & \text{if } x \in E \setminus A \\ 0, & \text{if } x = \Delta. \end{cases} \]

Since \( \Delta \) is an isolated point of \( E \setminus A \) and \( f, Lf \in C_0(E) \), we see that \( \mathcal{J}, Lf \in C_0(E) \). By theorem 6.14, it suffices to show that \( \mathcal{J}, Lf \) is a \( (\mathcal{F}_t)_{t \geq 0} \)-martingale under \( P_x \) for all \( x \in E \). If \( x = \Delta \), then

\[ X_t = \Delta \quad \forall t \geq 0 \quad P_x \text{-a.s.} \]

and so

\[ \mathcal{J}(X_t) = Lf(X_t) = 0 \quad \forall t \geq 0 \quad P_x \text{-a.s.} \]

Thus \( \mathcal{J}(X_t) - \int_0^t Lf(X_s)ds \) is a zero process. Now, we suppose \( x \in E \setminus A \). Since \( f \) and \( Lf \) vanish on an open set containing \( A \), we see that

\[ f(X_{t \wedge T_A}) = Lf(X_{t \wedge T_A}) = 0 \quad \forall t \geq T_A. \]

Thus, we have

\[ \mathcal{J}(X_t) = f(X_{t \wedge T_A}) \quad \forall t \geq 0 \]

and

\[ \int_0^t Lf(X_s)ds = \int_0^t Lf(X_{s \wedge T_A})ds = \int_0^{t \wedge T_A} Lf(X_s)ds \quad \forall t \geq 0. \]

Since \( f(X_t) - \int_0^t Lf(X_s)ds \) is a \( (\mathcal{F}_t)_{t \geq 0} \)-martingale under \( P_x \), we get

\[ \mathcal{J}(X_t) - \int_0^t Lf(X_s)ds \geq 0 \]

is a \( (\mathcal{F}_t)_{t \geq 0} \)-martingale under \( P_x \). Thus \( \mathcal{J} \in D(T) \) and

\[ T \mathcal{J}(x) = Lf(x) = \begin{cases} Lf(x), & \text{if } x \in E \setminus A \\ 0, & \text{if } x = \Delta. \end{cases} \]

\[ \square \]

6.7 Exercise 6.29 (Dynkin’s formula)

1. Let \( g \in C_0(E) \) and \( x \in E \), and let \( T \) be a stopping time. Justify the equality

\[ E_x[1_{\{T < \infty\}}e^{-\lambda T} \int_0^{T \wedge T_A} e^{-\lambda t} g(X_{T+t})dt] = E_x[1_{\{T < \infty\}}e^{-\lambda T} R_\lambda g(X_T)] \]  (29)
2. Infer that
\[ R_\lambda g(x) \mid T < \infty = E_x \left[ e^{-\lambda t} g(X_t) dt \right] + E_x \left[ 1_{\{T < \infty\}} e^{-\lambda T} R_\lambda g(X_T) \right]. \quad (30) \]

3. Show that, if \( f \in D(L) \),
\[ f(x) = E_x \left[ e^{-\lambda t} (Lf)(X_t) dt \right] + E_x \left[ 1_{\{T < \infty\}} e^{-\lambda T} f(X_T) \right]. \]

4. Assuming that \( E_x[T] < \infty \), infer from the previous question that
\[ E_x \left[ e^{-\lambda t} Lf(X_t) dt \right] = E_x[f(X_T)] - f(x). \quad (Dynkin's formula) \quad (31) \]

How could this formula have been established more directly?

5. For every \( \epsilon > 0 \), we set \( T_{\epsilon,x} = \inf \{ t > 0 \mid d(x, X_t) > \epsilon \} \). Assume that \( E_x[T_{\epsilon,x}] < \infty \), for every sufficiently small \( \epsilon \). Show that (still under the assumption \( f \in D(L) \)) one has
\[ Lf(x) = \lim_{\epsilon \downarrow 0} \frac{E_x[f(X_{T_{\epsilon,x}})] - f(x)}{E_x[T_{\epsilon,x}]} \]

6. Show that the assumption \( E_x[T_{\epsilon,x}] < \infty \) for every sufficiently small \( \epsilon \) holds if the point \( x \) is not absorbing, that is, if there exists a \( t > 0 \) such that \( Q_t(x, \{x\}) < 1 \). (Hint: Observe that there exists a nonnegative function \( h \in C_0(E) \) which vanishes on a ball centered at \( x \) and is such that \( Q_t h(x) > 0 \). Infer that one can choose \( \alpha > 0 \) and \( \eta \in (0, 1) \) such that \( P_x(T_{\alpha,x} > nt) \leq (1 - \eta)^n \) for every integer \( n \geq 1 \).)

**Proof.**

1. By Fubini’s theorem and the strong Markov property, we get
\[ E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} g(X_t) dt \right] = \int_0^\infty E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} e^{-\lambda t} g(X_{T+t}) dt \right] \]
\[ = \int_0^\infty E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} e^{-\lambda t} E_x[g(X_{T+t}) \mid \mathcal{F}_t] dt \right] \]
\[ = \int_0^\infty E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} e^{-\lambda t} E_{X_T}[g(X_t)] dt \right] \]
\[ = \int_0^\infty E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} e^{-\lambda t} \int_0^\infty e^{-\lambda t} Q_t g(X_T) dt \right] \]
\[ = E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} \int_0^\infty e^{-\lambda t} Q_t g(X_T) dt \right] \]
\[ = E_x\left[ 1_{\{T < \infty\}} e^{-\lambda t} R_\lambda g(X_T) \right]. \]

2. By (29), we get
\[ E_x\left[ \int_0^T e^{-\lambda t} g(X_t) dt \right] + E_x\left[ 1_{\{T < \infty\}} e^{-\lambda T} R_\lambda g(X_T) \right] \]
\[ = E_x\left[ \int_0^T e^{-\lambda t} g(X_t) dt \right] + E_x\left[ 1_{\{T < \infty\}} e^{-\lambda T} \int_0^\infty e^{-\lambda t} g(X_{T+t}) dt \right] \]
\[ = E_x\left[ \int_0^T e^{-\lambda t} g(X_t) dt \right] + E_x\left[ 1_{\{T < \infty\}} \int_T^\infty e^{-\lambda t} g(X_t) dt \right] \]
\[ = E_x\left[ \int_0^\infty e^{-\lambda t} g(X_t) dt \right] = \int_0^\infty e^{-\lambda t} E_x[g(X_t)] dt = \int_0^\infty e^{-\lambda t} Q_t g(X_t) dt = R_\lambda g(x). \]
3. Fix \( f \in D(L) \). By proposition 6.12, there exists \( g \in C_0(E) \) such that \( f = R_\lambda g \in D(L) \) and \( (\lambda - L)f = g \). By (30), we get
\[
f(x) = E_x\left[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt\right] + E_x[1_{\{T < \infty\}}e^{-\lambda T}f(X_T)].
\]

4. Note that \( f, L(f) \) are bounded and \( E_x[T] < \infty \). By Lebesgue’s dominated convergence theorem, we get
\[
\lim_{\lambda \to 0} E_x\left[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt\right] = \lim_{\lambda \to 0} E_x[1_{\{T < \infty\}}\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] = E_x[1_{\{T < \infty\}}\lim_{\lambda \to 0} \int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] = E_x[1_{\{T < \infty\}}\int_0^T Lf(X_t)dt]
\]
and therefore
\[
f(x) = \lim_{\lambda \to 0} E_x\left[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt\right] + \lim_{\lambda \to 0} E_x[1_{\{T < \infty\}}e^{-\lambda T}f(X_T)] = -E_x[\int_0^T Lf(X_t)dt] + E_x[f(X_T)].
\]

Next, we prove (31) directly. By theorem 6.14, we see that \((M_t)_{t \geq 0} \equiv (f(X_t) - \int_0^t Lf(X_s)ds)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-martingale. Let \( K > 0 \). Then \((M_t \wedge K)_{t \geq 0}\) is a uniformly integrable martingale. By optional stopping theorem, we have
\[
E_x[f(X_{T \wedge K}) - \int_0^{T \wedge K} Lf(X_s)ds] = f(x).
\]
Since \( E_x[T] < \infty \), we see that
\[
\lim_{K \to \infty} f(X_{T \wedge K}) = f(X_T) \quad P_x\text{-a.s.}
\]

By Lebesgue’s dominated convergence theorem, we get
\[
f(x) = E_x[f(X_T)] - E_x[\int_0^T Lf(X_s)ds].
\]

5. Fix \( f \in D(L) \). Given \( \eta > 0 \). Since \( Lf \) is continuous at \( x \), there exists \( \delta > 0 \) such that \( |Lf(y) - Lf(x)| < \eta \) whenever \( d(y, x) < \delta \). For sufficiently small \( \epsilon \) such that \( E_x[T_{\epsilon, x}] < \infty \) and \( \epsilon < \delta \), we have
\[
|Lf(X_t) - Lf(x)| < \eta \quad \forall 0 \leq t \leq T_{\epsilon, x}, P_x\text{-a.s.}
\]
and therefore
\[
\left|\frac{E_x[\int_0^{T_{\epsilon, x}} Lf(X_t)dt]}{E_x[T_{\epsilon, x}]} - Lf(x)\right| = \left|\frac{E_x[\int_0^{T_{\epsilon, x}} Lf(X_t)dt]}{E_x[T_{\epsilon, x}]} - \frac{E_x[\int_0^{T_{\epsilon, x}} Lf(X_t)dt]}{E_x[T_{\epsilon, x}]}\right|
\]
\[
= \frac{E_x[\int_0^{T_{\epsilon, x}} |Lf(X_t) - Lf(x)|dt]}{E_x[T_{\epsilon, x}]}
\]
\[
< \frac{E_x[T_{\epsilon, x}]}{E_x[T_{\epsilon, x}]} \eta = \eta
\]

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By (31), we get
\[
\lim_{\epsilon \downarrow 0} \frac{E_x[f(X_{T,\epsilon})] - f(x)}{E_x[T_{\epsilon,x}]} = \lim_{\epsilon \downarrow 0} \frac{E_x[T_{\epsilon,x} Lf(X_t)dt]}{E_x[T_{\epsilon,x}]} = Lf(x).
\]

6. Since \(Q_t(x, \{x\}) < 1\), there exists \(r > 0\) such that \(Q_t(x, B(x,r)) < 1\). Then \(E \setminus B(x,r)\) is an open set and \(Q_t(x, E \setminus B(x,r)) > 0\). Choose \(z \in E \setminus B(x,r)\). Then there exists \(R > 0\) such that \(Q_t(x, (E \setminus B(x,r)) \cap B(z,R)) > 0\). Set \(G = (E \setminus B(x,r)) \cap B(z,R)\). Then \(G\) is an bounded open set and \(Q_t 1_G(x) = Q_t(x,G) > 0\). Set
\[
f_k(y) = (\frac{d(y,E \setminus G)}{1 + d(y,E \setminus G)})^+ \quad \forall k \geq 1.
\]

Then
\[
0 \leq f_k(y) \uparrow 1_G(y) \quad \forall y \in E
\]
and \(f_k \in C_0(E)\) for all \(k \geq 1\). Since \((Q_t)_{t \geq 0}\) is Feller,
\[
Q_t f_k \in C_0(E) \quad \forall k \geq 1
\]
and
\[
Q_t f_k \xrightarrow{k \to \infty} Q_t(x,G).
\]
Choose large \(k\) such that \(Q_t f_k(x) > 0\) and set \(h = f_k\). Then \(0 < Q_t h(x) \leq 1\) and, hence, there exists \(0 < \alpha < r\) and \(0 < \eta < 1\) such that
\[
Q_t(y,G) \geq Q_t h(y) > \eta > 0 \quad \forall y \in B(x,\alpha).
\]
Thus,
\[
Q_t(y,E \setminus G) \leq (1 - \eta) \quad \forall y \in B(x,\alpha).
\]

For \(n \geq 1\), by the simple Markov property, we get
\[
P_x(T_{\alpha,x} > nt)
\leq E_x[1_{\{X_t \in B(x,\alpha)\}} \cdots 1_{\{X_{(n-1)t} \in B(x,\alpha)\}} 1_{\{X_{nt} \in B(x,\alpha)\}}]
= E_x[1_{\{X_t \in B(x,\alpha)\}} \cdots 1_{\{X_{(n-1)t} \in B(x,\alpha)\}} E_{X_{(n-1)t}}[1_{X_t \in B(x,\alpha)}]]
= E_x[1_{\{X_t \in B(x,\alpha)\}} \cdots 1_{\{X_{(n-1)t} \in B(x,\alpha)\}} Q_t(X_{(n-1)t}, B(x,\alpha))] 
\leq E_x[1_{\{X_t \in B(x,\alpha)\}} \cdots 1_{\{X_{(n-1)t} \in B(x,\alpha)\}} Q_t(X_{(n-1)t}, E \setminus G)] 
\leq E_x[1_{\{X_t \in B(x,\alpha)\}} \cdots 1_{\{X_{(n-1)t} \in B(x,\alpha)\}}] \{(1 - \eta)\}
\ldots
\leq (1 - \eta)^n.
\]

Therefore
\[
E_x[T_{\epsilon,x}] \leq E_x[T_{\alpha,x}] = \sum_{n=1}^{\infty} \int_{(n-1)t}^{nt} P_x(T_{\alpha,x} > t) dt \leq \sum_{n=1}^{\infty} (1 - \eta)^n < \infty
\]
for all \(\epsilon < \alpha\).
Chapter 7
Brownian Motion and Partial Differential Equations

7.1 Exercise 7.24
Let $B(0,1)$ be the open ball of $\mathbb{R}^d$ ($d \geq 2$), and $B(0,1)^* \equiv B(0,1) \setminus \{0\}$. Let $g$ be the continuous function defined on $\partial B(0,1)^*$ by
\[ g(x) = \begin{cases} 0, & \text{if } |x| = 1 \\ 1, & \text{if } x = 0. \end{cases} \]
Prove that the Dirichlet problem in $B(0,1)^*$ with boundary condition $g$ has no solution.

Proof. We prove this by contradiction. Assume that there exists a $u \in C^2(B(0,1)^*) \cap C(B(0,1))$ such that
\[ \begin{align*} \Delta u(x) &= 0, & \text{if } x \in B(0,1)^* \\ \lim_{y \in B(0,1)^* \to x \in \partial B(0,1)^*} u(y) &= g(x), & \text{if } x \in \partial B(0,1)^*. \end{align*} \]
By proposition 7.7, we see that $u(x) = E_x[g(B_T)] \quad \forall x \in B(0,1)^*$, where $T = U_0 \wedge U_1$ and $U_a = \inf\{t \geq 0 \mid |B_t| = a\}$. By proposition 7.16, we see that
\[ P_x(U_0 < U_1) = \lim_{\epsilon \downarrow 0} P_x(U_\epsilon < U_1) = \begin{cases} \lim_{\epsilon \downarrow 0} \frac{0 - \log(|x|)}{0 - \log(r)}, & \text{if } d = 2 \\ \lim_{\epsilon \downarrow 0} \frac{1 - |x|^2 - \epsilon^2}{1 - |x|^2 - \epsilon}, & \text{if } d \geq 3 \end{cases} = 0 \]
and, hence,
\[ u(x) = E_x[g(B_T)] = E_x[g(B_{U_1})1_{\{U_1 < U_0\}}] = 0 \quad \forall x \in B(0,1)^* \]
which contradict to
\[ \lim_{y \in B(0,1)^* \to 0} u(y) = 0 \neq 1 = g(0). \]

7.2 Exercise 7.25 (Polar sets)
Throughout this exercise, we consider a nonempty compact subset $K$ of $\mathbb{R}^d$ ($d \geq 2$). We set $T_K = \inf\{t \geq 0 \mid T_t \in K\}$. We say that $K$ is polar if there exists an $x \in K^c$ such that $P_x(T_K < \infty) = 0$.

1. Using the strong Markov property as in the proof of Proposition 7.7 (ii), prove that the function $x \mapsto P_x(T_K < \infty)$ is harmonic on every connected component of $K^c$.

2. From now on until question 4., we assume that $K$ is polar. Prove that $K^c$ is connected, and that the property $P_x(T_K < \infty) = 0$ holds for every $x \in K^c$. Hint: Observe that $\{x \in K^c \mid P_x(T_K < \infty) = 0\}$ is both open and closed.

3. Let $D$ be a bounded domain containing $K$, and $D' = D \setminus K$. Prove that any bounded harmonic function $h$ on $D'$ can be extended to a harmonic function on $D$. Does this remain true if the word “bounded” is replaced by “positive”?
4. Define
\[ g(x) = \begin{cases} 
0, & \text{if } x \in \partial D \\
1, & \text{if } x \in \partial D' \setminus \partial D. 
\end{cases} \]
Prove that the Dirichlet problem in $D'$ with boundary condition $g$ has no solution. (Note that this generalizes the result of Exercise 7.24.)

5. If $\alpha \in (0,d]$, we say that the compact set $K$ has zero $\alpha$-dimensional Hausdorff measure if, for every $\epsilon > 0$, we can find an integer $N_\epsilon \geq 1$ and $N_\epsilon$ open balls $B(c_k,r_k)$, $k = 1,2,...,N_\epsilon$, such that
\[ K \subseteq \bigcup_{k=1}^{N_\epsilon} B(c_k,r_k) \text{ and } \sum_{k=1}^{N_\epsilon} r_k^\alpha \leq \epsilon. \]
Prove that if $d \geq 3$ and $K$ has zero $d-2$-dimensional Hausdorff measure then $K$ is polar.

Proof.
We define $T_A = \inf\{t \geq 0 \mid B_t \in A\}$ for all closed subset $A$ of $\mathbb{R}^d$.

1. Define $\varphi : K^c \to \mathbb{R}$ by $\varphi(x) = P_x(T_K < \infty)$. To show that $\varphi$ is harmonic on every connected component of $K^c$, it suffices to show that $\varphi$ satisfies the mean value property for every $x \in K^c$. Fix $x \in K^c$. Let $r > 0$ such that $B(x,r) \subseteq K^c$. Set $T_{x,r} = \inf\{t \geq 0 \mid |B_t - x| = r\}$. Then
\[ T_{x,r} < T_K, \quad T_{x,r} < \infty \quad P_x\text{-a.s.} \]
By the strong Markov property, we get
\[ \varphi(x) = E_x[1_{\{T_K < \infty\}}] = E_x[E_{B_{T_{x,r}}}[1_{\{T_K < \infty\}}]] = E_x[\varphi(B_{T_{x,r}})]. \]
Since the distribution of $B_{T_{x,r}}$ under $P_x$ is the uniform probability measure $\sigma_{x,r}$ on the $\partial B(x,r)$, we have
\[ \varphi(x) = E_x[\varphi(B_{T_{x,r}})] = \int_{\partial B(x,r)} \varphi(y)\sigma_{x,r}(dy). \]

2. First, we show that $K^c$ is connected. We prove this by contradiction. Assume that $K^c = \bigcup_{n=1}^m G_n$, where $G_n$ is a connected component of $K^c$ and $2 \leq m \leq \infty$. Then
\[ \bigcup_{n=1}^m \partial G_n \subseteq K. \]
For $x \in G_i$, choose $y \in G_j$, where $i \neq j$, and $r > 0$ such that $\overline{B(y,r)} \subseteq G_j$. By proposition 7.16, we get
\[ P_x(T_K < \infty) \geq P_x(T_{\partial G_i} < \infty) \geq P_x(T_{\overline{B(y,r)}} < \infty) > 0. \]
Thus, we get
\[ P_x(T_K < \infty) > 0 \quad \forall x \in K^c \]
which contradict to $K$ is polar.

Next, we show that
\[ P_x(T_K < \infty) = 0 \quad \forall x \in K^c. \]
Since $K^c$ is connected, it suffices to show that
\[ \Gamma \equiv \{x \in K^c \mid P_x(T_K < \infty) = 0\} \]
is both open and closed in $K^c$. Indeed, since K is polar, we see that $\Gamma$ is nonempty and, hence, $\Gamma = K^c$. By problem 1., we see that $\varphi(z) = P_z(T_K < \infty)$ is continuous in $K^c$ and so
\[ \Gamma = \varphi^{-1}({0}). \]
is closed in $K^c$. Now, we show that $\Gamma$ is open in $K^c$. Fix $x \in \Gamma$. We choose $r > 0$ such that $B(x, r) \subseteq K^c$. Assume that there exists $y \in B(x, r)$ such that $P_y(T_K < \infty) > \eta$ for some $\eta > 0$. Since $\varphi(z) = P_z(T_K < \infty)$ is continuous in $K^c$, there exists $r' > 0$ such that $B(y, r') \subseteq B(x, r)$ and

$$P_z(T_K < \infty) > \eta \frac{1}{2} \quad \forall z \in B(y, r').$$

By the strong Markov property, we get

$$P_x(T_K < \infty) \geq P_x(T_{B(y,r')} < T_K < \infty) = E_x[E_{B(y,r')}[1_{\{T_K < \infty\}}]] \geq \frac{\eta}{2} > 0$$

which is a contradiction. Thus, $B(x, r) \subseteq \Gamma$ and therefore $\Gamma$ is open in $K^c$.

3. (a) Choose a sequence of bounded domains $\{\Gamma_n\}$ such that $K \subseteq \Gamma_n$, $\Gamma_n \subseteq \Gamma_{n+1}$ \quad \forall n \geq 1$, and $\Gamma_n \uparrow D$.

Define $u : D \mapsto \mathbb{R}$ by

$$u(x) = \lim_{n \to \infty} E_x[h(B_{\partial \Gamma_n})].$$

Now we show that $u$ satisfy

$$\begin{cases}
\Delta u(x) = 0, & \text{if } x \in D \\
u(x) = h(x), & \text{if } x \in D'.
\end{cases}$$

First, we show that $u = h$ in $D'$ and $u$ is well-defined.

i. Fix $x \in D'$. Choose large $n$ such that $x \in \Gamma_n$. Since $x \in K^c$ and $K$ is polar, we get $T_K = \infty P_x$-(a.s.) and so

$$B_{\partial \Gamma_n \wedge t} \in D' \quad \forall t \geq 0 \quad P_x$-(a.s.).

By Itô’s formula, we have

$$h(B_{t \wedge \partial \Gamma_n}) = h(x) + \int_0^{t \wedge \partial \Gamma_n} \nabla h(B_s) \cdot dB_s \quad \forall t \geq 0 \quad P_x$-(a.s.)

and therefore $(h(B_{t \wedge \partial \Gamma_n}))_{t \geq 0}$ is a continuous local martingale. Since $h$ is bounded in $D'$, $(h(B_{t \wedge \partial \Gamma_n}))_{t \geq 0}$ is a uniformly integrable martingale and, hence,

$$h(x) = E_x[h(B_{\partial \Gamma_n})].$$

Therefore, if $x \in \Gamma_m$ for some $m \geq 1$, then

$$E_x[h(B_{\partial \Gamma_n})] = h(x) \quad \forall n \geq m.$$  

Moreover,

$$u(x) = \lim_{n \to \infty} E_x[h(B_{\partial \Gamma_n})] = h(x).$$

ii. Fix $x \in K$. We show that

$$E_x[h(B_{\partial \Gamma_n})] = E_x[h(B_{\partial \Gamma_m})] \quad \forall n > m \geq 1.$$  

Fix $n > m$. Then $\Gamma_m \subseteq \Gamma_n$. By the strong Markov property, we get

$$E_x[h(B_{\partial \Gamma_n})] = E_x[E_{B_{\partial \Gamma_m}}[h(B_{\partial \Gamma_n})]].$$

By (32), we have

$$E_{B_{\partial \Gamma_m}}[h(B_{\partial \Gamma_n})] = h(B_{\partial \Gamma_m}) P_x$-(a.s.)
and so 

$$E_x[h(B_{T_{0\infty}})] = E_x[h(B_{T_{0\infty}})].$$

Moreover,

$$\lim_{n \to \infty} E_x[h(B_{T_{0\infty}})] = E_x[h(B_{T_{0\infty}})]$$

and, hence, $u$ is well-defined.

Next, we show that $u$ is harmonic on $D$. It suffices to show that $u$ satisfies the mean value property. Fix $x \in D$ and $r > 0$ such that $B(x, r) \subseteq D$. Choose $n \geq 1$ such that $B(x, r) \subseteq \Gamma_n$. Set $T_{x, r} = \inf\{t \geq 0 \mid |B_t - x| = r\}$. By (32) and (33), we have

$$E_x[h(B_{T_{0\infty}})] = u(z) \quad \forall z \in \Gamma_n.$$ 

By the strong Markov property, we get

$$u(x) = E_x[h(B_{T_{0\infty}})] = E_x[E_{B_{x, r}}[h(B_{T_{0\infty}})]] = E_x[u(B_{x, r})].$$

Since the distribution of $B_{x, r}$ under $P_x$ is the uniform probability measure $\sigma_{x, r}$ on the $\partial B(x, r)$, we have

$$u(x) = \int_{\partial B(x, r)} u(y)\sigma_{x, r}(dy).$$

Therefore $u$ is a harmonic function on $D$ such that $u(x) = h(x)$ for all $x \in D'$.

(b) Now we show that boundedness is necessary for this statement. Set $K = \{0\}$. By proposition 7.16, $K$ is a polar. Choose $D = B(0, r)$ for some $0 < r < 1$. Then $D' = B(0, r) \setminus \{0\}$. Define $\Phi$ to be the fundamental solution of Laplace equation. That is,

$$\Phi(x) = \begin{cases} \frac{-1}{2\pi} \log(|x|), & \text{if } d = 2 \\ \frac{1}{n(n-2)\pi n} \frac{1}{|x|^{d-2}}, & \text{if } d \geq 3. \end{cases}$$

Then $\Phi$ is a unbounded, positive harmonic function on $D'$ and $\Phi$ can’t be extended to a harmonic function on $D$.

4. We prove this by contradiction. Assume that there exists a $u \in C^2(D') \cap C(\overline{D'})$ such that

$$\begin{cases} \Delta u(x) = 0, & \text{if } x \in D' \\ \lim_{y \to x \in \partial D'} u(y) = g(x), & \text{if } x \in \partial D'. \end{cases}$$

By proposition 7.7, we see that

$$u(x) = E_x[g(B_T)] \quad \forall x \in D',$$

where $T = T_{\partial D} \wedge T_{\partial D' \setminus \partial D}$. Note that

$$T_{\partial D' \setminus \partial D} = T_K \quad P_x\text{-a.s.} \quad \forall x \in D'.$$

Fix $x \in D'$. Since $T_K = \infty P_x$-(a.s.), we see that $T = T_{\partial D} P_x$-(a.s.) and, hence,

$$u(x) = E_x[g(B_T)] = E_x[g(B_{T_{\partial D}})] = 0.$$

Thus, we see that

$$u(x) = 0 \quad \forall x \in D'$$

which contradict to

$$\lim_{x \in D', y \to \partial D' \setminus \partial D} u(x) = 0 \neq 1 = g(y) \quad \forall y \in \partial D' \setminus \partial D.'
5. To show that \( K \) is polar, we show that \( P_x(T_K < \infty) = 0 \) for all \( x \in K^c \). Fix \( x \in K^c \). Then
\[
h_{x,K} \equiv \inf \{ |x - z| | z \in K \} > 0.
\]
Given \( \epsilon > 0 \). There exists \( N_\epsilon \geq 1 \) and \( N_\epsilon \) open balls \( B(c_k, r_k) \), \( k = 1, 2, ..., N_\epsilon \), such that
\[
K \subseteq \bigcup_{k=1}^{N_\epsilon} B(c_k, r_k) \text{ and } \sum_{k=1}^{N_\epsilon} r_k^{d-2} \leq \epsilon.
\]
Without loss of generality, we assume that
\[
B(c_k, r_k) \bigcap K \neq \emptyset \quad \forall k = 1, 2, ..., N_\epsilon.
\]
Choose \( \tilde{c}_k \in B(c_k, r_k) \bigcap K \) and set \( \tilde{r}_k = 2r_k \) for all \( k = 1, 2, ..., N_\epsilon \). Then
\[
K \subseteq \bigcup_{k=1}^{N_\epsilon} B(\tilde{c}_k, \tilde{r}_k) \text{ and } \sum_{k=1}^{N_\epsilon} \tilde{r}_k^{d-2} \leq 2^{d-2} \epsilon.
\]
Set \( T_k = \inf \{ t \geq 0 | |B_t - \tilde{c}_k| = \tilde{r}_k \} \) for all \( k = 1, 2, ..., N_\epsilon \). Then
\[
P_x(T_K < \infty) \leq P_x(\bigwedge_{k=1}^{N_\epsilon} T_k < \infty) \leq \sum_{k=1}^{N_\epsilon} P_x(T_k < \infty).
\]
By proposition 7.16, we get
\[
P_x(T_k < \infty) = \left( \frac{\tilde{r}_k}{|x - \tilde{c}_k|} \right)^{d-2} \quad \forall k = 1, 2, ..., N_\epsilon
\]
and, hence,
\[
P_x(T_K < \infty) \leq \sum_{k=1}^{N_\epsilon} \left( \frac{\tilde{r}_k}{|x - \tilde{c}_k|} \right)^{d-2} \leq \sum_{k=1}^{N_\epsilon} \left( \frac{\tilde{r}_k}{h_{x,K}} \right)^{d-2} < \frac{2^{d-2}}{h_{x,K}^{d-2}} \epsilon.
\]
By letting \( \epsilon \downarrow 0 \), we have \( P_x(T_K < \infty) = 0 \).

\[\square\]

### 7.3 Exercise 7.26

In this exercise, \( d \geq 3 \). Let \( K \) be a compact subset of the open unit ball of \( \mathbb{R}^d \), and \( T_K = \inf \{ t \geq 0 : B_t \in K \} \). We assume that \( D := \mathbb{R}^d \setminus K \) is connected. We also consider a function \( g \) defined and continuous on \( K \). The goal of the exercise is to determine all functions \( u : D \rightarrow \mathbb{R} \) that satisfy:

(P) \( u \) is bounded and continuous on \( D \), harmonic on \( D \), and \( u(y) = g(y) \) if \( y \in \partial D \).

(This is the Dirichlet problem in \( D \), but in contrast with Sect. 7.3 above, \( D \) is unbounded here.) We fix an increasing sequence \( \{R_n\}_{n \geq 1} \) of reals, with \( R_1 \geq 1 \) and \( R_n \uparrow \infty \) as \( n \to \infty \). For every \( n \geq 1 \), we set \( T_n = \inf \{ t \geq 0 | |B_t| \geq R_n \} \).

1. Suppose that \( u \) satisfies (P). Prove that, for every \( n \geq 1 \) and every \( x \in D \) such that \( |x| < R_n \),
\[
u(x) = E_x[g(B_{T_K}) 1\{T_K \leq T_n\}] + E_x[u(B_{T_n}) 1\{T_n \leq T_K\}].
\]

2. Show that, by replacing the sequence \( \{R_n\} \) with a subsequence if necessary, we may assume that there exists a constant \( \alpha \in \mathbb{R} \) such that, for every \( x \in D \),
\[
\lim_{n \to \infty} E_x[u(B_{T_n})] = \alpha,
\]
and that we then have
\[
\lim_{|x| \to \infty} u(x) = \alpha.
\]
3. Show that, for every $x \in D$,
\[ u(x) = E_x[g(B_{T_K})1_{T_K < \infty}] + \alpha P_x(T_K = \infty). \]

4. Assume that $D$ satisfies the exterior cone condition at every $y \in \partial D$ (this is defined in the same way as when $D$ is bounded). Show that, for any choice of $\alpha \in \mathbb{R}$ the formula of question 3. gives a solution of the problem (P).

**Proof.**

We define $T_A := \inf \{ t \geq 0 : B_t \in A \}$ for all closed subset $A$ of $\mathbb{R}^d$.

1. Fix $n \geq 1$. Set continuous function
\[ f(x) = \begin{cases} 
    u(x), & \text{if } y \in \partial B(0, R_n) \\
    g(x), & \text{if } y \in \partial K,
\end{cases} \]

By using proposition 7.7 on the bounded domain $B(0, R_n) \setminus K$, we get
\[ u(x) = E_x[g(B_{T_K})1_{T_K \leq T_n}] + E_x[u(B_{T_n})1_{T_n \leq T_K}] \quad \forall x \in D \bigcap B(0, R_n). \]

2. Denote $M := \sup_{z \in T} |u(z)|$.

(a) We show that there exists $1 \leq n_1 < n_2 < n_3 < ...$ such that $\lim_{k \to \infty} E_x[u(B_{T_{n_k}})]$ converges uniformly on every compact subset $K \subseteq \mathbb{R}^d$ for every $x \in \mathbb{R}^d$. Denote
\[ f_n(x) := E_x[u(B_{T_n})] \quad \forall x \in B(0, R_n), \quad n \geq 1. \]

By the strong Markov property, we get $f_n$ is harmonic on $B(0, R_n)$ for every $n \geq 1$.

First, we show that $\{f_n\}$ is equicontinuous on $\bar{B}(p, r)$ for every $p \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$. Fix $p \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$. Choose $N \geq 1$ such that $B(p, r) \subseteq B(0, R_N)$ and $\eta := d(B(p, r), \partial B(0, R_N)) > 0$. By local estimates for harmonic function, there exists $C_1 > 0$ such that
\[ |Df_n(x)| \leq \frac{C_1}{(\eta/2)^d+1} \sup_n ||f_n||_{L^1(B(x, \eta/2))} \leq \frac{C_1M}{\eta/2} \quad \forall x \in B(p, r + \eta/2), \quad n \geq N. \]

Fix $\epsilon > 0$. Let $x, y \in \bar{B}(p, r)$ such that $|x - y| < \frac{\eta}{2C_1M} \epsilon$. Then
\[ |f_n(x) - f_n(y)| \leq \sup_{z \in B(p, r + \eta/2)} |Df_n(z)||x - y| < \epsilon \quad \forall n \geq N. \]

Moreover, by Arzelà–Ascoli theorem, there exists a subsequence $N \leq n_1 < n_2 < n_3 < ...$ such that $f_{n_k}(x)$ converges uniformly on $\bar{B}(p, r)$.

Next, by a standard diagonalization procedure, there exists $1 \leq n_1 < n_2 < n_3 < ...$ such that $f_{n_k}(x)$ converges uniformly on $\bar{B}(p_i, r_i)$ for each $i \geq 1$, where $Q^d = \{p_i\}_{i \geq 1}$ and $Q_+ = \{r_i\}_{i \geq 1}$, and so, $\lim_{k \to \infty} f_{n_k}(x)$ uniformly on every compact subset $K$ of $\mathbb{R}^d$.

(b) We show that there exists $\alpha \in \mathbb{R}$ such that
\[ \lim_{k \to \infty} E_x[u(B_{T_{n_k}})] = \alpha \quad \forall x \in D. \]

Set
\[ f(x) := \lim_{k \to \infty} f_{n_k}(x) \quad \forall x \in \mathbb{R}^d. \]

By the strong Markov property, we get
\[ \int f(y) \sigma_{x,r}(dy) = \lim_{k \to \infty} \int E_y[u(B_{T_{n_k}})] \sigma_{x,r}(dy) = \lim_{k \to \infty} E_x[u(B_{T_{n_k}})] = f(x) \]

and so $f$ is a bounded, harmonic function. By Liouville's theorem, we see that $f = \alpha$ for some $\alpha \in \mathbb{R}$. 

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(c) We show that \( \lim_{|x| \to \infty} u(x) = \alpha \). Fix \( \epsilon > 0 \). Choose \( R > 0 \) such that \( \frac{1}{R^2} < \epsilon \). Let \( |x| \geq R \). Choose large \( j \geq 1 \) such that \( |x| \leq R_{n_j} \),

\[
|E_x[u(B_{T_{n_j}})] - \alpha| < \epsilon,
\]

and

\[
\frac{P_{n_j}^{2-d} - |x|^{2-d}}{R_{n_j}^{2-d} - 1} \leq |x|^{2-d} + \epsilon.
\]

Set \( B := \overline{B(0,1)} \). Then

\[
P_x(T_B < T_{n_j}) = \frac{P_{n_j}^{2-d} - |x|^{2-d}}{R_{n_j}^{2-d} - 1} \leq |x|^{2-d} + \epsilon \leq R^{2-d} + \epsilon < 2\epsilon
\]

and so

\[
|u(x) - \alpha| = |E_x[g(B_{T_K})1_{\{T_K \leq T_{n_j}\}}] - E_x[u(B_{T_{n_j}})1_{\{T_j > T_K\}}] + E_x[u(B_{T_{n_j}})] - \alpha| \\
\leq MP_x(T_{n_j} > T_K) + MP_x(T_{n_j} > T_K) + \epsilon \leq (4M + 1)\epsilon.
\]

3. Since \( \lim_{k \to \infty} |B_k| = \infty \) and \( u(x) \uparrow \infty \) a.s., we get \( T_{n_k} < \infty \) for every \( k \geq 1 \) (a.s.) and so

\[
E_x[u(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K\}}] = E_x[u(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K < \infty\}}] + E_x[u(B_{T_{n_k}})1_{\{T_{n_k} < \infty\}}]\]

\[
\leq MP_x(T_K = \infty) + MP_x(T_K = \infty).
\]

By problem 1 and problem 2, we have

\[
\lim_{k \to \infty} E_x[g(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K\}}] + \lim_{k \to \infty} E_x[u(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K < \infty\}}] = E_x[g(B_{T_K})1_{\{T_K < \infty\}}] + \alpha P_x(T_K = \infty).
\]

4. It suffices to show that \( \lim_{x \in D \to y} u(x) = g(y) \) for every \( y \in \partial D \). Denote \( M := \sup_{x \in K} |g(z)| \). Fix \( \epsilon > 0 \) and \( y \in \partial D \). Choose \( \delta > 0 \) such that

\[
|g(z) - g(y)| < \epsilon \quad \forall z \in K \cap B(y, \delta).
\]

Choose \( \eta > 0 \) such that

\[
P_0(\sup_{t \leq \eta} |B_t| \geq \frac{\delta}{2}) < \epsilon.
\]

Observe that

\[
\lim_{x \in D \to y} P_x(T_K > \eta) = 0
\]

(This proof is the same as the proof of lemma 7.9) and so there exists \( \delta' > 0 \) such that

\[
P_x(T_K > \eta) < \epsilon \quad \forall x \in D \cap B(y, \delta').
\]

Let \( x \in D \cap B(y, \delta' \wedge \frac{\delta}{2}) \). Then

\[
P_0(\sup_{t \leq \eta} |B_t - x| \geq \frac{\delta}{2}) = P_0(\sup_{t \leq \eta} |B_t| \geq \frac{\delta}{2}) < \epsilon
\]

and so

\[
|u(x) - g(y)| \\
\leq E_x[|g(B_{T_K}) - g(y)|1_{\{T_K \leq \eta\}}] + E_x[|g(B_{T_K}) - g(y)|1_{\{\eta < T_K \leq \infty\}}] + (g(y) + \alpha)P_x(T_K = \infty) \\
\leq E_x[|g(B_{T_K}) - g(y)|1_{\{\sup_{t \leq \eta} |B_t - x| < \frac{\delta}{2}\}}] + 2M P_x(\sup_{t \leq \eta} |B_t - x| \geq \frac{\delta}{2}) + \\
E_x[|g(B_{T_K}) - g(y)|1_{\{\eta < T_K \leq \infty\}}] + (g(y) + \alpha)P_x(T_K = \infty) \\
\leq \epsilon + 2M \epsilon + 2MP_x(\eta < T_K < \infty) + (g(y) + \alpha)P_x(T_K = \infty) \\
\leq \epsilon + 2M \epsilon + (3M + \alpha)P_x(T_K > \eta) < \epsilon + 2M \epsilon + (3M + \alpha)\epsilon.
\]
7.4 Exercise 7.27

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Use planar Brownian motion to prove that the set \( \{ f(x) : z \in \mathbb{C} \} \) is dense in \( \mathbb{C} \). (Much more is true, since Picard’s little theorem asserts that the complement of \( \{ f(x) : z \in \mathbb{C} \} \) in \( \mathbb{C} \) contains at most one point: This can also be proved using Brownian motion, but the argument is more involved)

**Proof.**

We prove this by contradiction. Assume that there exists \( z \in \mathbb{C} \) and \( r > 0 \) such that \( \overline{B(z, r)} \subseteq G^c \), where \( G = \{ f(z) : z \in \mathbb{C} \} \). For any filtration \((\mathcal{F}_t)_{t \geq 0}\) and \((\mathcal{G}_t)_{t \geq 0}\)-adapted process \((A_t)_{t \geq 0}\) on \( \mathbb{C} \), we define a stopping time

\[
T^f_0 = \inf\{ t \geq 0 : A_t \in F \}
\]

for closed subset \( F \) of \( \mathbb{C} \). Let \((B_t)_{t \geq 0}\) be a complex Brownian motion that starts from 0 under the probability measure \( P_0 \). Since \( \overline{B(z, r)} \subseteq G^c \), we get

\[
P_0(T^f_0 < \infty) = 0.
\]

By Theorem 7.18, there exists a complex Brownian motion \( \Gamma \) that starts from \( f(0) \) under \( P_0 \), such that

\[
f(B_t) = \Gamma_{C_t}, \quad \forall t \geq 0 \quad P_0\text{-}(a.s.),
\]

where

\[
C_t = \int_0^t |f'(B_s)|^2 ds \quad \forall t \geq 0.
\]

By Proposition 7.16, we see that

\[
P_0(T^0_{\overline{B(z, r)}} < \infty) = 1.
\]

Since \((C_t)_{t \geq 0}\) is a continuous increasing process and \( C_\infty = \infty \) \( P_0\text{-}(a.s.) \), we have

\[
P_0(T^f_0 < \infty) = P_0(T^0_{\overline{B(z, r)}} < \infty) = 1
\]

which is a contradiction. \( \square \)

7.5 Exercise 7.28 (Feynman–Kac formula for Brownian motion)

This is a continuation of Exercise 6.26 in Chap. 6. With the notation of this exercise, we assume that \( E = \mathbb{R}^d \) and \( X_t = B_t \). Let \( v \) be a nonnegative function in \( C_0(\mathbb{R}^d) \), and assume that \( v \) is continuously differentiable with bounded first derivatives. As in Exercise 6.26, set, for every \( \varphi \in B(\mathbb{R}^d) \),

\[
Q_t^* \varphi(x) = E_x[\varphi(X_t)e^{-\int_0^t v(X_s)ds}].
\]

1. Using the formula derived in question 2. of Exercise 6.26, prove that, for every \( t > 0 \), and every \( \varphi \in C_0(\mathbb{R}^d) \), the function \( Q_t^* \varphi \) is twice continuously differentiable on \( \mathbb{R}^d \), and that \( Q_t^* \varphi \) and its partial derivatives up to order 2 belong to \( C_0(\mathbb{R}^d) \). Conclude that \( Q_t^* \varphi \in D(L) \).

2. Let \( \varphi \in C_0(\mathbb{R}^d) \) and set \( u_t(x) = Q_t^* \varphi(x) \) for every \( t > 0 \) and \( x \in \mathbb{R}^d \). Using question 3. of Exercise 6.26, prove that, for every \( x \in \mathbb{R}^d \), the function \( t \mapsto u_t(x) \) is continuously differentiable on \( (0, \infty) \), and

\[
\frac{\partial}{\partial t} u_t = \frac{1}{2} \Delta u_t - v u_t.
\]

**Proof.**

1. For \( f : \mathbb{R}^d \to \mathbb{R} \), we set \( \|f\| = \sup_{x \in \mathbb{R}^d} |f(x)| \). Observe that we have the following facts:
(a) Fix $\varphi \in B(\mathbb{R}^d)$ and $t \geq 0$. By the definition of $Q_t^*\varphi$, we get

$$||Q_t^*\varphi|| \leq ||\varphi||.$$ 

(b) Fix $\varphi \in C_0(\mathbb{R}^d)$ and $t \geq 0$. By question 2. of Exercise 6.26, we get

$$Q_t^*\varphi(x) = Q_t\varphi(x) - \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds \quad \forall x \in \mathbb{R}^d,$$

where $\{Q_t\}$ is the semigroup of $(B_t)_{t \geq 0}$.

(c) Fix $f \in C_0(\mathbb{R}^d)$ and $t \geq 0$. Since $Q_tf(x) = f \ast k_s(x)$, where

$$k(x) := (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}$$

and

$$k_s(x) := (s)^{-\frac{d}{2}} k\left(\frac{x}{s}\right),$$

we see that $Q_tf \in C^\infty(\mathbb{R}^d)$, and that $Q_tf$ and all its partial derivatives belong to $C_0(\mathbb{R}^d)$. Moreover, if $t > 0$, then

$$||D_jQ_tf|| \leq \frac{1}{\sqrt{t}} ||D_jk||_{L^1(\mathbb{R}^d)}||f||.$$ 

Indeed, since

$$D_jQ_tf(x) = D_j(f \ast k_s)(x) = \int_{\mathbb{R}^d} (2\pi)^{-\frac{d}{2}} e^{-\frac{|y|^2}{2}} \left(\frac{x-y}{t}\right)^s f(y)dy = \frac{1}{\sqrt{t}}((D_jk_s) \ast f)(x),$$

we have

$$||D_jQ_tf(x)|| \leq \frac{1}{\sqrt{t}} ||((D_jk_s) \ast f)|| \leq \frac{1}{\sqrt{t}} ||D_jk||_{L^1(\mathbb{R}^d)}||f||.$$

(d) Let $s > 0$. Then

$$D_jk_s(x) = \frac{1}{\sqrt{s}} (D_jk)_s(x) \quad \forall x \in \mathbb{R}^d.$$ 

(e) Let $\varphi \in C_0(\mathbb{R}^d)$. Then

$$||Q_t^*\varphi|| \leq ||\varphi||$$

for all $r \geq 0$. We will show that $x \in \mathbb{R}^d \mapsto Q_t^*\varphi(x)$ is continuous for all $r \geq 0$. Therefore $vQ_t^*\varphi \in C_0(\mathbb{R}^d)$,

$$Q_s(vQ_t^*\varphi)(x) = ((vQ_t^*\varphi) \ast k_s)(x) \in C^\infty(\mathbb{R}^d)$$

and $Q_s(vQ_t^*\varphi)(x)$ and all its derivatives belong to $C_0(\mathbb{R}^d)$ for all $r, s \geq 0$. Moreover,

$$\int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds = \int_0^t ((vQ_{t-s}^*\varphi) \ast k_s)(x)ds \quad \forall x \in \mathbb{R}^d.$$ 

(f) Note that

$$\{h \in C^2(\mathbb{R}^d) \mid h \text{ and } \Delta h \in C_0(\mathbb{R}^d)\} \subseteq D(L),$$

where $L$ is the generator of $B$ and $D(L)$ is the domain of $L$.

Fix $\varphi \in C_0(\mathbb{R}^d)$. To prove problem 1, it suffices to show that $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds$ is twice continuously differentiable, and that $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds$ and its partial derivatives up to order 2 belong to $C_0(\mathbb{R}^d)$.
(a) We show that \( x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds \) belong to \( C_0(\mathbb{R}^d) \). It suffices to show that \( x \in \mathbb{R}^d \mapsto Q_r^*\varphi(x) \) is continuous for all \( r \geq 0 \). Indeed, since

\[
Q_s(vQ^*_{t-s}\varphi) \in C_0(\mathbb{R}^d) \quad \forall s \in [0, t]
\]

and

\[
||Q_s(vQ^*_{t-s}\varphi)|| \leq ||v||||\varphi|| \quad \forall s \in [0, t],
\]

we get

\[
\lim_{x \to a} \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds = \int_0^t \lim_{x \to a} Q_s(vQ^*_{t-s}\varphi)(x)ds = \begin{cases} 
\int_0^t Q_s(vQ^*_{t-s}\varphi)(a)ds, & \text{if } a \neq \infty \\
0, & \text{otherwise}
\end{cases}
\]

and, hence, \( x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds \) belong to \( C_0(\mathbb{R}^d) \).

Now we show that \( x \in \mathbb{R}^d \mapsto Q_r^*\varphi(x) \) is continuous for all \( r \geq 0 \). Fix \( r \geq 0 \). Observe that

\[
E_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u)}] \xrightarrow{n \to \infty} Q_r^*\varphi(x) := E_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] \text{ uniformly on } \mathbb{R}^d.
\]

Indeed, since

\[
E_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u)}] = E_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u+x)}] \quad \forall n \geq 1,
\]

\[
E_x[\varphi(X_r)e^{-r\sum_{i=1}^n v(X_k^u)}] = E_0[\varphi(X_r+x)e^{-\int_0^r v(X_s+ds)}] \quad \forall n \geq 1,
\]

and

\[
\frac{r}{n} \sum_{i=1}^n v(X_k^u) + x \xrightarrow{n \to \infty} \int_0^r v(X_s+ds) \text{ uniformly on } \mathbb{R}^d \quad P_0-(a.s.),
\]

we get

\[
\lim_{n \to \infty} E_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u)}] = \lim_{n \to \infty} E_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u+x)}]
\]

\[
= E_0[\varphi(X_r+x)e^{-\int_0^r v(X_s+ds)}]
\]

\[
= E_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] \text{ uniformly on } \mathbb{R}^d.
\]

By Lebesgue’s dominated convergence theorem, we get

\[
x \in \mathbb{R}^d \mapsto E_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u+x)}] = E_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_k^u)}]
\]

is continuous for all \( n \geq 1 \) and so

\[
x \in \mathbb{R}^d \mapsto E_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] = Q_r^*\varphi(x)
\]

is continuous.

(b) We show that

\[
D_i \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds = D_i \int_0^t ((vQ^*_{t-s}\varphi) \ast k_s)(x)ds = \int_0^t ((vQ^*_{t-s}\varphi) \ast (D_ik_s))(x)ds
\]

for all \( x \in \mathbb{R}^d \) and

\[
x \in \mathbb{R}^d \mapsto D_i \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds
\]

belong to \( C_0(\mathbb{R}^d) \) for all \( i = 1, 2, \ldots, d \). Since \( vQ^*_{t-s}\varphi \) is bounded, we have

\[
D_i((vQ^*_{t-s}\varphi) \ast k_s)(x) = ((vQ^*_{t-s}\varphi) \ast (D_ik_s))(x) \quad \forall x \in \mathbb{R}^d.
\]

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Note that, if \( s \in [0, t] \), then
\[
\| (vQ^*_t - s) \ast (D_i k_s) \| \leq \| vQ^*_t - s \| \times \| D_i k_s \|_{L^1(\mathbb{R}^d)}
\]
\[
\leq \| v \| \| \varphi \| \times \frac{1}{\sqrt{s}} \| (D_i k) \|_{L^1(\mathbb{R}^d)}
\]
\[
\leq \| v \| \| \varphi \| \times \frac{1}{\sqrt{s}} \| D_i k \|_{L^1(\mathbb{R}^d)} \in L^1([0, t]).
\]

By mean value theorem and Lebesgue’s dominated convergence theorem, we have
\[
D_i \int_0^t ((vQ^*_t - s) \ast k_s)(x)ds = \int_0^t D_i ((vQ^*_t - s) \ast k_s)(x)ds = \int_0^t ((vQ^*_t - s) \ast (D_i k_s))(x)ds
\]
for all \( x \in \mathbb{R}^d \). Given \( a \in \mathbb{R}^d \cup \{\infty\} \). By Lebesgue’s dominated convergence theorem, we have
\[
\lim_{x \to a} D_i \int_0^t ((vQ^*_t - s) \ast k_s)(x)ds = \lim_{x \to a} \int_0^t ((vQ^*_t - s) \ast (D_i k_s))(x)ds
\]
\[
= \int_0^t \lim_{x \to a} ((vQ^*_t - s) \ast (D_i k_s))(x)ds
\]
\[
= \int_0^t \lim_{x \to a} D_i((vQ^*_t - s) \ast (k_s))(x)ds
\]
\[
= \int_0^t \lim_{x \to a} D_i(Q_s(vQ^*_t - s))(x)ds.
\]
Since \( D_i Q_s(vQ^*_t - s) \in C_0(\mathbb{R}^d) \), we see that
\[
\int_0^t \lim_{x \to a} D_i(Q_s(vQ^*_t - s))(x)ds = \begin{cases} \int_0^t D_i(Q_s(vQ^*_t - s))(a)ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise} \end{cases}
\]
\[
= \begin{cases} D_i \int_0^t (Q_s(vQ^*_t - s))(a)ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise} \end{cases}
\]
and so
\[
x \in \mathbb{R}^d \mapsto D_i \int_0^t Q_s(vQ^*_t - s)(x)ds
\]
belong to \( C_0(\mathbb{R}^d) \).

(c) We show that
\[
D_{j,i} \int_0^t Q_s(vQ^*_t - s)(x)ds = D_{j,i} \int_0^t ((vQ^*_t - s) \ast k_s)(x)ds = \int_0^t ((D_j(vQ^*_t - s)) \ast (D_i k_s))(x)ds
\]
for all \( x \in \mathbb{R}^d \) and
\[
x \in \mathbb{R}^d \mapsto D_{j,i} \int_0^t Q_s(vQ^*_t - s)(x)ds
\]
belong to \( C_0(\mathbb{R}^d) \) for all \( i, j = 1, 2, ..., d \). Since we have shown that
\[
D_j Q^*_r \varphi(x) = D_j Q^*_r \varphi(x) - D_j \int_0^r Q_s(vQ^*_t - s)(x)ds
\]
and
\[
D_j Q^*_r \varphi(x), D_j \int_0^r Q_s(vQ^*_t - s)(x)ds \in C_0(\mathbb{R}^d)
\]
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for all \( r \geq 0 \) and \( j = 1, 2, \ldots, d \), we see that
\[
vQ^*_r \varphi \in C^1(\mathbb{R}^d) \text{ and } D_j(vQ^*_r \varphi) \in C_0(\mathbb{R}^d).
\]
Thus \( \int_0^t ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x) \, ds \) is well-defined.

Fix \( 0 < s < t \). First, we show that
\[
D_{j,i} Q_s(vQ^*_r \varphi)(x) = D_j((vQ^*_r \varphi) * (D_t k_s))(x) = ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x)
\]
for all \( x \in \mathbb{R}^d \). Note that \( D_t k_s \in L^1(\mathbb{R}^d) \) and
\[
\|D_j(vQ^*_r \varphi)\| = \|(D_j v)Q^*_r \varphi + vD_j Q^*_r \varphi\|
\[
= \|(D_j v)Q^*_r \varphi + vD_j Q^*_r \varphi - vD_j \int_0^{t-s} Q_u(vQ^*_r \varphi - u \varphi) \, du\|
\[
= \|(D_j v)Q^*_r \varphi + vD_j Q^*_r \varphi - v\int_0^{t-s} D_j Q_v (vQ^*_r \varphi - u \varphi) \, du\|
\[
= \|(D_j v)Q^*_r \varphi + vD_j Q^*_r \varphi - v\int_0^{t-s} (vQ^*_r \varphi - u \varphi) * (k_u) \, du\|
\[
\leq \|D_j v\| \|\varphi\| + \|v\| \|D_j Q^*_r \varphi\| + \int_0^t \|(vQ^*_r \varphi - u \varphi) * (D_j k_u)\| \, du\|
\[
\leq \|D_j v\| \|\varphi\| + \|v\| \|D_j Q^*_r \varphi\| + \int_0^t \|(vQ^*_r \varphi - u \varphi)\| L^1(\mathbb{R}^d) \, du\]
\[
\leq \|D_j v\| \|\varphi\| + \|v\| \|D_j Q^*_r \varphi\| + \int_0^t \|v\| \|\varphi\| \frac{1}{\sqrt{u}} \|D_j k\| L^1(\mathbb{R}^d) \, du.
\]

By (34), we get
\[
\|D_j(vQ^*_r \varphi)\| \leq C(1 + \frac{1}{\sqrt{t-s}}),
\]
where \( C \) is a constant independent of \( s \) and \( j \). We may set \( C = \max_{1 \leq i \leq d} C_i \) and so \( C \) is independent of \( i \). Fix \( x \in \mathbb{R}^d \). By mean value theorem, we get
\[
|D_i k_s(y)\left( \frac{(vQ^*_r \varphi)(x - y + h e_j) - (vQ^*_r \varphi)(x - y + h e_j)}{h} \right)| \leq C(1 + \frac{1}{\sqrt{t-s}}) |D_i k_s(y)| \in L^1(\mathbb{R}^d).
\]

By Lebesgue’s convergence theorem, we have
\[
D_{j,i} Q_s(vQ^*_r \varphi)(x) = D_j((vQ^*_r \varphi) * (D_t k_s))(x) = ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x).
\]
Next, we show that
\[
D_{j,i} \int_0^t Q_s(vQ^*_r \varphi)(x) \, ds = D_{j,i} \int_0^t ((vQ^*_r \varphi) * k_s)(x) \, ds = \int_0^t ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x) \, ds
\]
for all \( x \in \mathbb{R}^d \). Note that we already have
\[
D_i \int_0^t Q_s(vQ^*_r \varphi)(x) \, ds = \int_0^t ((vQ^*_r \varphi) * (D_t k_s))(x) \, ds.
\]
It suffices to show that
\[
D_j \int_0^t ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x) \, ds = \int_0^t ((D_j(vQ^*_r \varphi)) * (D_t k_s))(x) \, ds.
\]
Fix $x \in \mathbb{R}^d$. If $0 < s < t$, then
\[
\frac{\left| ((vQ_{t-s}^* \varphi) * (D_i k_s))(x + h e_j) - ((vQ_{t-s}^* \varphi) * (D_i k_s))(x) \right|}{h} \\
\leq \| (D_j (vQ_{t-s}^* \varphi)) * (D_i k_s) \| \\
\leq \| D_j (vQ_{t-s}^* \varphi) \| \| D_i k_s \|_{L^1(\mathbb{R}^d)} \\
\leq C(1 + \frac{1}{\sqrt{t-s}}) \frac{1}{\sqrt{s}} \| D_i k_s \|_{L^1(\mathbb{R}^d)} \\
= C(1 + \frac{1}{\sqrt{t-s}}) \frac{1}{\sqrt{s}} \| D_i k_s \|_{L^1(\mathbb{R}^d)} \in L^1((0, t)).
\]

By Lebesgue's dominated convergence theorem, we have
\[
D_j D_i \int_0^t Q_s(vQ_{t-s}^* \varphi)(x) ds = D_j \int_0^t ((vQ_{t-s}^* \varphi) * (D_i k_s))(x) ds = \int_0^t ((D_j (vQ_{t-s}^* \varphi)) * (D_i k_s))(x) ds.
\]

Given $a \in \mathbb{R}^d \cup \{ \infty \}$. Note that
\[
D_{j,i} \int_0^t Q_s(vQ_{t-s}^* \varphi)(x) ds = \int_0^t ((D_j (vQ_{t-s}^* \varphi)) * (D_i k_s))(x) ds \\
= \int_0^t D_{j,i}(vQ_{t-s}^* \varphi)(k_s) ds \\
= \int_0^t D_{j,i} Q_s(vQ_{t-s}^* \varphi)(x) ds
\]

and
\[
D_{j,i} Q_s(vQ_{t-s}^* \varphi) \in C_0(\mathbb{R}^d) \quad \forall s \in (0, t).
\]

By Lebesgue's dominated convergence theorem, we have
\[
\lim_{x \to a} D_{j,i} \int_0^t Q_s(vQ_{t-s}^* \varphi)(x) ds \\
= \int_0^t \lim_{x \to a} D_{j,i} Q_s(vQ_{t-s}^* \varphi)(x) ds \\
= \begin{cases} \\
\int_0^t D_{j,i} Q_s(vQ_{t-s}^* \varphi)(a) ds, & \text{if } a \neq \infty \\
0, & \text{otherwise.} \\
\end{cases} \\
= \begin{cases} \\
D_{j,i} \int_0^t Q_s(vQ_{t-s}^* \varphi)(a) ds, & \text{if } a \neq \infty \\
0, & \text{otherwise.} \\
\end{cases}
\]

2. Since $u_t(x) = Q_t \varphi(x) - \int_0^t Q_s(vQ_{t-s}^* \varphi)(x) ds$, we show that
\[
\frac{\partial}{\partial t} (Q_t \varphi - \int_0^t Q_s(vQ_{t-s}^* \varphi) ds) = \frac{1}{2} \Delta u_t - vu_t
\]
and
\[
t \in [0, \infty) \mapsto \frac{1}{2} \Delta u_t(x) - v(x) u_t(x)
\]
is continuous for all $x \in \mathbb{R}^d$. Note that
\[
u_t(x) = Q_t \varphi - \int_0^t Q_s(vQ_{t-s}^* \varphi) ds = Q_t \varphi - \int_0^t Q_t(vQ_{t-s}^* \varphi) ds.
\]
By Theorem 7.1 and Leibniz integral rule, we get
\[
\frac{\partial}{\partial t} u_t(x) = \frac{\partial}{\partial t} Q_t \varphi(x) - v(t) Q_t \varphi(x) - \int_0^t \frac{\partial}{\partial t} Q_{t-s} (v Q_s^* \varphi) ds.
\]
\[
= \frac{1}{2} \Delta Q_t \varphi(x) - v(t) Q_t^* \varphi(x) - \int_0^t \frac{1}{2} \Delta Q_{t-s} (v Q_s^* \varphi) ds.
\]
Since we have shown that
\[
D_{i,j} \int_0^t Q_{t-s} (v Q_s^* \varphi) ds = D_{i,j} \int_0^t Q_s (v Q_s^* \varphi^2) ds = \int_0^t D_{i,j} Q_s (v Q_s^* \varphi^2) ds = \int_0^t D_{i,j} Q_{t-s} (v Q_s^* \varphi) ds,
\]
we get
\[
\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} \Delta (Q_t \varphi(x)) - \int_0^t Q_{t-s} (v Q_s^* \varphi(x)) ds - v Q_t^* \varphi(x) = \frac{1}{2} \Delta u_t(x) - v(x) u_t(x).
\]
Now we show that
\[
t \in [0, \infty) \mapsto \frac{1}{2} \Delta u_t(x) - v(x) u_t(x)
\]
is continuous for all \(x \in \mathbb{R}^d\). Fix \(x \in \mathbb{R}^d\). By Lebesgue's dominated convergence theorem, we see that
\[
t \in [0, \infty) \mapsto u_t(x) = Q_t^* (x) = E_x[\varphi(X_t) e^{-\int_0^t v(X_s) ds}]
\]
is continuous. It remains to show that \(t \in [0, \infty) \mapsto \Delta u_t(x)\) is continuous. Let \(h > 0\). Because
\[
D_{i,k} u_t(x) = \int_0^t (((D_{j} (v Q_{t-s}^* \varphi^2)) * (D_{i} k_s))(x) ds \quad \forall t \geq 0,
\]
we get
\[
|D_{i,k} u_{t+h}(x) - D_{i,k} u_t(x)|
\]
\[
\leq \left| \int_0^{t+h} (((D_{j} (v Q_{t+s-h}^* \varphi^2)) * (D_{i} k_s))(x) ds - \int_0^t (((D_{j} (v Q_{t+s}^* \varphi^2)) * (D_{i} k_s))(x) ds
\]
\[
+ \int_0^t (((D_{j} (v Q_{t+s}^* \varphi^2)) * (D_{i} k_s))(x) ds - \int_0^t (((D_{j} (v Q_{t+s}^* \varphi^2)) * (D_{i} k_s))(x) ds |.
\]
\[
\leq \int_0^{t+h} |((D_{j} (v Q_{t+s}^* \varphi^2)) * (D_{i} k_s)| ds + \int_0^t |((D_{j} (v Q_{t+s}^* \varphi^2)) - (D_{j} (v Q_{t-s}^* \varphi^2)) | * (D_{i} k_s))(x) ds
\]
\[
= \alpha + \beta.
\]
Note that
\[
\alpha \leq \int_0^{t+h} ||(D_{j} (v Q_{t+s}^* \varphi^2)) * (D_{i} k_s)||_{L^1(\mathbb{R}^d)} ds
\]
\[
\leq 1 \int_0^{t+h} C(1 + \frac{1}{t+h-s}) \frac{1}{\sqrt{s}} ||D_{i} k||_{L^1(\mathbb{R}^d)} ds \xrightarrow{h \to 0} 0.
\]
Now we show that \(\beta \xrightarrow{h \to 0} 0\). Fix \(0 < s < t\). First, we show that
\[
|((D_{j} (v Q_{t+s}^* \varphi^2)) - (D_{j} (v Q_{t-s}^* \varphi^2))) * (D_{i} k_s))(x)| \xrightarrow{h \to 0} 0
\]
for all \(x \in \mathbb{R}^d\). Note that
\[
|((D_{j} (v Q_{t+s}^* \varphi^2))(x) - (D_{j} (v Q_{t-s}^* \varphi^2))(x - y)) * (D_{i} k_s))(y)|
\]
\[
\leq ((|D_{j} (v Q_{t+s}^* \varphi^2)|) + ||D_{j} (v Q_{t-s}^* \varphi^2)||) ||(D_{i} k_s))(y)|
\]
\[
\leq (C(1 + \frac{1}{t+h-s}) + C(1 + \frac{1}{t-s}))(D_{i} k_s))(y)|
\]
\[
\leq 2C(1 + \frac{1}{t-s}) ||(D_{i} k_s))(y)|| \in L^1(\mathbb{R}^d).
\]
By Lebesgue convergence theorem, we have

\[ \|(D_j(vQ_{t+h-s}^\varphi)) - (D_j(vQ_{t-s}^\varphi)) \ast (D_i k_s))\| \xrightarrow{h \to 0} 0. \]

Next, we show that \( \beta \xrightarrow{h \to 0} 0 \). Note that

\[ \|(D_j(vQ_{t+h-s}^\varphi)) - (D_j(vQ_{t-s}^\varphi)) \ast (D_i k_s)\| \leq \|(D_j(vQ_{t+h-s}^\varphi)) - (D_j(vQ_{t-s}^\varphi))\| \times \| (D_i k_s)\|_{L^1(\mathbb{R}^d)} \leq (C(1 + \frac{1}{\sqrt{t+h-s}}) + C(1 + \frac{1}{\sqrt{t-s}})) \times \frac{1}{\sqrt{t}} \|D_i k\|_{L^1(\mathbb{R}^d)} \leq 2C(1 + \frac{1}{\sqrt{t-s}}) \times \frac{1}{\sqrt{t}} \|D_i k\|_{L^1(\mathbb{R}^d)} \in L^1(0, t)). \]

By Lebesgue’s convergence theorem, we have \( \beta \xrightarrow{h \to 0} 0 \) and so \( t \in [0, \infty) \mapsto \Delta u_t(x) \) is right continuous. By using similar way, we get \( t \in [0, \infty) \mapsto \Delta u_t(x) \) is left continuous and, hence, \( t \in [0, \infty) \mapsto \Delta u_t(x) \) is continuous which complete the proof.

\[ \square \]

### 7.6 Exercise 7.29

In this exercise \( d = 2 \) and \( \mathbb{R}^2 \) is identified with the complex plane \( \mathbb{C} \). Let \( \alpha \in (0, 2\pi) \), and consider the open cone

\[ \mathcal{C}_\alpha = \{ re^{i\theta} : r > 0, \theta \in (-\alpha, \alpha) \}. \]

Set \( T := \inf \{ t \geq 0 : B_t \notin \mathcal{C}_\alpha \} \).

1. Show that the law of \( \log |B_T| \) under \( P_1 \) is the law of \( \beta_{\inf \{ t \geq 0 : |\gamma_t| = \alpha \}} \), where \( \beta \) and \( \gamma \) are two independent linear Brownian motions started from 0.

2. Verify that, for every \( \lambda \in \mathbb{R} \),

\[ E_1[e^{i\lambda \log |B_T|}] = \frac{1}{\cosh(\alpha \lambda)}. \]

**Proof.**

1. By the skew-product representation (Theorem 7.19), there exist two independent linear Brownian motions \( \beta \) and \( \gamma \) that start from 0 under \( P_1 \) such that

\[ B_t = e^{\beta H_t + i \gamma H_t}, \quad \forall t \geq 0 \quad P_1\text{-}(a.s.), \]

where \( H_t = \int_0^t \frac{1}{|\beta|} ds \). Set \( S := \inf \{ t \geq 0 : |\gamma_t| = \alpha \} \). Since \( (H_t)_{t \geq 0} \) is a continuous increasing process and \( H_\infty = \infty \) \( P_1\text{-}(a.s.) \), we have

\[ H_T = H_{\inf \{ t \geq 0 : |\gamma_t| = \alpha \}} = \inf \{ t \geq 0 : |\gamma_t| = \alpha \} = S \]

and so \( \log |B_T| = \beta_{H_T} = \beta_S = \beta_{\inf \{ t \geq 0 : |\gamma_t| = \alpha \}} P_1\text{-}(a.s.). \)

2. Note that \( \cosh(x) \) is an even function. By taking complex conjugate in both side of the identity, we may assume that \( \lambda \geq 0 \). By problem 1., we get

\[ E_1[e^{i\lambda \log |B_T|}] = E_1[e^{i\lambda \beta_S}] = E_1[E_1[e^{i\lambda \beta_S} | \sigma(\gamma_t, t \geq 0)]]. \]

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Recall that, if $X \sim \mathcal{N}(\mu, \sigma)$, then the characteristic function of $X$ is

$$E[e^{i\xi X}] = e^{i\mu \xi - \frac{\sigma^2}{2} \xi^2}. $$

Since $\beta$ and $\gamma$ are independent, we get

$$E_1[E_1[e^{i\lambda \beta} \mid \sigma(\gamma_t, t \geq 0)]] = E_1[\int e^{i\lambda y} \frac{1}{\sqrt{2\pi S}} e^{-\frac{y^2}{2S}} dy] = E_1[e^{-\frac{\lambda^2}{2} S}].$$

Since $(e^{\lambda \gamma_t S - \frac{\lambda^2}{2} (t \wedge S)})_{t \geq 0}$ is an uniformly integrable martingale, we see that

$$E_1[e^{\lambda \gamma_S - \frac{\lambda^2}{2} S}] = 1.$$ and so

$$e^{\lambda \alpha} E_1[e^{-\frac{\lambda^2}{2} S 1\{\gamma_S = \alpha\}}] + e^{-\lambda \alpha} E_1[e^{-\frac{\lambda^2}{2} S 1\{\gamma_S = -\alpha\}}] = 1.$$ By symmetry ($-\gamma$ is a Brownian motion), we have

$$E_1[e^{-\frac{\lambda^2}{2} S 1\{\gamma_S = \alpha\}}] = E_1[e^{-\frac{\lambda^2}{2} S 1\{\gamma_S = -\alpha\}}] = \frac{1}{2} E_1[e^{-\frac{\lambda^2}{2} S}]$$ and, hence,

$$E_1[e^{-\frac{\lambda^2}{2} S}] = \frac{1}{\cosh(\alpha \lambda)}. $$

\[\square\]
Chapter 8
Stochastic Differential Equations

8.1 Exercise 8.9 (Time change method)

We consider the stochastic differential equation

\[ E(\sigma, 0) : \quad dX_t = \sigma(X_t)dB_t \]

where the function \( \sigma : \mathbb{R} \to \mathbb{R} \) is continuous and there exist constants \( \epsilon > 0 \) and \( M \) such that \( \epsilon \leq \sigma \leq M \).

1. In this question and the next one, we assume that \( X \) solves \( E(\sigma, 0) \) with \( X_0 = x \), for every \( t \geq 0 \),

\[ A_t = \int_0^t \sigma(X_s)^2ds, \quad \tau_t = \inf\{s \geq 0 \mid A_s > t\}. \]

Justify the equalities

\[ \tau_t = \int_0^t \frac{1}{\sigma(X_{\tau_s})^2}dr, \quad A_t = \inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau_r})^2}dr > t\}. \]

2. Show that there exists a real Brownian motion \( \beta = (\beta_t)_{t \geq 0} \) started from \( x \) such that, a.s. for every \( t \geq 0 \),

\[ X_t = \beta_{\inf\{s \geq 0 \mid \int_0^s \sigma(\beta_r)^{-2}dr > t\}}. \]

3. Show that weak existence and weak uniqueness hold for \( E(\sigma, 0) \). (Hint: For the existence part, observe that, if \( X \) is defined from a Brownian motion \( \beta \) by the formula of question 2., \( X \) is (in an appropriate filtration) a continuous local martingale with quadratic variation \( \langle X, X \rangle_t = \int_0^t \sigma(X_r)^2dr \).

**Proof.**

For the sake of simplicity, sometimes we denote \( A_t \) and \( \tau_t \) as \( A(t) \) and \( \tau(t) \), respectively.

1. Since \( \sigma \in C(\mathbb{R}) \) and \( A'(t) = \sigma(X_t)^2 \geq \epsilon^2 > 0 \), we see that \( A(t) \) is strictly increasing and so \( A(t) \) is injective. Because \( A(\tau(t)) = t \) for all \( t \geq 0 \), we see that \( \tau(t) = A^{-1}(t) \) and, hence, \( \tau(t) \in C^1(\mathbb{R}) \). By setting \( s = \tau(r) \), we get \( r = A(s), dr = A'(s)ds \), and so

\[ \int_0^t \frac{1}{\sigma(X_{\tau(t)})^2}dr = \int_0^{\tau(t)} A'(r)^{-1}dr = \int_0^{\tau(t)} A'(s)^{-1}A'(s)ds = \tau(t). \]

Moreover,

\[ A(t) = \inf\{s \geq 0 \mid s > A(t)\} = \inf\{s \geq 0 \mid \tau(s) > t\} = \inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau(r)})^2}dr > t\}. \]

2. Note that \( X_t = X_0 + \int_0^t \sigma(X_s)dB_s \) is a continuous local martingale and

\[ \langle X, X \rangle_t = \int_0^t \sigma(X_s)^2ds = A(t) \quad \forall t \geq 0. \]

Since \( \sigma \geq \epsilon > 0 \), we see that \( \langle X, X \rangle_\infty = \infty \) and, hence, there exists a Brownian motion \( \beta = (\beta_t)_{t \geq 0} \) such that

\[ X_t = \beta_{\langle X, X \rangle_t} = \beta_{A(t)} \quad \forall t \geq 0 \text{ (a.s.)}. \]

By problem 1., we get \( X_{\tau(r)} = \beta_r \) and

\[ X_t = \beta_{A(t)} = \beta_{\inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau_r})^2}dr > t\}} = \beta_{\inf\{s \geq 0 \mid \int_0^s \sigma(\beta_r)^{-2}dr > t\}}. \]
3. (a) We prove that weak existence hold for $E(\sigma, 0)$. Fix $x \in \mathbb{R}$. We show that there exists a solution $(X, B, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \setminus E(\sigma, 0)$, $\mathcal{F}_t)_{t \geq 0}$ be a filtered probability space ($(\mathcal{F}_t)_{t \geq 0}$ is complete) and $(\beta_t)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion such that $\beta_0 = x$. Define

$$\tau(t) := \int_0^t \sigma(\beta_r)^{-2}dr \quad \text{and} \quad A(t) := \inf\{s \geq 0 \mid \tau(s) > t\}.$$ 

As the proof in problem 1., we have $\tau(A(t)) = t$ for all $t \geq 0$ and $A(t), \tau(t) \in C^1(\mathbb{R})$. Moreover, since $A'(\tau(t)) = \tau'(t)^{-1} = \sigma(\beta_t)^2$, we see that

$$A(t) = \int_0^t \sigma(\beta_r)^2dr.$$ 

Set

$$X_t := \beta_{A(t)} \quad \text{and} \quad \mathcal{G}_t := \mathcal{F}_{A_t}.$$ 

Then $X$ is continuous. Because $(\mathcal{F}_t)_{t \geq 0}$ is complete, we see that $(\mathcal{G}_t)_{t \geq 0}$ is complete. Since $A_t < \infty$ (a.s.) and $A_t$ is a $(\mathcal{F}_t)_{t \geq 0}$-stopping time for all $t \geq 0$, we see that $X_t$ is $\mathcal{G}_t$-measurable for all $t \geq 0$. Define

$$Y_t := \int_0^t \sigma(\beta_s)^{-1}d\beta_s, \quad B_t := Y_{A_t}.$$ 

Then $B_0 = 0$ and $B_t$ is $\mathcal{G}_t$-measurable for all $t \geq 0$. Now, we show that $(B_t)_{t \geq 0}$ is a $(\mathcal{G}_t)_{t \geq 0}$-Brownian motion such that $B_0 = 0$. It suffices to show that $(B_t)_{t \geq 0}$ is a $(\mathcal{G}_t)_{t \geq 0}$-martingale and $(B_t)_{t \geq 0}$ is $t$ for all $t \geq 0$. Fix $s \leq r < t$. Since $Y$ is a $(\mathcal{F}_t)_{t \geq 0}$-continuous local martingale, $Y^{A_t}$ is a $(\mathcal{F}_t)_{t \geq 0}$-continuous local martingale. Moreover, since

$$\langle Y^{A_t}, Y^{A_t} \rangle_{\infty} = \int_0^{A_t} \sigma(X_r)^{-2}dr \leq \delta^2 A_t \leq \delta^{-2} M^2 t < \infty,$$ 

we see that $Y^{A_t}$ is a uniform integrable $(\mathcal{F}_t)_{t \geq 0}$-martingale. By optional stopping theorem, we get

$$E[B_r \mid \mathcal{G}_s] = E[Y^{A_t} \mid \mathcal{F}_{A_s}] = Y^{A_t}_{A_s} = Y_{A_s} = B_s$$

and so $(B_t)_{t \geq 0}$ is a $(\mathcal{G}_t)_{t \geq 0}$-martingale. Moreover, since $\langle Y, Y \rangle_t = \tau(t)$, we get

$$(B, B)_t = \langle Y, Y \rangle_{A_t} = \tau(A(t)) = t \quad \forall t \geq 0$$

and, hence, $(B_t)_{t \geq 0}$ is a $(\mathcal{G}_t)_{t \geq 0}$-Brownian motion. Observe that

$$\int_0^t \sigma(\beta_s) dY_{A_s} = \int_0^{A_t} \sigma(\beta_s) dY_s.$$ 

Indeed, since

$$\sum_{i=0}^{n-1} \sigma(\beta_{\frac{A_{i+1}}{n}})(Y_{\frac{A_{i+1}}{n}} - Y_{\frac{A_i}{n}}) \xrightarrow{P} \int_0^t \sigma(\beta_s) dY_s \quad \text{as} \quad n \to \infty,$$

there exists $\{n_k\}$ such that

$$\sum_{i=0}^{n_k-1} \sigma(\beta_{\frac{A_{i+1}}{n_k}})(Y_{\frac{A_{i+1}}{n_k}} - Y_{\frac{A_i}{n_k}}) \xrightarrow{(a.s.)} \int_0^t \sigma(\beta_s) dY_s \quad \text{as} \quad n \to \infty.$$ 

Because

$$\sum_{i=0}^{n_k-1} \sigma(\beta_{\frac{A_{i+1}}{n_k}})(Y_{\frac{A_{i+1}}{n_k}} - Y_{\frac{A_i}{n_k}}) \xrightarrow{(a.s.)} \int_0^{A_t} \sigma(\beta_s) dY_s \quad \text{as} \quad n \to \infty,$$ 

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we have
\[ \int_0^t \sigma(\beta_s) dY_s = \int_0^{A_t} \sigma(\beta_s) dY_s \text{ (a.s.)} \]
and so
\[ \int_0^t \sigma(X_s) dB_s = \int_0^t \sigma(\beta_s) dY_s = \int_0^{A_t} \sigma(\beta_s) dY_s = \int_0^{A_t} \sigma(\beta_s) \sigma(\beta_s)^{-1} dB_s = \beta_{A_t} - \beta_0 = X_t - x. \]

Therefore \((X, B), (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) is a solution of \(E_{x}(\sigma, 0)\).

(b) We prove that weak uniqueness hold for \(E(\sigma, 0)\). Fix \(x \in \mathbb{R}\). Let \((X, B), (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a solution of \(E_{x}(\sigma, 0)\). By problem 2., there exists a Brownian motion \((\beta_t)_{t \geq 0}\) such that
\[ X_t = \beta_{\inf\{s \geq 0 | f_s(\beta_s) - 2dr > t\}} \text{ (a.s.) } \forall t \geq 0. \]
Define \(\Phi_t : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}\) by
\[ \Phi_t(b) := b(\inf\{s \geq 0 | \int_0^s \sigma(b(r))^{-2} dr > t\}). \]
Let \(f_i : \mathbb{R} \to \mathbb{R}\) be bounded measurable functions for \(i = 1, 2, ..., m\) and \(0 \leq t_1 < t_2 < ... < t_m\). Then
\[ E[f_1(X_{t_1}) f_2(X_{t_2}) ... f_m(X_{t_m})] = E[f_1(\Phi_{t_1}(\beta)) f_2(\Phi_{t_2}(\beta)) ... f_m(\Phi_{t_m}(\beta))] \]
\[ = \int f_1(\Phi_{t_1}(w)) f_2(\Phi_{t_2}(w)) ... f_m(\Phi_{t_m}(w)) W(dw), \]
where \(W(dw)\) is the Wiener measure on \(C(\mathbb{R}_+, \mathbb{R})\). Thus, weak uniqueness hold for \(E_{x}(\sigma, 0)\).

\[ \square \]

### 8.2 Exercise 8.10

We consider the stochastic differential equation
\[ E(\sigma, b) : \quad dX_t = \sigma(X_t) dB_t + b(X_t) dt \]
where the function \(\sigma, b : \mathbb{R} \to \mathbb{R}\) are bounded and continuous, and such that \(\int_\mathbb{R} |b(x)| dx < \infty\) and \(\sigma \geq \epsilon\) for some \(\epsilon > 0\).

1. Let \(X\) be a solution of \(E(\sigma, b)\). Show that there exists a monotone increasing function \(F : \mathbb{R} \to \mathbb{R}\), which is also twice continuously differentiable, such that \(F(X_t)\). Give an explicit formula for \(F\) in terms of \(\sigma\) and \(b\).

2. Show that the process \(Y_t = F(X_t)\) solves a stochastic differential equation of the form \(dY_t = \sigma'(Y_t) dB_t\), with a function \(\sigma'\) to be determined.

3. Using the result of the preceding exercise, show that weak existence and weak uniqueness hold for \(E(\sigma, b)\). Show that pathwise uniqueness also holds if \(\sigma\) is Lipschitz.

**Proof.**

For the sake of simplicity, we define \(||f||_u := \sup_{x \in \mathbb{R}} |f(x)|\) and \(||f||_{C^1(\mathbb{R})} := \int_\mathbb{R} |f(x)| dx\).

1. Suppose \(F \in C^2(\mathbb{R})\). By Itô’s formula, we get
\[ F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int_0^t F''(X_s) d\langle X, X \rangle_s \]
\[ = F(X_0) + \int_0^t F'(X_s) \sigma(X_s) dB_s + \int_0^t F'(X_s) b(X_s) ds + \frac{1}{2} \int_0^t F''(X_s) \sigma(X_s)^2 ds. \]
Define $F : \mathbb{R} \mapsto \mathbb{R}$ by
\[
F(x) := \int_0^x e^{-\int_0^r \frac{2b(s)}{\sigma(s)^2}ds}dr.
\]
Note that
\[
F'(x) = e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds}, \quad F''(x) = -e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \frac{2b(x)}{\sigma(x)^2},
\]
and
\[
2F'(x)b(x) + F''(x)\sigma(x)^2 = 0.
\]
Then $F$ is a monotone increasing, twice continuously differentiable function and
\[
F(X_t) = F(X_0) + \int_0^t F'(X_s)\sigma(X_s)dB_s
\]
is a continuous local martingale. Since
\[
\mathbb{E}[(F(X), F(X))_t] = \mathbb{E}\left[\int_0^t F'(X_s)^2\sigma(X_s)^2ds \right] \leq t \times \|F'(x)^2\|_u \|\sigma^2\|_u \leq t \times e^{\frac{1}{2} \int_0^t \|b(r)\|^2dr} \|\sigma^2\|_u < \infty,
\]
we see that $(F(X_t))_{t \geq 0}$ is a martingale.

2. Since $F'(x) > 0$ for all $x \in \mathbb{R}$, $F$ is strictly increasing and so $F^{-1}$ exist. Observe that
\[
e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \geq e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \geq e^{-\frac{2}{2} \|b\|_{L^1(\mathbb{R})}} > 0.
\]
Then
\[
\lim_{x \to \pm \infty} F(x) = \lim_{x \to \pm \infty} \int_0^x e^{-\int_0^r \frac{2b(s)}{\sigma(s)^2}ds}dr = \pm \infty
\]
and so the domain of $F^{-1}$ is $\mathbb{R}$. Moreover, since $F \in C^2(\mathbb{R})$, we see that $F^{-1} \in C^2(\mathbb{R})$. Set
\[
H(x) := F'(x)\sigma(x) \quad \text{and} \quad \sigma'(y) := H(F^{-1}(y)).
\]
Then
\[
E'(\sigma') : \quad dY_t = H(X_t)dB_t = H(F^{-1}(Y_t))dB_t = \sigma'(Y_t)dB_t.
\]

3. First, we show that weak existence and weak uniqueness hold for $E'(\sigma')$. By Exercise 8.9, it suffices to show that $\sigma' : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and the exist $\epsilon, M > 0$ such that $\delta \leq \sigma'(y) \leq M$ for all $y \in \mathbb{R}$. Since $F^{-1}$ and $H$ are continuous,
\[
H(x) = e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \sigma(x) \geq e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \sigma(x) \geq e^{-\frac{2}{2} \|b\|_{L^1(\mathbb{R})}} \epsilon \geq 0 \quad \forall x \in \mathbb{R},
\]
and
\[
H(x) = e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \sigma(x) \leq e^{\int_0^x \frac{2b(s)}{\sigma(s)^2}ds} \sigma(x) \leq e^{\frac{2}{2} \|b\|_{L^1(\mathbb{R})}} M < \infty \quad \forall x \in \mathbb{R},
\]
we see that $\sigma'(y) = H(F^{-1}(y))$ is continuous and $\delta \leq \sigma'(x) \leq M$ for all $x \in \mathbb{R}$. Thus, weak existence and weak uniqueness hold for $E'(\sigma')$.

Now, we show that weak existence hold for $E(\sigma, b)$. Fix $x \in \mathbb{R}$. Set $y = F(x)$. There exists a solution $(Y, B), (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ of $E'(\sigma')$. Define
\[
X_t := F^{-1}(Y_t).
\]
By Itô’s formula, we get
\[
X_t = x + \int_0^t \frac{dF^{-1}}{dy}(Y_s)dY_s + \frac{1}{2} \int_0^t \frac{d^2F^{-1}}{dy^2}(Y_s)d(Y_s,Y)_s.
\]
By $F^{-1}(F(x)) = x$, we get
\[ \frac{dF^{-1}}{dy}(F(x)) \frac{dF}{dx}(x) = 1 \] and
\[ \frac{d^2F^{-1}}{dy^2}(F(x))\left(\frac{dF}{dx}(x)\right)^2 + \frac{dF^{-1}}{dy}(F(x)) \frac{d^2F}{dx^2}(x) = 0. \]

Thus,
\[ \frac{dF^{-1}}{dy}(Y_s) = \frac{dF^{-1}}{dy}(F(X_s)) = \left(\frac{dF}{dx}(X_s)\right)^{-1} = e^{\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} \]
and
\[ \frac{d^2F^{-1}}{dy^2}(Y_s) = \frac{d^2F^{-1}}{dy^2}(F(X_s)) = \left(-\frac{dF^{-1}}{dy}(F(X_s)) \frac{d^2F}{dx^2}(X_s) \right) \times \left(\frac{dF}{dx}(X_s)\right)^{-2} \]
\[ = -e^{\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} \times -e^{\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} \times e^{2\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} \frac{2b(X_s)}{\sigma(X_s)^2} \]
\[ = \frac{2b(X_s)}{\sigma(X_s)^2} d\int_0^s \frac{2b(r)}{\sigma(r)^2} dr. \]

By
\[ dY_t = \sigma(Y_t) dB_t = H(F^{-1}(Y_t)) dB_t = H(X_t) dB_t = e^{-\int_0^t \frac{2b(r)}{\sigma(r)^2} dr} \sigma(X_t) dB_t, \]
we get
\[ X_t = x + \int_0^t \frac{dF^{-1}}{dy}(Y_s) dY_s + \frac{1}{2} \int_0^t \frac{d^2F^{-1}}{dy^2}(Y_s) d(Y_s)_s \]
\[ = x + \int_0^t e^{\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} d\int_0^s \frac{2b(r)}{\sigma(r)^2} dr \sigma(X_s) dB_s + \frac{1}{2} \int_0^t \frac{2b(X_s)}{\sigma(X_s)^2} e^{2\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} d\int_0^s \frac{2b(r)}{\sigma(r)^2} dr \sigma(X_s)^2 ds \]
\[ = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds \]
and so $(X, B, \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a solution of $E_x(\sigma,b)$.

Now, we show that weak uniqueness hold for $E(\sigma,b)$. Fix $x \in \mathbb{R}$ and $y = F(x)$. Let $(X, B, \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and $(X', B', \Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, P')$ be solutions of $E_x(\sigma,b)$. By problem 2., we see that $(Y_t)_{t \geq 0} := (F(X_t))_{t \geq 0}$ and $(Y'_t)_{t \geq 0} := (F(X'_t))_{t \geq 0}$ are solutions of $E_y(\sigma')$. Since weak uniqueness hold for $E'(\sigma')$ and $F$ is injective, we get
\[ E[1_{X_t \in \Gamma_1} \ldots 1_{X_k \in \Gamma_k}] = E[1_{Y_t \in F(\Gamma_1)} \ldots 1_{Y_h \in F(\Gamma_h)}] \]
\[ = E'[1_{Y'_t \in F(\Gamma_1)} \ldots 1_{Y'_k \in F(\Gamma_k)}] \]
\[ = E'[1_{X'_t \in \Gamma_1} \ldots 1_{X'_k \in \Gamma_k}] \]
and, hence, weak uniqueness hold for $E(\sigma,b)$.

Finally, we show that pathwise uniqueness hold for $E(\sigma,b)$ whenever $\sigma$ is Lipshitz. To show this, it suffices to show that $\sigma'$ is Lipshitz. Indeed, by Theorem 8.3 and $\sigma'$ is Lipshitz, we see that pathwise uniqueness hold for $E'(\sigma')$. Let $X$ and $X'$ are solutions of $E(\sigma,b)$ under $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $(B_t)_{t \geq 0}$ started from $0$ such that $P(X_0 = X'_0) = 1$. By problem 2., we get $(Y_t)_{t \geq 0} := (F(X_t))_{t \geq 0}$ and $(Y'_t)_{t \geq 0} := (F(X'_t))_{t \geq 0}$ are solutions of $E'(\sigma')$ such that $P(Y_0 = Y'_0) = 1$ and so
\[ F(X_t) = Y_t = Y'_t = F(X'_t) \quad \forall t \geq 0 \quad P-(a.s.). \]

Since $F$ is injective, we get
\[ X_t = X'_t \quad \forall t \geq 0 \quad P-(a.s.). \]

Now, we show that $\sigma'(y) := H(F^{-1}(y))$ is Lipshitz whenever $\sigma$ is Lipshitz. Choose $C > 0$ such that
\[ |\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2|. \]
Fix real numbers $y_1$ and $y_2$. Set $x_i = F^{-1}(y_i)$ for $i = 1, 2$. Note that
\[ \|F'\|_u \leq e^{\frac{\lambda}{2} \|b\|_{L^1(P)}} < \infty. \]
and
\[ \|F''\|_u \leq \frac{2 \|b\|_u e^{\frac{\lambda}{2} \|b\|_{L^1(P)}}}{\epsilon^2} < \infty. \]
By mean value theorem, we get
\[ |\sigma'(y_1) - \sigma'(y_2)| = |H(x_1) - H(x_2)| = |F'(x_1)\sigma(x_1) - F'(x_2)\sigma(x_2)| \leq |F'(x_1)\sigma(x_1) - F'(x_1)\sigma(x_2)| + |F'(x_1)\sigma(x_2) - F'(x_2)\sigma(x_2)| \leq |F'\|_u C|x_1 - x_2| + |\sigma\|_u \|F''\|_u |x_1 - x_2| = C'|x_1 - x_2|, \]
where $C' := (\|F'\|_u C) \vee (\|\sigma\|_u \|F''\|_u)$. Because
\[ \left| \frac{dF^{-1}}{dy}(y) \right| = |F'(F^{-1}(y))| \leq \|(F')^{-1}\|_u = \sup_{x \in \mathbb{R}} e^{\int_{0}^{x} \frac{\lambda(r)}{\lambda(r)} dr} \leq e^{\frac{\lambda}{2} \|b\|_{L^1(P)}} < \infty, \]
we get
\[ |x_2 - x_1| = |F^{-1}(y_2) - F^{-1}(y_1)|^{-1} \leq \left| \frac{dF^{-1}}{dy} \right|_u |y_2 - y_1| \]
and so
\[ |\sigma'(y_1) - \sigma'(y_2)| \leq C|y_1 - y_2|, \]
where $C := \|\frac{dF^{-1}}{dy}\|_u C'$.

\[ \square \]

8.3 Exercise 8.11

We suppose that, for every $x \in \mathbb{R}_+$, one can construct on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ a process $X^x$ taking nonnegative values, which solves the stochastic differential equation
\[
\begin{cases}
  dX_t = \sqrt{2X_t}dB_t \\
  X_0 = x.
\end{cases}
\]
and that the processes $X^x$ are Markov processes with values in $\mathbb{R}_+$, with the same semigroup $(Q_t)_{t \geq 0}$, with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ (This is, of course, close to Theorem 8.6, which however cannot be applied directly because the function $\sqrt{2x}$ is not Lipschitz.)

1. We fix $x \in \mathbb{R}_+$, and real $T > 0$. We set, for every $t \in [0, T]$
\[ M_t = e^{-\frac{\lambda x_t}{1 + \lambda x_T - 1}}. \]

   Show that the process $(M_{t \wedge T})$ is a martingale.

2. Show that $(Q_t)_{t \geq 0}$ is the semigroup of Feller’s branching diffusion (see the end of Chap. 6).

Proof.

Note that $\lambda \geq 0$. 

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1. Fix $T > 0$. By Itô’s formula, we get
\[
M_t = e^{-\lambda X_T^2} \\
= e^{-\lambda X_T^2} + \int_0^t \frac{-\lambda}{1 + \lambda(T - s)} e^{-\lambda X_s^2} dB_s + \int_0^t \frac{-\lambda X_s^2}{1 + \lambda(T - s)} e^{-\lambda X_s^2} ds \\
+ \frac{1}{2} \int_0^t \frac{-\lambda^2}{(1 + \lambda(T - s))^2} e^{-\lambda X_s^2} d\langle X_s, X_s \rangle + \int_0^t \frac{-\lambda X_s^2}{1 + \lambda(T - s)} e^{-\lambda X_s^2} ds \\
= e^{-\lambda X_T^2} + \int_0^t \frac{-\lambda}{1 + \lambda(T - s)} e^{-\lambda X_s^2} dB_s + \int_0^t \frac{-\lambda X_s^2}{1 + \lambda(T - s)} e^{-\lambda X_s^2} ds \\
+ \frac{1}{2} \int_0^t \frac{-\lambda^2}{(1 + \lambda(T - s))^2} e^{-\lambda X_s^2} dB_s.
\]
is a continuous local martingale. Since $x \leq e^x$ for all $x \geq 0$, we have
\[
E[(M, M)_T] = E\left[\int_0^T \frac{\lambda^2 X_s^2}{(1 + \lambda(T - s))^2} e^{-\lambda X_s^2} ds\right] \leq E\left[\int_0^T \frac{\lambda}{1 + \lambda(T - s)} ds\right] \\
= \int_0^T \frac{\lambda}{1 + \lambda(T - s)} ds < \infty
\]
and so $(M_{\lambda T})_{t \geq 0}$ is a uniformly integrable martingale.

2. Fix $T > 0$. By optional stopping theorem and problem 1., we get
\[
e^{\frac{\lambda}{1 + \lambda T^2}} = E[M_{0, T}] = E[M_{\infty, T}] = E[e^{-\lambda X_T^2}] = \int e^{-\lambda y} Q_T(x, dy).
\]
Thus, we have
\[
\int e^{-\lambda y} Q_t(x, dy) = e^{-x \psi_t(\lambda)},
\]
where $\psi_t(\lambda) := \frac{\lambda}{1 + \lambda T^2}$ and $t > 0$. By the last example in chapter 6., we see that $(Q_t)_{t \geq 0}$ is the semigroup of Feller’s branching diffusion.

\[\square\]

8.4 Exercise 8.12

We consider two sequences $(\sigma_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ of real functions defined on $\mathbb{R}$. We assume that:

1. There exists a constant $C > 0$ such that $|\sigma_n(x)| \vee |b_n(x)| \leq C$ for every $n \geq 1$ and $x \in \mathbb{R}$.
2. There exists a constant $K > 0$ such that, for every $n \geq 1$ and $x, y \in \mathbb{R}$,
\[
|\sigma_n(x) - \sigma_n(y)| \vee |b_n(x) - b_n(y)| \leq K|x - y|.
\]

Let $B$ be an $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion and, for every $n \geq 1$, let $X^n$ be the unique adapted process satisfying
\[
X^n_t = \int_0^t \sigma_n(X^n_s) dB_s + \int_0^t b_n(X^n_s) ds.
\]

1. Let $T > 0$. Show that there exists a constant $A > 0$ such that, for every real $M > 0$ and for every $n \geq 1$,
\[
P(\sup_{t \leq T} |X^n_t| \geq M) \leq \frac{A}{M^2}.
\]

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2. We assume that the sequences \( \{\sigma_n\} \) and \( \{b_n\} \) converge uniformly on every compact subset of \( \mathbb{R} \) to limiting functions denoted by \( \sigma \) and \( b \) respectively. Justify the existence of an adapted process \( X = (X_t)_{t \geq 0} \) with continuous sample paths, such that

\[
X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds,
\]

then show that there exists a constant \( A' \) such that, for every real \( M > 0 \), for every \( t \in [0, T] \) and \( n \geq 1 \),

\[
E[\sup_{s \leq t} |X^n_s - X_s|^2] \leq 4(4 + T)K^2 \int_0^t E[|X^n_s - X_s|^2]ds + \frac{A'}{M^2}
\]

\[
+ 4T(4 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + T \sup_{|x| \leq M} |b_n(x) - b(x)|^2).
\]

3. Infer from the preceding question that

\[
\lim_{n \to \infty} E[\sup_{s \leq T} |X^n_s - X_s|^2] = 0.
\]

**Proof.**

1. Fix \( T > 0 \) and \( M > 0 \). By Burkholder–Davis–Gundy inequalities (Theorem 5.16), we get

\[
P(\sup_{t \leq T} |X^n_t| \geq M) \leq \frac{1}{M^2} E[\sup_{t \leq T} |X^n_t|^2] \leq \frac{C_2}{M^2} E[(X^n, X^n)_T]
\]

\[
= \frac{C_2}{M^2} E[\int_0^T \sigma_n(X^n_s)^2ds] \leq \frac{C_2TC^2}{M^2} := \frac{A}{M^2},
\]

where \( A = A(T) := C_2TC^2 \).

2. Since \( \sigma_n \to \sigma \) and \( b_n \to b \) uniformly on every compact subset of \( \mathbb{R} \), we get

\[
|\sigma(x) - \sigma(y)| \vee |b(x) - b(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R},
\]

and

\[
|\sigma(x)| \vee |b(x)| \leq C \quad \forall x \in \mathbb{R}.
\]

By Theorem 8.5, there exists an adapted process \( X = (X_t)_{t \geq 0} \) with continuous sample paths, such that

\[
X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \quad \forall t \geq 0 \quad \mathcal{P}\text{-a.s.}
\]

By similar argument, we have

\[
P(\sup_{t \leq T} |X_t| \geq M) \leq \frac{A(T)}{M^2} \quad \forall T > 0 \text{ and } M > 0.
\]

Fix \( T > 0 \), \( t \in [0, T] \), and \( M > 0 \). Now, we show that

\[
E[\sup_{s \leq t} |X^n_s - X_s|^2] \leq 2 \times 4^2 K^2 (4 + T) \int_0^t E[|X^n_s - X_s|^2]ds + \frac{(4 + T)4^3C^22A(T)}{M^2}
\]

\[
+ 4T(4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2)
\]
for all \( n \geq 1 \). (Note that this upper bound is larger than the upper bound in problem 2. However, this doesn’t affect of the proof of problem 3.) Let \( n \geq 1 \). Then

\[
E[\sup_{s \leq t} |X^n_s - X_s|^2] \leq 4E[\sup_{s \leq t} |\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r|^2] + 4E[\sup_{s \leq t} |\int_0^s b_n(X^n_r) - b(X_r)dr|^2].
\]

Since \( |\sigma_n(x)| \leq C \) for all \( x \in \mathbb{R} \), we see that \( (\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r)_{s \geq 0} \) is a martingale. By Doob’s inequality in \( L^2 \) and Hölder’s inequality, we have

\[
4E[\sup_{s \leq t} |\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r|^2] + 4E[\sup_{s \leq t} |\int_0^s b_n(X^n_r) - b(X_r)dr|^2] \\
\leq 4 \times 4E[|\int_0^t \sigma_n(X^n_s) - \sigma(X_s)dB_s|^2] + 4TE[|\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds] \\
\leq 4 \times 4E[|\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds] + 4TE[|\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds] \\
\leq 4 \times 4E[|\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \geq M\}} \cup \{\sup_{s \leq t} |X_s| \geq M\}] \\
+ 4 \times 4E[|\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}] \\
+ 4 \times T E[|\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \geq M\}} \cup \{\sup_{s \leq t} |X_s| \geq M\}] \\
+ 4 \times T E[|\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}] \\
\leq 4 \times 4E[|\int_0^t 4|\sigma_n(X^n_s) - \sigma(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \geq M\}} \cup \{\sup_{s \leq t} |X_s| \geq M\}] \\
+ 4 \times 4E[|\int_0^t 4|\sigma_n(X^n_s) - \sigma(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}] \\
+ 4 \times T E[|\int_0^t 4|b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \geq M\}} \cup \{\sup_{s \leq t} |X_s| \geq M\}] \\
+ 4 \times T E[|\int_0^t 4|b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}] \\
+ 4 \times T E[|\int_0^t 4|b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}] \\
+ 4 \times T E[|\int_0^t 4|b_n(X^n_s) - b(X_s)|^2 ds_1_{\{\sup_{s \leq t} |X^n_s| \leq M\}} \cap \{\sup_{s \leq t} |X_s| \leq M\}].
\]
\[\begin{align*}
&\leq 4^2 E\left[ \int_0^t 4K^2|X^n_s - X_s|^2 ds \right] + 4^3 (T4C^2 P(\{s \leq T \sup_{s \leq T} |X^n_s| \geq M\} \cup \{s \leq T \sup_{s \leq T} |X_s| \geq M\}) \\
&+ 4^2 E\left[ \int_0^t 4K^2|X^n_s - X_s|^2 ds \right] + 4^3 T \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 \\
&+ 4T E\left[ \int_0^t 4K^2|X^n_s - X_s|^2 ds \right] + 4^2 T (T4C^2 P(\{s \leq T \sup_{s \leq T} |X^n_s| \geq M\} \cup \{s \leq T \sup_{s \leq T} |X_s| \geq M\}) \\
&+ 4T E\left[ \int_0^t 4K^2|X^n_s - X_s|^2 ds \right] + 4^2 T \times T \sup_{|x| \leq M} |b_n(x) - b(x)|^2 \\
&= 2 \times 4^2 K^2 (4 + T) \int_0^t E[|X^n_s - X_s|^2] ds + (4 + T) T4^3 C^2 P(\{s \leq T \sup_{s \leq T} |X^n_s| \geq M\} \cup \{s \leq T \sup_{s \leq T} |X_s| \geq M\}) \\
&+ 4T (4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2) \\
&= 2 \times 4^2 K^2 (4 + T) \int_0^t E[|X^n_s - X_s|^2] ds + (4 + T) T4^3 C^2 (P(\sup_{s \leq T} |X^n_s| \geq M) \cup P(\sup_{s \leq T} |X_s| \geq M)) \\
&+ 4T (4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2) \\
&= 2 \times 4^2 K^2 (4 + T) \int_0^t E[|X^n_s - X_s|^2] ds + (4 + T) T4^3 C^2 (2 \frac{A(T)}{M^2}) \\
&+ 4T (4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2) \\
&\quad \text{for all } t \in [0, T]. \text{ Define } g : [0, T] \to \mathbb{R}_+ \text{ by} \\
g(t) := E[\sup_{s \leq t} |X^n_s - X_s|^2].
\end{align*}\]

Set positive real numbers

\[a := (4 + T) T4^3 C^2 (2 \frac{A(T)}{M^2}) + 4T (4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2)\]

and

\[b := 2 \times 4^2 K^2 (4 + T).\]

Then we have

\[g(t) \leq b \int_0^t g(s) ds + a \quad \forall t \in [0, T],\]
By Burkholder–Davis–Gundy inequalities (Theorem 5.16) and Hölder’s inequality, we get
\[
|g(t)| = E[\sup_{s \leq t} |X^n_s - X_s|^2] \\
\leq 4E[\sup_{s \leq t} |\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r|^2] + 4E[\sup_{s \leq t} |\int_0^s b_n(X^n_r) - b(X_r)dr|^2] \\
\leq 4C_2E[\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds + 4tE[||\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds]] \\
\leq 4C_2(4C^2T) + 4T(4C^2T) < \infty
\]
and so \( g \) is bounded. By Gronwall’s lemma (Lemma 8.4), we have
\[
E[\sup_{s \leq T} |X^n_s - X_s|^2] = g(T) \leq a \times e^{bT} \\
\leq ((4 + T)4^3C^2(2\frac{A(T)}{M^2}) + 4T(4^2 \sup_{|x| \leq M} |\sigma_n(x) - \sigma(x)|^2 + 4T \sup_{|x| \leq M} |b_n(x) - b(x)|^2)) \\
\times \exp(2 \times 4^2K^2(4 + T) \times T)
\]
and so
\[
\limsup_{n \to \infty} E[\sup_{s \leq T} |X^n_s - X_s|^2] \leq (4 + T)4^3C^2(2\frac{A(T)}{M^2}) \exp(2 \times 4^2K^2(4 + T) \times T).
\]
By letting \( M \to \infty \), we get
\[
\lim_{n \to \infty} E[\sup_{s \leq T} |X^n_s - X_s|^2] = 0.
\]

8.5 Exercise 8.13

Let \( \beta = (\beta_t)_{t \geq 0} \) be an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion started from 0. We fix two real parameters \( \alpha \) and \( r \), with \( \alpha > \frac{1}{2} \) and \( r > 0 \). For every integer \( n \geq 1 \) and every \( x \in \mathbb{R} \), we set
\[
f_n(x) = \frac{1}{|x|} \land n.
\]

1. Let \( n \geq 1 \). Justify the existence of unique semimartingale \( Z^n \) that solves the equation
\[
Z^n_t = r + \beta_t + \alpha \int_0^t f_n(Z^n_s)ds.
\]

2. We set \( S_n := \inf\{t \geq 0 \mid Z^n_t \leq \frac{1}{n}\} \). After observing that, for \( t \leq S_{n+1} \land S_n \),
\[
Z^{n+1}_t - Z^n_t = \alpha \int_0^t \frac{1}{Z^{n+1}_s} - \frac{1}{Z^n_s}ds,
\]
show that \( Z^{n+1}_t = Z^n_t \) for every \( t \in [0, S_{n+1} \land S_n] \) (a.s.). Infer that \( S_{n+1} \geq S_n \).

3. Let \( g \) be a twice continuously differentiable function on \( \mathbb{R} \). Show that the process
\[
g(Z^n_t) - g(r) - \int_0^t (\alpha g'(Z^n_s))f_n(Z^n_s) + \frac{1}{2} g''(Z^n_s))ds
\]
is a continuous local martingale.
4. We set \( h(x) = x^{1-2\alpha} \) for every \( x > 0 \). Show that, for every integer \( n \geq 1 \), \( h(Z^n_{t_0,S_n}) \) is a bounded martingale. Infer that, for every \( t' \geq 0 \), \( P(S_n \leq t') \to 0 \) as \( n \to \infty \), and consequently \( S_n \to \infty \) as \( n \to \infty \) \( P \)-(a.s.).

5. Infer from questions 2. and 4. that there exists a unique positive semimartingale \( Z \) such that, for every \( t \geq 0 \),

\[
Z_t = r + \beta t + \alpha \int_0^t ds Z_s
\]

6. Let \( d \geq 3 \) and let \( B \) be a \( d \)-dimensional Brownian motion started from \( y \in \mathbb{R}_d \setminus \{0\} \). Show that \( Y_t = |B_t| \) satisfies the stochastic equation in question 5. (with an appropriate choice of \( \beta \)) with \( r = |y| \) and \( \alpha = \frac{d-1}{2} \).

One may use the results of Exercise 5.33.

**Proof.**

1. To prove the existence of unique of soltion of

\[
E^n_t: \quad dZ^n_t = d\beta t + \alpha f_n(Z^n_t) dt
\]

it suffices to show that \( f_n \) is Lipschitz. Observe that, if \( |x|, |y| \geq \frac{1}{n} \), and if \( |v| < \frac{1}{n} \leq |u| \), then

\[
|f_n(x) - f_n(y)| = |\frac{1}{|x|} - \frac{1}{|y|}| = |\frac{|x| - |y|}{|x||y|}| \leq n^2|x - y|
\]

and

\[
|f_n(v) - f_n(u)| = n - \frac{1}{|u|} = \frac{|u| - |v + \frac{1}{n}|}{\frac{1}{n}|u|} \leq n^2(|u + \frac{1}{n}| \wedge |u - \frac{1}{n}|) \leq n^2|u - v|.
\]

Hence \( f_n \) is Lipschitz. By Theorem 8.5.(iii), there exists a unique solution of \( E^n_t \).

2. Observe that, if \( 0 \leq t \leq S_{n+1} \wedge S_n \), then

\[
Z^k_t = r + \beta t + \alpha \int_0^t \frac{1}{Z^{k}_s} ds \quad \forall k = n, n + 1
\]

and

\[
Z^{n+1}_t - Z^n_t = \alpha \int_0^t \frac{1}{Z^{n+1}_s} - \frac{1}{Z^n_s} ds.
\]

Then \( Z^n_t \geq \frac{1}{n} > 0 \) and \( Z^{n+1}_t \geq \frac{1}{n+1} > 0 \) for every \( 0 \leq t \leq S_n \wedge S_{n+1} \). Fix \( 0 \leq t \leq S_n \wedge S_{n+1} \). Note that \( \frac{1}{a} \leq \frac{1}{b} \)

whenever \( 0 < b \leq a \). Suppose \( Z^{n+1}_s \geq Z^n_s \) for all \( s \in [0, t] \). Then

\[
0 \leq Z^{n+1}_s - Z^n_s = \alpha \int_0^s \frac{1}{Z^{n+1}_r} - \frac{1}{Z^n_r} dr \leq 0
\]

and so \( Z^{n+1}_s = Z^n_s \) for all \( s \in [0, t] \). Similarly, if \( Z^{n+1}_s \leq Z^n_s \) for all \( s \in [0, t] \), then \( Z^{n+1}_s = Z^n_s \) for all \( s \in [0, t] \). Thus, we get

\[
Z^{n+1}_t = Z^n_t \quad \forall t \in [0, S_n \wedge S_{n+1}] \quad P-(a.s.).
\]

Now, we show that \( S_{n+1} \geq S_n \) for every \( n \geq 1 \) by contradiction. Fix \( n \geq 1 \). Assume that \( P(S_{n+1} < S_n) > 0 \). Then

\[
P(S_{n+1} < S_n, Z^{n+1}_t = Z^n_t \quad \forall t \in [0, S_n \wedge S_{n+1}]) > 0.
\]

Fix \( w \in \{S_{n+1} < S_n\} \cap \{Z^{n+1}_t = Z^n_t \quad \forall t \in [0, S_n \wedge S_{n+1}]\} \). Set \( \lambda = S_{n+1}(w) \). Since \( Z^{n+1}_t(w) = Z^n_t(w) \) for all \( 0 \leq t \leq S_n(w) \wedge S_{n+1}(w) = S_{n+1}(w) = \lambda \), we get

\[
Z^n_t(w) = Z^{n+1}_t = \frac{1}{n+1} < \frac{1}{n}
\]

and so \( S_{n+1}(w) = \lambda \geq S_n(w) \) which contradicts \( S_{n+1}(w) < S_n(w) \). Therefore, we have

\[
S_{n+1} \geq S_n \quad \forall n \geq 1 \quad P-(a.s.).
\]
3. By Itô’s formula, we get

\[ g(Z^n_t) = g(r) + \int_0^t g'(Z^n_s)\,dZ^n_s + \frac{1}{2} \int_0^t g''(Z^n_s)\,d\langle Z^n, Z^n \rangle_s \]

\[ = g(r) + \int_0^t g'(Z^n_s)\,d\beta_s + \int_0^t g'(Z^n_s)\,d\beta_s + \frac{1}{2} \int_0^t g''(Z^n_s)\,ds \]

and so

\[ g(Z^n_t) - g(r) = \int_0^t (\alpha g'(Z^n_s) f_n(Z^n_s) + \frac{1}{2} g''(Z^n_s))\,ds = \int_0^t g'(Z^n_s)\,d\beta_s \]

is a continuous local martingale.

4. Fix large \( n \geq 1 \) such that \( n > \frac{1}{r} \). Then \( S_n > 0 \). Since \( Z^n_{t\wedge S_n} > \frac{1}{n} \) for every \( t \geq 0 \), we have \( f_n(Z^n_{t\wedge S_n}) = \frac{1}{Z^n_{t\wedge S_n}} \)

for every \( t \geq 0 \) and so

\[ \int_0^t 1(s \leq S_n)\,dZ^n_s = \int_0^t 1(s \leq S_n)\,d\beta_s + \alpha \int_0^t \frac{1}{Z^n_{s\wedge S_n}} 1(s \leq S_n)\,ds. \]

By Itô’s formula, we get

\[ M_t := h(Z^n_{t\wedge S_n}) \]

\[ = r^{1-2\alpha} + \int_0^t (1 - 2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha} 1(s \leq S_n)\,dZ^n_s \]

\[ + \frac{(-2\alpha)(1 - 2\alpha)}{2} \int_0^t (Z^n_{s\wedge S_n})^{-2\alpha-1} 1(s \leq S_n)\,d\langle Z^n, Z^n \rangle_s \]

\[ = r^{1-2\alpha} + \int_0^t (1 - 2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha} 1(s \leq S_n)\,d\beta_s + \int_0^t (1 - 2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha} \frac{1}{Z^n_{s\wedge S_n}} 1(s \leq S_n)\,ds \]

\[ + \frac{(-2\alpha)(1 - 2\alpha)}{2} \int_0^t (Z^n_{s\wedge S_n})^{-2\alpha-1} 1(s \leq S_n)\,ds \]

\[ = r^{1-2\alpha} + \int_0^t (1 - 2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha} 1(s \leq S_n)\,d\beta_s \]

is a continuous local martingale. Moreover, since

\[ E[(M,M)_t] = E[(1 - 2\alpha)^2 \int_0^t (Z^n_{s\wedge S_n})^{-4\alpha} 1(s \leq S_n)\,ds] \leq (1 - 2\alpha)^2 \times t \times n^{4\alpha} < \infty \]

for every \( t \geq 0 \), we see that \( h(Z^n_{t\wedge S_n}) \) is a martingale. Because

\[ 0 < M_t = h(S^n_{t\wedge S_n}) = (Z^n_{t\wedge S_n})^{1-2\alpha} \leq n^{2\alpha-1} < \infty \]

for every \( t \geq 0 \), we get \( h(Z^n_{t\wedge S_n}) \) is a bounded martingale.

Now, we show that \( \lim_{n \to \infty} P(S_n \leq t') = 0 \) for every \( t' \geq 0 \). Fix \( t' \geq 0 \). Choose large \( n \geq 1 \) such that \( n > \frac{1}{r} \).

Since \( h(Z^n_{t\wedge S_n}) \) is a bounded martingale and \( h \) is positive, we get

\[ r^{1-2\alpha} = h(r) = E[h(Z^n_{0\wedge S_n})] = E[h(Z^n_{t\wedge S_n})] \]

\[ = P(S_n \leq t')n^{2\alpha-1} + E[h(Z^n_{t\wedge S_n})1_{\nu < S_n}] \]

\[ \geq P(S_n \leq t')n^{2\alpha-1} \]

and, hence,

\[ P(S_n \leq t') \leq (\frac{1}{nr})^{2\alpha-1} \to 0 \text{ as } n \to \infty. \]
Moreover, since $S_{n+1} \geq S_n$ for every $n \geq 1$, $S := \lim_{n \to \infty} S_n$ exist and so

$$P(S \leq t) = \lim_{n \to \infty} P(S_n \leq t) = 0$$

for every $t \geq 0$. Thus,

$$\lim_{n \to \infty} S_n = S = \infty \quad P-(a.s.).$$

5. (a) We show that there exists a positive semimartingale $Z$ such that, for every $t \geq 0$,

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s}.$$

By problem 2., we have

$$Z_t^{n+1} = Z_t^n \quad \forall t \in [0, S_n] \text{ and } n \geq 1 \text{ outside a zero set } N.$$

For the sake of simplicity, we redefine $N$ as

$$N \bigcup \bigcap_{n \geq 1} \{Z^n_t = r + \beta_t + \alpha \int_0^t f_n(Z^n_s) ds \quad \forall t \geq 0\}^c.$$

Define

$$Z_t(w) = \begin{cases} Z^n_t(w), & \text{if } w \notin N \text{ and } t \leq S_n(w) \\ 0, & \text{otherwise.} \end{cases}$$

Then $Z$ is a positive, adapted, continuous process. Fix $w \notin N$ and $t \geq 0$. Choose large $n \geq 1$ such that $S_n(w) \geq t$. Then

$$Z_t(w) = Z^n_t(w) = r + \beta_t(w) + \int_0^t f_n(Z^n_s(w)) ds = r + \beta_t(w) + \int_0^t \frac{1}{Z^n_s(w)} ds = r + \beta_t(w) + \int_0^t \frac{1}{Z_s(w)} ds.$$

Thus, $Z$ is a positive semimartingale such that

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s} \quad \forall t \geq 0 \quad P-(a.s.).$$

(b) Let $Z$ and $Z'$ are postive semimartingales such that

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s} \quad \forall t \geq 0 \quad P-(a.s.)$$

and

$$Z'_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z'_s} \quad \forall t \geq 0 \quad P-(a.s.)$$

under filered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and Brownian motion $\beta$ started from 0. Note that $\frac{1}{a} \leq \frac{1}{b}$ whenever $0 < b \leq a$. Fix $w \in \Omega$. Observe that, if there exists real number $T > 0$ such that

$$Z_t \geq Z'_t \quad \forall t \in [0, T],$$

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The goal of the exercise is to get pathwise uniqueness for the one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt$$

when the functions $\sigma$ and $b$ satisfy the conditions

$$|\sigma(x) - \sigma(y)| \leq K \sqrt{|x-y|}, \quad |b(x) - b(y)| \leq K |x-y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$.

1. Preliminary question. Let $Z$ be a semimartingale such that $\langle Z, Z \rangle_t = \int_0^t h_s ds$, where $0 \leq h_s \leq C |Z_s|$, with a constant $C < \infty$. Show that, for every $t \geq 0$,

$$\lim_{n \to \infty} n E \left[ \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} d\langle Z, Z \rangle_s \right] = 0.$$

(Hint: Observe that, $E[\int_0^t |Z_s|^{-1} 1_{\{0 < |Z_s| \leq 1\}} d\langle Z, Z \rangle_s] \leq C t < \infty$.)

2. For every $n \geq 1$, let $\varphi_n$ be the function defined on $\mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} 0, & \text{if } |x| \geq \frac{1}{n} \\ 2n(1-nx), & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2n(1+nx), & \text{if } -\frac{1}{n} \leq x \leq 0. \end{cases}$$

Also write $F_n$ for the unique twice continuously differentiable function on $\mathbb{R}$ such that $F_n(0) = F_n'(0) = 0$ and $F_n'' = \varphi_n$. Note that, for every $x \in \mathbb{R}$, one has $F_n(x) \to |x|$ and $F_n'(x) \to sgn(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$ when

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{1}{Z_s} ds \leq r + \beta_t + \alpha \int_0^t \frac{1}{Z_s} ds = Z'_t$$

for all $t \in [0, T]$ and so $Z_t = Z'_t$ for all $t \in [0, T]$. Similarly, if there exists real number $T > 0$ such that

$$Z_t \leq Z'_t \quad \forall t \in [0, T],$$

then $Z_t = Z'_t$ for all $t \in [0, T]$. This shows that

$$Z_t = Z'_t \quad \forall t \geq 0 \quad P-(a.s.).$$

6. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space and $B$ be a $d$-dimensional Brownian motion started from $y \in \mathbb{R}^d \setminus \{0\}$. By Exercise 5.33, we get

$$|B_t| = |y| + \beta_t + \frac{d-1}{2} \int_0^t \frac{ds}{|B_s|},$$

where

$$\beta_t = \frac{d}{2} \int_0^t \frac{B^i_s}{|B_s|^2} dB^i_s$$

is a $(\mathcal{F}_t)_{t \geq 0}$ 1-dimensional Brownian motion started from 0. Thus, $(|B|, \beta, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a solution of the stochastic equation in question

$$Z_t = |y| + \beta_t + \frac{d-1}{2} \int_0^t \frac{ds}{Z_s}.$$

8.6 Exercise 8.14 (Yamada–Watanabe uniqueness criterion)

The goal of the exercise is to get pathwise uniqueness for the one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt$$

when the functions $\sigma$ and $b$ satisfy the conditions

$$|\sigma(x) - \sigma(y)| \leq K \sqrt{|x-y|}, \quad |b(x) - b(y)| \leq K |x-y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$. 

1. Preliminary question. Let $Z$ be a semimartingale such that $\langle Z, Z \rangle_t = \int_0^t h_s ds$, where $0 \leq h_s \leq C |Z_s|$, with a constant $C < \infty$. Show that, for every $t \geq 0$,

$$\lim_{n \to \infty} n E \left[ \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} d\langle Z, Z \rangle_s \right] = 0.$$

(Hint: Observe that, $E[\int_0^t |Z_s|^{-1} 1_{\{0 < |Z_s| \leq 1\}} d\langle Z, Z \rangle_s] \leq C t < \infty$.)

2. For every $n \geq 1$, let $\varphi_n$ be the function defined on $\mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} 0, & \text{if } |x| \geq \frac{1}{n} \\ 2n(1-nx), & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2n(1+nx), & \text{if } -\frac{1}{n} \leq x \leq 0. \end{cases}$$

Also write $F_n$ for the unique twice continuously differentiable function on $\mathbb{R}$ such that $F_n(0) = F_n'(0) = 0$ and $F_n'' = \varphi_n$. Note that, for every $x \in \mathbb{R}$, one has $F_n(x) \to |x|$ and $F_n'(x) \to sgn(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$ when
\( n \to \infty. \)

Let \( X \) and \( X' \) be two solutions of \( E(\sigma, b) \) on the same filtered probability space and with the same Brownian motion \( B \). Infer from question 1. that

\[
\lim_{n \to \infty} E\left[ \int_0^t \varphi_n(X_s - X_s') d(X - X', X - X')_s \right] = 0.
\]

3. Let \( T \) be a stopping time such that the semimartingale \( X_{t\wedge T} - X'_{t\wedge T} \) is bounded. By applying Itô’s formula to \( F_n(X_{t\wedge T} - X'_{t\wedge T}) \), show that

\[
E[|X_{t\wedge T} - X'_{t\wedge T}|] = E[|X_0 - X'_0|] + E\left[ \int_0^{t\wedge T} (b(X_s) - b(X'_s)) \text{sgn}(X_s - X'_s) ds \right].
\]

4. Using Gronwall’s lemma, show that, if \( X_0 = X'_0 \), one has \( X_t = X'_t \) for every \( t \geq 0 \) (a.s.).

**Proof.**

1. Note that

\[
E\left[ \int_0^t |Z_s|^{-1}1_{\{0 < |Z_s| \leq 1\}} d(Z, Z)_s \right] = E\left[ \int_0^t |Z_s|^{-1}1_{\{0 < |Z_s| \leq 1\}} h_s ds \right] \\
= E\left[ \int_0^t |Z_s|^{-1}1_{\{0 < |Z_s| \leq 1\}} 1_{\{h_s > 0\}} h_s ds \right] \\
\leq E\left[ \int_0^t C 1_{\{0 < |Z_s| \leq 1\}} 1_{\{h_s > 0\}} h_s ds \right] \\
\leq Ct
\]

and

\[
\int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} d(Z, Z)_s \leq \int_0^t |Z_s|^{-1}1_{\{0 < |Z_s| \leq 1\}} d(Z, Z)_s \quad \forall n \geq 1.
\]

By Lebesgue’s dominated convergence theorem, we get

\[
\lim_{n \to \infty} E\left[ \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} d(Z, Z)_s \right] = E\lim_{n \to \infty} \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} d(Z, Z)_s \\
= E\lim_{n \to \infty} \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} h_s ds \\
\leq E\lim_{n \to \infty} \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} C|Z_s| ds \\
\leq E\lim_{n \to \infty} \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} C\frac{1}{n} ds \\
= E\lim_{n \to \infty} \int_0^t 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} C ds \\
= E\int_0^t \lim_{n \to \infty} 1_{\{0 < |Z_s| \leq \frac{1}{n}\}} C ds = 0
\]

2. Since \( \varphi_n \in C(\mathbb{R}) \), we get \( F_n \in C^2(\mathbb{R}) \). Note that

\[
F_n'(x) = \int_0^x \varphi_n(t) dt = \begin{cases} 
(2nx - n^2x)1_{[0, \frac{1}{n}]}(x) + \frac{1}{2}x_+ - \frac{1}{2}x_-, & \text{if } x \geq 0 \\
(2nx + n^2x)1_{[-\frac{1}{n}, 0]}(x) - \frac{1}{2}x_+ - \frac{1}{2}x_-, & \text{if } x \leq 0
\end{cases}
\]

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and

\[ F_n(x) = \int_0^x F'_n(t)dt = \begin{cases} (x - \frac{1}{n})1_{[\frac{1}{n}, \infty)}(x) + (n(x \wedge \frac{1}{n})^2 - \frac{n^2}{4}(x \wedge \frac{1}{n})^3), & \text{if } x \geq 0 \\ -(x + \frac{1}{n})1_{(-\infty, -\frac{1}{n})}(x) + (n(x \vee \frac{1}{n})^2 + \frac{n^2}{4}(x \vee \frac{1}{n})^3), & \text{if } x \leq 0. \end{cases} \]

Then \( F'_n(x) \to \text{sgn}(x) \) and \( F_n(x) \to |x| \) as \( n \to \infty \). Indeed, if \( x > 0 \) and \( y < 0 \), choose large \( N \geq 1 \) such that \( \frac{1}{N} \leq x \) and \( -\frac{1}{N} \geq y \), we have

\[ F_n(x) = x - \frac{1}{n} + (n - \frac{n^2}{3n^3}) = x - \frac{1}{3n} \quad \forall n \geq N, \]

\[ F_n(y) = -y - \frac{1}{n} + (n - \frac{n^2}{3n^3}) = -y - \frac{1}{3n} \quad \forall n \geq N \]

and so \( F_n(x) \to x \) and \( F_n(y) \to -y \) as \( n \to \infty \).

Let \( X \) and \( X' \) be two solutions of \( E(\sigma, b) \) on the same filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with the same Brownian motion \((B_t)_{t \geq 0}\). Then

\[ X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \]

and

\[ X'_t = X'_0 + \int_0^t \sigma(X'_s)dB_s + \int_0^t b(X'_s)ds \]

for all \( t \geq 0 \). Set \( Z_t := X_t - X'_t \) and \( h_t := (\sigma(X_t) - \sigma(X'_t))^2 \) for all \( t \geq 0 \). Then

\[ \langle Z, Z \rangle_t = \int_0^t h_sds \]

and

\[ 0 \leq h_t \leq K^2|X_t - X'_t| = K^2|Z_t| \]

for all \( t \geq 0 \). By problem 1., we get

\[ \lim_{n \to \infty} \mathbb{E}\left[ \int_0^t \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right] \]

\[ = \lim_{n \to \infty} \mathbb{E}\left[ \int_0^t \varphi_n(X_s - X'_s)1_{0<|X_s - X'_s|\leq \frac{1}{n}}(s)d(X - X', X - X')_s \right] \]

\[ \leq \lim_{n \to \infty} \mathbb{E}\left[ \int_0^t (2n + 2n^2|Z_s|)1_{0<|Z_s|\leq \frac{1}{n}}(s)d(Z, Z)_s \right] \]

\[ \leq \lim_{n \to \infty} 2n\mathbb{E}\left[ \int_0^t 1_{0<|Z_s|\leq \frac{1}{n}}(s)d(Z, Z)_s \right] + \lim_{n \to \infty} \mathbb{E}\left[ \int_0^t 2n^2 \times 1_{0<|Z_s|\leq \frac{1}{n}}(s)d(Z, Z)_s \right] = 0. \]

3. Fix \( M > 0 \). Define \( T_M := \inf\{t \geq 0 \mid |X_t| + |X'_t| \geq M\} \). For the sake of simplicity, we denote \( T \) as \( T_M \). Then \((X_{t\wedge T} - X'_{t\wedge T})_{t \geq 0}\) is a bounded martingale. Fix \( t \geq 0 \). By Itô’s formula, we get

\[ F_n(X_{t\wedge T} - X'_{t\wedge T}) = F_n(X_0 - X'_0) + \int_0^{t\wedge T} F'_n(X_s - X'_s)(\sigma(X_s) - \sigma(X'_s))dB_s := Y_t \]

\[ + \int_0^{t\wedge T} F'_n(X_s - X'_s)b(X_s) - b(X'_s))ds \]

\[ + \frac{1}{2} \int_0^{t\wedge T} \varphi_n(X_s - X'_s)d(X - X', X - X')_s. \]
Since

\[
E[(Y, Y)_t] = E\left[ \int_0^t |F_n'(X_s - X'_s)|^2 |\sigma(X_s) - \sigma(X'_s)|^2 ds \right]
\]

\[
\leq E\left[ \int_0^t 1 \times K^2 |X_s - X'_s| ds \right] \quad (|F_n'(x)| \leq 1)
\]

\[
\leq K^2 2Mt < \infty \quad \forall t \geq 0,
\]

we see that \( Y \) is a martingale and so

\[
E[F_n(X_{t\wedge T} - X'_{t\wedge T})] = E[F_n(X_0 - X'_0)]
\]

\[
+ E\left[ \int_0^{t\wedge T} F_n'(X_s - X'_s)(b(X_s) - b(X'_s))ds \right]
\]

\[
+ E\left[ \frac{1}{2} \int_0^{t\wedge T} \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right].
\]

Note that \(|X_{t\wedge T} \vee X'_{t\wedge T}| \leq M\), \(\sup_{|x| \leq M} |b(x)| < \infty\), and \(F_n(x)\) are uniformly bounded over \([-2M, 2M]\). By Lebesgue’s dominated theorem, we get

\[
E[|X_{t\wedge T} - X'_{t\wedge T}|] = \lim_{n \to \infty} E[F_n(X_{t\wedge T} - X'_{t\wedge T})]
\]

\[
= \lim_{n \to \infty} E[F_n(X_0 - X'_0)]
\]

\[
+ \lim_{n \to \infty} E\left[ \int_0^{t\wedge T} F_n'(X_s - X'_s)(b(X_s) - b(X'_s))ds \right]
\]

\[
+ \lim_{n \to \infty} E\left[ \frac{1}{2} \int_0^{t\wedge T} \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right]
\]

\[
= E[|X_0 - X'_0|] + E\left[ \int_0^{t\wedge T} \text{sgn}(X_s - X'_s)(b(X_s) - b(X'_s))ds \right]
\]

\[
+ \lim_{n \to \infty} E\left[ \frac{1}{2} \int_0^{t\wedge T} \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right].
\]

By problem 2., we get

\[
\lim_{n \to \infty} E\left[ \frac{1}{2} \int_0^{t\wedge T} \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right] = \lim_{n \to \infty} E\left[ \frac{1}{2} \int_0^{t} \varphi_n(X_s - X'_s)d(X - X', X - X')_s \right] = 0
\]

and so

\[
E[|X_{t\wedge T} - X'_{t\wedge T}|] = E[|X_0 - X'_0|] + E\left[ \int_0^{t\wedge T} \text{sgn}(X_s - X'_s)(b(X_s) - b(X'_s))ds \right].
\]

4. Fix \( t_0 \geq 0, t_0 \leq L, \) and \( M > 0 \). Define \( g : [0, L] \to \mathbb{R}_+ \) by

\[
g(t) := E[|X_{t\wedge TM} - X'_{t\wedge TM}|].
\]

Then \( 0 \leq g(t) \leq 2M \). By problem 3., we get

\[
g(t) \leq E\left[ \int_0^{t\wedge TM} \text{sgn}(X_s - X'_s)(b(X_s) - b(X'_s))ds \right]
\]

\[
\leq E\left[ \int_0^{t} |\text{sgn}(X_{s\wedge TM} - X'_{s\wedge TM})(b(X_{s\wedge TM}) - b(X'_{s\wedge TM}))|ds \right]
\]

\[
\leq E\left[ \int_0^{t} K^2 |X_{s\wedge TM} - X'_{s\wedge TM}|ds \right] = K^2 \int_0^{t} g(s)ds.
\]
By Gronwall’s lemma, we get $g = 0$ and so

$$E[|X_{t0} \wedge T_M - X'_{t0} \wedge T_M|] = 0.$$ 

By letting $M \to \infty$, we get $E[|X_{t0} - X'_{t0}|] = 0$ and, hence, $X_{t0} = X'_{t0}$ (a.s.). Since $X$ and $X'$ have continuous sample path, we get

$$X_t = X'_t \quad \forall t \geq 0 \quad P-(a.s.).$$
Chapter 9

Local Times

9.1 Exercise 9.16

Let \( f : \mathbb{R} \to \mathbb{R} \) be a monotone increasing function, and assume that \( f \) is a difference of convex functions. Let \( X \) be a semimartingale and consider the semimartingale \( Y_t = f(X_t) \). Prove that, for every \( a \in \mathbb{R} \),

\[
L^a_t(Y) = f'_+(a)L^a_t(X) \quad \text{and} \quad L^a_t(Y) = f'_-(a)L^a_t(X).
\]

In particular, if \( X \) is a Brownian motion, the local times of \( f(X) \) are continuous in the space variable if and only if \( f \) is continuously differentiable.

Remark.

Note that \((L^a(X), a \in \mathbb{R})\) is the càdlàg modification of local time of \( X \). The formula

\[
L^a_t(Y) = f'_+(a)L^a_t(X)
\]

doesn’t hold for all increasing function \( f = \varphi_1 - \varphi_2 \), where \( \varphi_i \) is a convex function on \( \mathbb{R} \). For example, if \( \varphi_1(x) = 2e^x \) and \( \varphi_2(x) = e^x \), and if \( X \) is a continuous semimartingale such that \( \mathbb{P}(L^a(X) \neq 0) > 0 \) for some \( a < 0 \) and \( t > 0 \), then \( f(x) = e^x \) and so

\[
L^a_t(Y) = L^a_t(f(X)) = 0 \neq e^aL^a_t(X) = f'(a)L^a_t(X)
\]
on \( \{L^a_t(X) \neq 0\} \).

To avoid this problem, we restatement Exercise 9.16 as following: Let \( f : \mathbb{R} \to \mathbb{R} \) be a strictly increasing function such that \( f = \varphi_1 - \varphi_2 \), where \( \varphi_i \) is a convex function on \( \mathbb{R} \). Let \( X \) be a semimartingale and consider the semimartingale \( Y_t = f(X_t) \). Prove that, a.s.

\[
L^{f(a)}_t(Y) = f'_+(a)L^a_t(X) \quad \text{and} \quad L^{f(a)}_t(Y) = f'_-(a)L^a_t(X) \quad \forall a \in \mathbb{R}, \ t \geq 0
\]

In particular, if \( X \) is a Brownian motion and \( (u, v) \subseteq R(f) := \{a \in \mathbb{R} \mid f(a)\} \), we have, a.s. \( a \in (u, v) \mapsto L^a(Y) \) is continuous if and only if \( a \in (u, v) \mapsto f(a) \) is continuously differentiable.

Proof.

1. Since \( f = \varphi_1 - \varphi_2 \), we see that \( f \) is continuous and \( f'_+ \) is right continuous. We show that, a.s.

\[
L^{f(a)}_t(Y) = f'_+(a)L^a_t(X) \quad \forall t > 0, a \in \mathbb{R}.
\]

To show this, it suffices to show that \( \mathbb{P}(L^{f(a)}_t(Y) = f'_+(a)L^a_t(X)) = 1 \) for all \( t \geq 0 \) and \( a \in \mathbb{R} \). Indeed, since \( a \in \mathbb{R} \mapsto f'_+(a)L^a_t(X) \) is right continuous for \( t \geq 0 \) and

\[
E_a := \{L^{f(a)}_t(Y) = f'_+(a)L^a_t(X) \quad \forall t \geq 0\} = \bigcap_{s \in \mathbb{Q}^+} E_{a,s} \quad \forall a \in \mathbb{R},
\]

where

\[
E_{a,s} := \{L^{f(a)}_s(Y) = f'_+(a)L^a_s(X)\} \quad \forall a \in \mathbb{R}, s > 0,
\]

we see that

\[
\mathbb{P}(L^{f(a)}_t(Y) = f'_+(a)L^a_t(X) \quad \forall a \in \mathbb{R}, t \geq 0) = \mathbb{P}(\bigcap_{q \in \mathbb{Q}} E_q) = 1.
\]

Fix \( a \in \mathbb{R} \) and \( t > 0 \). Now, we show that \( \mathbb{P}(L^{f(a)}_t(Y) = f'_+(a)L^a_t(X)) = 1 \). By generalized Itô formula, we see that

\[
d(Y,Y)_s = f'_-(X_s)^2d(X,X)_s.
\]
By Proposition 9.9 and Corollary 9.7, we have, a.s.

\[ L_t^{f(a)}(Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\{f(a) \leq f(X,s) \leq f(a) + \epsilon\}} f'_-(X,s)^2 d<X, X>_s \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{f(a) \leq f(b) \leq f(a) + \epsilon\}} f'_-(b)^2 L_t^b(X) db \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{f(a) \leq f(b) \leq f(a) + \epsilon\}} f'_-(b)^2 L_t^b(X) db. \]

We show that, a.s.

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{f(a) \leq f(b) \leq f(a) + \epsilon\}} f'_+(b)^2 L_t^b(X) db = f'_+(a)L_t^a(X). \]

Fix \( w \). Given \( \eta > 0 \). Choose \( h > 0 \) such that

\[ |f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta \]

whenever \( a \leq b < a + h \). Note that \( f \) is a continuous strictly increasing function. For \( \epsilon > 0 \), define

\[ a_\epsilon := \inf\{b \in \mathbb{R} \mid f(b) = f(a) + \epsilon\}. \]

Choose \( j > 0 \) such that \( a < a_\epsilon < a + h \) for every \( 0 < \epsilon < j \). Let \( 0 < \epsilon < j \). Then \(-\infty < a < a_\epsilon < \infty, f(a_\epsilon) = f(a) + \epsilon,\]

\[ |f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta \quad \forall b \in [a,a_\epsilon], \]

and so

\[ \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{f(a) \leq f(b) \leq f(a) + \epsilon\}} f'_+(b) db = \frac{1}{\epsilon} \int_{a}^{a_\epsilon} f'_+(b) db = \frac{f(a_\epsilon) - f(a)}{\epsilon} = 1. \]

Thus,

\[ \left| \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{a \leq f(b) \leq a + \epsilon\}} f'_+(b)^2 L_t^b(X) db - f'_+(a)L_t^a(X) \right| \]

\[ = \frac{1}{\epsilon} \int_{a}^{a_\epsilon} f'_+(b)^2 L_t^b(X) db \]

\[ \leq \frac{1}{\epsilon} \int_{a}^{a_\epsilon} f'_+(b)^2 \left| f'_+(b) L_t^b(X) - f'_+(a)L_t^a(X) \right| db \]

\[ < \eta \frac{1}{\epsilon} \int_{a}^{a_\epsilon} f'_+(b) db = \frac{1}{\epsilon} \left(f(a_\epsilon) - f(a)\right) = \eta \frac{1}{\epsilon} = \eta. \]

Therefore, we have, a.s.

\[ L_t^{f(a)}(Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \mathbf{1}_{\{f(a) \leq f(b) \leq f(a) + \epsilon\}} f'_+(b)^2 L_t^b(X) db = f'_+(a)L_t^a(X). \]

2. We show that, a.s.

\[ L_t^{f(a)-}(Y) = f'_-(a)L_t^{a-}(X) \quad \forall t > 0, a \in \mathbb{R}. \]

To show this, it suffices to show that \( \lim_{t \to a} f'_+(b) = f'_-(a) \) for every \( a \in \mathbb{R} \). Indeed, if \( w \in E \), where \( E = \{ L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall a \in \mathbb{R}, t \geq 0 \} \), then

\[ L_t^{f(a)-}(Y) = \lim_{b \to a} L_t^{f(b)}(Y) = \lim_{b \to a} f'_+(b)L_t^b(X) = f'_-(a)L_t^{a-}(X) \quad \forall a \in \mathbb{R}, t \geq 0. \]
Fix $a \in \mathbb{R}$. Now, we show that $\lim_{t \to a} f'_+(b) = f'_+(a)$. Since $f = \varphi_1 - \varphi_2$, it suffices to show that $\lim_{t \to a} \varphi'_{i,+}(b) = \varphi'_{i,-}(a)$ for $i = 1, 2$. We denote $\varphi_i$ as $\varphi$. It’s clear that

$$\varphi'_+(b) \leq \varphi'_-(a) \quad \forall b < a.$$ 

Given $\eta > 0$. There exists $c < a$ such that

$$\varphi'_-(a) - \eta \leq \varphi(a) - \varphi(c) \over a - c.$$ 

By continuity, there exists $c < d < a$ such that

$$\varphi(a) - \varphi(c) \over a - c - \eta \leq \varphi(d) - \varphi(c) \over d - c$$

and so

$$\varphi'_-(a) - 2\eta \leq \varphi(d) - \varphi(c) \over d - c \leq \varphi'_+(b) \quad \forall d < b < a.$$ 

Thus, we get

$$\varphi'_-(a) - 2\eta \leq \varphi'_+(b) \leq \varphi'_-(a) \quad \forall d < b < a$$

and, hence, $\lim_{t \to a} f'_+(b) = f'_-(a)$.

3. Assume that $X$ is a Brownian motion and $(u, v) \subseteq R(f)$. Then $a \mapsto L^a_{\cdot}(X)$ is continuous and so, a.s.

$$L^a_{\cdot}(X) = L^a_{\cdot}(X) \quad \forall a \in \mathbb{R}, t \geq 0.$$ 

Note that, a.s.

$$a \in (u, v) \mapsto L^a_{\cdot}(Y) \text{ is continuous if and only if } L^a_{\cdot}(Y) = L^a_{\cdot}(Y) \quad \forall a \in (u, v), t \geq 0.$$ 

Thus, if $f$ is continuously differentiable, then we have, a.s.

$$L^a_{\cdot}(Y) = f'(f^{-1}(a))L^a_{\cdot}(f^{-1}(X)) = f'(f^{-1}(a))L^a_{\cdot}(f^{-1}(X)) = L^a_{\cdot}(Y) \quad \forall a \in (u, v), t \geq 0.$$ 

Now, we suppose $a \in (u, v) \mapsto L^a_{\cdot}(Y)$ is continuous. Note that $-\infty = \liminf_{t \to \infty} X_t$ and $\limsup_{t \to \infty} X_t = \infty$. By Theorem 9.12, we get, a.s.

$$\forall a \in \mathbb{R} \quad \exists t_a > 0 \quad \forall t > t_a \quad L^a_{\cdot}(X) > 0$$

($t_a$ also depend on $w$). Fix $a \in (u, v)$. Choose $w$ and $t > 0$ such that $L^a_{\cdot}(X) > 0$, $L^a_{t}(Y) = f'_+(a)L^a_{t}(X)$ and, $L^a_{t}(Y) = f'_+(a)L^a_{t}(X)$ for all $a \in \mathbb{R}$. Thus,

$$f'_+(a)L^a_{t}(X) = L^a_{t}(Y) = f'_+(a)L^a_{t}(X) = f'_+(a)L^a_{t}(X)$$

and so $f'_+(a) = f'_+(a)$ Therefore $f$ is differentiable at $a$. Moreover, since $(a, s) \mapsto L^a_{\cdot}(X)$ is continuous, there exists $\delta > 0$ such that

$$L^a_{\cdot}(X) > 0 \quad \forall (a, s) \in (\alpha - \delta, \alpha + \delta) \times (t - \delta, t + \delta)$$

and so $a \in (\alpha - \delta, \alpha + \delta) \mapsto f'(a) = \frac{L^a_{\cdot}(Y)}{L^a_{t}(X)}$ is continuous.
9.2 Exercise 9.17

Let $M$ be a continuous local martingale such that $\langle M, M, \rangle = \infty$ (a.s.) and let $B$ be the Brownian motion associated with $M$ via the Dambis–Dubins–Schwarz theorem (Theorem 5.13). Prove that, a.s. for every $a \geq 0$ and $t \geq 0$,

$$L^a_t(M) = L^a_{\langle M, M, \rangle_t}(B).$$

Proof.

Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of continuous semimartingale $X$. Set

$$E_{a,t} := \{L^a_t(M) = L^a_{\langle M, M, \rangle_t}(B)\} \quad \forall t > 0, a \in \mathbb{R}.$$  

Then it suffices to show that $P(E_{a,t}) = 1$ for all $t > 0$ and $a \in \mathbb{R}$. Indeed, since

$$E_a := \{L^a_t(M) = L^a_{\langle M, M, \rangle_t}(B) \quad \forall t \geq 0\} \quad \forall a \in \mathbb{R},$$

we see that $P(E) = 1$. Fix $t > 0$ and $a \in \mathbb{R}$. Now, we show that $P(E_{a,t}) = 1$. Note that

$$M_s = B_{\langle M, M, \rangle_s} \quad \forall s \geq 0 \quad \text{(a.s.)}.$$  

By Tanaka's formula, we get, a.s.

$$\int_0^t sgn(M_s - a) \, dM_s + L^a_t(M)$$

and

$$\int_0^t sgn(B_{\langle M, M, \rangle_s} - a) \, dB_s + L^a_{\langle M, M, \rangle_t}(B).$$

By Proposition 5.9, there exists $\{n_k\}$ such that, a.s.

$$\int_0^t sgn(M_s - a) \, dM_s = \lim_{k \to \infty} \sum_{i=0}^{n_k-1} sgn(M_{\frac{\alpha}{n_k}} - a)(M_{\frac{i+1}{n_k}} - M_{\frac{i}{n_k}})$$

$$= \lim_{k \to \infty} \sum_{i=0}^{n_k-1} sgn(B_{\langle M, M, \rangle_{\frac{\alpha}{n_k}}} - a)(B_{\langle M, M, \rangle_{\frac{i+1}{n_k}}} - B_{\langle M, M, \rangle_{\frac{i}{n_k}}}).$$

Since $s \in \mathbb{R}_+ \mapsto \langle M, M, \rangle_s$ is increasing continuous function, we have, a.s.

$$\lim_{k \to \infty} \sum_{i=0}^{n_k-1} sgn(B_{\langle M, M, \rangle_{\frac{\alpha}{n_k}}} - a)(B_{\langle M, M, \rangle_{\frac{i+1}{n_k}}} - B_{\langle M, M, \rangle_{\frac{i}{n_k}}}) = \int_0^{\langle M, M, \rangle_t} sgn(B_s - a) \, dB_s$$

and so

$$\int_0^t sgn(M_s - a) \, dM_s = \int_0^{\langle M, M, \rangle_t} sgn(B_s - a) \, dB_s.$$  

Thus, we have, a.s.

$$L^a_t(M) = L^a_{\langle M, M, \rangle_t}(B).$$
9.3 Exercise 9.18

Let $X$ be a continuous semimartingale, and assume that $X$ can be written in the form

$$X_t = X_0 + \int_0^t \sigma(w, s)dB_s + \int_0^t b(w, s)ds,$$

where $B$ is a Brownian motion and $\sigma$ and $b$ are progressive and locally bounded. Assume that $\sigma(w, s) \neq 0$ for Lebesgue a.e. $s \geq 0$ a.s. Show that the local times $L^a_t(X)$ are jointly continuous in the pair $(a, t)$.

Proof.
By the proof of theorem 9.4, it suffices to show that

$$\int_0^t 1_{\{X_s = a\}} b(w, s)ds = 0 \quad \forall t \geq 0, a \in \mathbb{R} \quad (a.s.)$$

and so we show that $1_{\{X_s = a\}} = 0$ for almost every $s \geq 0$ and for every $a \in \mathbb{R}$ (a.s.). By density of occupation time formula (Corollary 9.7), we have

$$\int_0^t \varphi(X_s)\sigma(w, s)^2ds = \int_\mathbb{R} \varphi(a)L^a_t(X)da$$

for all nonnegative measurable function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ and $t \geq 0$ (a.s.) and so

$$\int_0^t 1_{\{X_s = a\}} \sigma(w, s)^2ds = 0 \quad \forall t \geq 0, a \in \mathbb{R} \quad (a.s.).$$

Since $\sigma(w, s) \neq 0$ for almost every $s \geq 0$ (a.s.), we get $1_{\{X_s = a\}} = 0$ for almost every $s \geq 0$ and for every $a \in \mathbb{R}$ (a.s.).

9.4 Exercise 9.19

Let $X$ be a continuous semimartingale. Show that the property

$$supp(d_sL^a_s(X)) \subseteq \{s \geq 0 \mid X_s = a\}$$

holds simultaneously for all $a \in \mathbb{R}$, outside a single set of probability zero.

Proof.
Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of $X$. Set

$$E_a := \{w \in \Omega \mid supp(d_sL^a_s(X)) \subseteq \{s \geq 0 \mid X_s = a\}\} \quad \forall a \in \mathbb{R}$$

and

$$E = \bigcap_{q \in \mathbb{Q}} E_q.$$

By Proposition 9.3, $P(E) = 1$ and so it suffices to show that

$$supp(d_sL^a_s(X)) \subseteq \{s \geq 0 \mid X_s = a\} \quad \forall a \in \mathbb{R}\text{ on } E.$$

Fix $w \in E$. Assume that there exists $b \in \mathbb{R}$ and $0 \leq s < t$ such that $L^b_s(X)(w) < L^b_t(X)(w)$ and $X_r(w) \neq b$ for all $s \leq r \leq t$. Suppose that $b < \min_{s \leq r \leq t} X_r(w)$. Choose $\epsilon > 0$ such that

$$L^b_s(X)(w) + \epsilon < L^b_t(X)(w) - \epsilon.$$

Since $a \mapsto L^a(X)(w)$ is right continuous, there exists $q \in \mathbb{Q}$ such that $b < q < \min_{s \leq r \leq t} X_r(w)$ and

$$|L^q_s(X)(w) - L^b_s(X)(w)| \vee |L^q_t(X)(w) - L^b_t(X)(w)| < \epsilon.$$

Thus, we get $X_r(w) \neq q$ for all $s \leq r \leq t$ and $L^q_s(X)(w) < L^q_t(X)(w)$ which is a contradiction. By similar argument, we see that $b > \max_{s \leq r \leq t} X_r(w)$ is a contradiction and so

$$supp(d_sL^a_s(X))(w) \subseteq \{s \geq 0 \mid X_s(w) = a\} \quad \forall a \in \mathbb{R}.$$

\qed
9.5 Exercise 9.20

Let $B$ be a Brownian motion started from 0. Show that a.s. there exists an $a \in \mathbb{R}$ such that the inclusion $\text{supp}(d_s L^a_s (X)) \subseteq \{ s \geq 0 \mid X_s = a \}$ is not an equality. (Hint: Consider the maximal value of $B$ over $[0, 1]$.)

Proof.

We denote $B$ as $X$. Note that $(L^a_t (B), a \in \mathbb{R})$ is the càdlàg modification of local time of $B$. First, we show that, a.s.

$$\max_{0 \leq t \leq 1} B_t > B_1.$$ 

Note that

$$P(B_1 \geq \max_{0 \leq t \leq 1} B_t) = P(\min_{0 \leq t \leq 1} B_1 - B_t \geq 0) = P(\min_{0 \leq t \leq 1} B_1 - B_{1-t} \geq 0).$$

Define

$$B'_t = B_1 - B_{1-t} \quad \forall t \in [0, 1].$$

By Exercise 2.31, we see that $(B'_t |_{[0, 1]})$ and $(B_t |_{[0, 1]})$ have the same law and so

$$P(\min_{0 \leq t \leq 1} B_1 - B_{1-t} \geq 0) = P(\min_{0 \leq t \leq 1} B_t \geq 0).$$

By Proposition 2.14, we get

$$P(\max_{0 \leq t \leq 1} B_t > B_1) = 1 - P(B_1 \geq \max_{0 \leq t \leq 1} B_t) = 1 - P(\min_{0 \leq t \leq 1} B_t \geq 0) = 1 - P(\min_{0 \leq t \leq 1} B_t \geq 0).$$

Next, we show that a.s. there exists an $a \in \mathbb{R}$ such that the inclusion

$$\text{supp}(d_s L^a_s (X)) \subseteq \{ s \geq 0 \mid X_s = a \}$$

is not an equality. Fix

$$w \in \{ \max_{0 \leq t \leq 1} B_t > B_1 \} \cap \{ L^a_t (B) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{\{a \leq B_s \leq a + \epsilon\}} ds \quad \forall a \in \mathbb{R}, t > 0\}. $$

Choose $a = \max_{0 \leq t \leq 1} B_s$. Since $\max_{0 \leq t \leq 1} B_t > B_1$, there exists $t \in (0, 1)$ such that $B_t = a$. Let $b > a$. Then

$$L^b_t (B) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{\{b \leq B_s \leq b + \epsilon\}} ds = 0.$$ 

By right continuity, we get

$$L^b_t (B) = \lim_{b \to a} L^b_t (B) = 0$$

and so

$$t \in \{ s \geq 0 \mid B_s = a\} \cap (\text{supp}(d_s L^a_s (B)))^c. $$

\[\square\]

9.6 Exercise 9.21

Let $B$ be a Brownian motion started from 0. Note that

$$\int_0^\infty 1_{\{B_s > 0\}} ds = \infty$$

a.s. and set, for every $t \geq 0$,

$$A_t = \int_0^t 1_{\{B_s > 0\}} ds, \quad \sigma_t = \inf\{s \geq 0 \mid A_s > t\}. $$

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1. Verify that the process 
\[ \gamma_t = \int_0^t 1_{\{B_s > 0\}} dB_s \]

is a Brownian motion in an appropriate filtration.

2. Show that the process \( \Lambda_t = L^{\sigma_0}_t(B) \) has nondecreasing and continuous sample paths, and that the support of the measure \( d_s \Lambda_s \) is contained in \( \{ s \geq 0 \mid B_s = 0 \} \).

3. Show that the process \( (B_{\sigma_t})_{t \geq 0} \) has the same distribution as \( (|B_t|)_{t \geq 0} \).

**Proof.**

1. Since \( \limsup_{t \to \infty} B_s = \infty \), we see that \( \int_0^\infty 1_{\{B_s > 0\}} ds = \infty \) (a.s.) and so
\[ \sigma_t < \infty \quad \forall t \geq 0 \quad (a.s.) \]
Note that \( \gamma_t \) is \( \mathcal{F}_{\sigma_t} \)-measurable for every \( t \geq 0 \) and \( (\sigma_t)_{t \geq 0} \) is nondecreasing. It’s clear that \( t \to \sigma_t \) is right continuous and so \( (\gamma_t)_{t \geq 0} \) has a right continuous sample path. Observe that
\[ B_s \leq 0 \quad \forall s \in (\sigma_{t-}, \sigma_t), \quad \forall t > 0 \quad (a.s.) \]
Then
\[ \lim_{t \uparrow u} \gamma_t = \lim_{t \uparrow u} \int_0^{\sigma_t} 1_{\{B_s > 0\}} dB_s = \int_0^{\sigma_u} 1_{\{B_s > 0\}} dB_s = \int_0^{\sigma_u} 1_{\{B_s > 0\}} dB_s = \gamma_u \quad \forall u > 0 \quad (a.s.) \]
and so \( (\gamma_t)_{t \geq 0} \) has a continuous sample path.

Now, we show that \( (\gamma_t)_{t \geq 0} \) is a \( (\mathcal{F}_{\sigma_t})_{t \geq 0} \)-martingale. Fix \( s_1 < s_2 \). Since
\[ E[\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s, \int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s] \leq E[\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} ds] = E[A_{\sigma_{s_2}}] = s_2, \]
we get \( (\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s)_{t \geq 0} \) is a \( L^2 \)-bounded \( (\mathcal{F}_t)_{t \geq 0} \)-martingale and so \( (\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s)_{t \geq 0} \) is an uniformly integrable \( (\mathcal{F}_t)_{t \geq 0} \)-martingale. By optional stopping theorem, we get
\[ E[\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s \mid \mathcal{F}_{\sigma_{s_1}}] = \int_0^{\sigma_{s_1}} 1_{\{B_s > 0\}} dB_s \]
and so \( (\int_0^{\sigma_{s_2}} 1_{\{B_s > 0\}} dB_s)_{t \geq 0} \) is a \( (\mathcal{F}_t)_{t \geq 0} \)-martingale. Moreover, since
\[ \langle \gamma, \gamma \rangle_{\infty} = \int_0^\infty 1_{\{B_s > 0\}} ds = \infty \quad \text{and} \quad \langle \gamma, \gamma \rangle_t = t \quad \forall t \geq 0, \]
we see that \( (\gamma_t)_{t \geq 0} \) is a \( (\mathcal{F}_{\sigma_t})_{t \geq 0} \)-Brownian motion.

2. It’s clear that \( (\Lambda_t)_{t \geq 0} = (L^{\sigma_0}_t(B))_{t \geq 0} \) has nondecreasing and right continuous sample paths. Note that
\[ B^+_{\sigma_t} = \int_0^{\sigma_t} 1_{\{B_s > 0\}} dB_s + \frac{1}{2} L^{\sigma_0}_t(B) = \gamma_t + \frac{1}{2} L^{\sigma_0}_t(B) \quad \forall t \geq 0 \quad (a.s.). \]
Recall that
\[ B_s \leq 0 \quad \forall s \in (\sigma_{t-}, \sigma_t), \quad \forall t > 0 \quad (a.s.). \]
Observe that if \( \sigma_{t-} < \sigma_t \), then \( \lim_{u \uparrow t} B^+_{\sigma_t} = 0 = B^+_{\sigma_{t-}} \) and so \( (L^{\sigma_0}_t(B))_{t \geq 0} \) has a continuous sample path. Now, we show that \( \text{supp}(d_s \Lambda_s) \subseteq \{ s \geq 0 \mid B_s = 0 \} \). Recall that
\[ \text{supp}(d_s L^{\sigma_0}_s(B)) = \{ s \geq 0 \mid B_s = 0 \} \quad (a.s.). \]
Fix $w \in \{\text{supp}(d_sL^0_s(B)) = \{s \geq 0 \mid B_s = 0\}\}$. Let $t \in \text{supp}(d_s\Lambda_s)$. If $\sigma_{t-} < \sigma_t$, it’s clear that $B_{\sigma_t} = 0$. Now, we assume that $(\sigma_t)_{t \geq 0}$ is continuous at $t$. Let $\alpha < \sigma_t < \beta$. Then there exists $u < t < v$ such that $(\sigma_u, \sigma_v) \subseteq (\alpha, \beta)$,

$$L^0_\alpha(B) \leq L^0_{\sigma_u}(B) < L^0_{\sigma_v}(B) \leq L^0_\beta(B),$$

and so $\sigma_t \in \text{supp}(d_sL^0_s(B)) = \{s \geq 0 \mid B_s = 0\}$.

3. Observe that $B_{\sigma_t} \geq 0 \ \forall t \geq 0 \ (a.s.)$ and so $B_{\sigma_t} = B_{\sigma_t}^+ \ \forall t \geq 0 \ (a.s.)$. Then

$$B_{\sigma_t} = B_{\sigma_t}^+ = \gamma_t + \frac{1}{2}L^0_{\sigma_t}(B) \ \forall t \geq 0 \ (a.s.)$$

By Skorokhod’s Lemma (Appendices), we see that

$$\sup_{s \leq t}(-\gamma_s) = \frac{1}{2}L^0_{\sigma_t}(B) \ \forall t \geq 0 \ (a.s.).$$

By Theorem 9.14, we get

$$B_{\sigma_t} = \sup_{s \leq t}(-\gamma_s) + \gamma_t = \sup_{s \leq t}(-\gamma_s) - (-\gamma_t) \overset{d}{=} |\gamma_t| \overset{d}{=} |B_t| \ \forall t \geq 0$$

and so

$$(B_{\sigma_t})_{t \geq 0} \overset{d}{=} (|B_t|)_{t \geq 0}. \quad \Box$$

9.7 Exercise 9.22

9.8 Exercise 9.23

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a real integrable function ($\int_{\mathbb{R}} |g(x)|dx < \infty$). Let $B$ be a Brownian motion started from 0, and set

$$A_t = \int_0^t g(B_s)ds.$$

1. Justify the fact that the integral defining $A_t$ makes sense, and verify that, for every $c > 0$ and every $u \geq 0$, $A_{c^2u}$ has the same distribution as

$$c^2\int_0^u g(cB_s)ds.$$

2. Prove that

$$\frac{A_t}{\sqrt{t}} \overset{d}{\to} (\int_{\mathbb{R}} g(x)dx)|N| \text{ as } t \to \infty,$$

where $N$ is $\mathcal{N}(0, 1)$.

Proof.

1. Let $t > 0$. Then

$$E[\int_0^t |g(B_s)|ds] = \int_{\mathbb{R}} \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right)ds|g(x)|dx \leq \int_{\mathbb{R}} \int_0^t \frac{1}{\sqrt{2\pi s}} \times 1ds|g(x)|dx$$

$$= \sqrt{\frac{2t}{\pi}} \int_{\mathbb{R}} |g(x)|dx < \infty$$
and so $\int_0^t |g(B_s)|ds < \infty$ (a.s.). Since
\[
\int_0^t |g(B_s)|ds < \infty \quad \forall t \in \mathbb{Q}_+ \quad (a.s.),
\]
we see that
\[
\int_0^t |g(B_s)|ds < \infty \quad \forall t \in \mathbb{R} \quad (a.s.)
\]
and so $(A_t)_{t \geq 0}$ is well-defined. Moreover, by changing of variable, we get
\[
A_{czu} = \int_0^u c^2 g(B_s)ds = c^2 \int_0^u g(cB_{czs})ds = c^2 \int_0^u g(c\frac{1}{c}B_{czs})ds \Rightarrow c^2 \int_0^u g(cB_s)ds.
\]

2. By Density of occupation time formula, we get
\[
\frac{A_u}{\sqrt{u}} = \int_\mathbb{R} g(a) \frac{1}{\sqrt{u}} L_u^a(B) da \quad (a.s.)
\]
for every $u > 0$. First, we show that
\[
\left( \frac{1}{\sqrt{u}} L_u^a(B) \right)_{a \in \mathbb{R}} \overset{d}{=} \left( L_1^{\frac{a}{\sqrt{u}}}(B) \right)_{a \in \mathbb{R}} \quad \forall u > 0.
\]

Fix $u > 0$ and $a \in \mathbb{R}$. Define Brownian motion $\tilde{B}$ by $\tilde{B}_t = \frac{1}{\sqrt{u}} B_{tu}$. By Tanaka’s formula, we get
\[
|\tilde{B}_1 - \frac{a}{\sqrt{u}}| = \left| \frac{a}{\sqrt{u}} \right| + \frac{1}{\sqrt{u}} \int_0^u \text{sgn}(B_s - a)dB_s + \frac{1}{\sqrt{u}} L_u^a(B) \quad (a.s.).
\]

Choose increasing sequence $\{n_k\}_{k \geq 1}$ such that (1),(2) hold (a.s.):
\[
\frac{1}{\sqrt{u}} \int_0^u \text{sgn}(B_s - a)dB_s \overset{(1)}{=} \frac{1}{\sqrt{u}} \lim_{k \to \infty} \sum_{i=0}^{n_k-1} \text{sgn}(B_{\frac{i+1}{n_k}u} - a)(B_{\frac{i+1}{n_k}u} - B_{\frac{i}{n_k}u}) = \lim_{k \to \infty} \sum_{i=0}^{n_k-1} \text{sgn}(\tilde{B}_{\frac{i+1}{n_k}} - \frac{a}{\sqrt{u}})(\tilde{B}_{\frac{i+1}{n_k}} - \tilde{B}_{\frac{i}{n_k}}) = \left( \int_0^1 \text{sgn}(\tilde{B}_s - a)d\tilde{B}_s. \right)
\]
Thus,
\[
|\tilde{B}_1 - \frac{a}{\sqrt{u}}| = \left| \frac{a}{\sqrt{u}} \right| + \int_0^1 \text{sgn}(\tilde{B}_s - a)d\tilde{B}_s + \frac{1}{\sqrt{u}} L_u^a(B) \quad (a.s.)
\]
and so $\frac{1}{\sqrt{u}} L_u^a(B) = L_1^{\frac{a}{\sqrt{u}}}(\tilde{B}) \quad (a.s.)$. By right continuity, we get
\[
\frac{1}{\sqrt{u}} L_u^a(B) = L_1^{\frac{a}{\sqrt{u}}}(\tilde{B}) \quad \forall a \in \mathbb{R} \quad (a.s.)
\]
and so
\[
\left( \frac{1}{\sqrt{u}} L_u^a(B) \right)_{a \in \mathbb{R}} \overset{d}{=} \left( L_1^{\frac{a}{\sqrt{u}}}(B) \right)_{a \in \mathbb{R}} \quad \forall u > 0.
\]

Next, we show that
\[
\frac{A_u}{\sqrt{u}} \overset{d}{\to} \left( \int_\mathbb{R} \text{sgn}(x)dx \right)N \quad \text{as} \quad u \to \infty.
\]
Note that
\[ E[\exp (i\xi \frac{A_u}{\sqrt{u}})] = E[\exp (i\xi \int_\mathbb{R} g(a) \frac{1}{\sqrt{u}} L_\sigma^u(B) da)] = E[\exp (i\xi \int_\mathbb{R} g(a) L_\sigma^u (B) da)]. \]

Since
\[ L_1^u(B) = 0 \quad \forall a \notin [ \min_{0 \leq s \leq 1} B_s, \max_{0 \leq s \leq 1} B_s ] \quad (a.s.), \]
we get
\[ |L_1^u(B)| \leq M \text{ for some } M = M(w) < \infty \quad (a.s.) \]
and so
\[ |L_1^\omega(B)| \leq M(w) < \infty \quad \forall a \in \mathbb{R}, u \in \mathbb{R}_+ \quad (a.s.). \]

By dominated convergence theorem and right continuity, we get
\[ \lim_{u \to \infty} E[\exp (i\xi \frac{A_u}{\sqrt{u}})] = \lim_{u \to \infty} E[\exp (i\xi \int_\mathbb{R} g(a) L_\sigma^u (B) da)] = E[\exp (i\xi \int_\mathbb{R} g(a) L_\sigma^1 (B) da)]. \]

By Theorem 9.14 and Theorem 2.21, we have
\[ L_1^0(B) \overset{d}{=} \sup_{0 \leq s \leq 1} B_s \overset{d}{=} |B_1| \]
and so
\[ \lim_{u \to \infty} E[\exp (i\xi \frac{A_u}{\sqrt{u}})] = E[\exp (i\xi \int_\mathbb{R} g(a) L_1^0 (B) da)] = E[\exp (i\xi \int_\mathbb{R} g(a) da |B_1|)]. \]

\[ \square \]

9.9 Exercise 9.24

Let \( \sigma \) and \( b \) be two locally bounded measurable functions on \( \mathbb{R}_+ \times \mathbb{R} \), and consider the stochastic differential equation
\[ E(\sigma, b) : \quad dX_t = \sigma(t, X_t) dB_t + b(t, X_t) dt. \]

Let \( X \) and \( X' \) be two solutions of \( E(\sigma, b) \) on the same filtered probability space and with the same Brownian motion \( B \).

1. Suppose that \( L_1^0(X - X') = 0 \) for every \( t \geq 0 \). Show that both \( X \lor X' \) and \( X \land X' \) are solutions of \( E(\sigma, b) \).
   (Hint: Write \( X_t \lor X_t' = X_t + (X_t' - X_t)^+ \), and use Tanaka’s formula.)

2. Suppose that \( \sigma(t, x) = 1 \) for all \( t, a \). Show that the assumption in question 1 holds automatically. Suppose in addition that weak uniqueness holds for \( E(\sigma, b) \). Show that, if \( X_0 = X_0' = x \in \mathbb{R} \), the two processes \( X \) and \( X' \) are indistinguishable.

\textit{Proof.}

1. Note that
\[ X_t \lor X_t' = X_t + (X_t' - X_t)^+. \]

By Tanaka’s formula, we get
\[ (X_t' - X_t)^+ = (X_0' - X_0)^+ + \int_0^t 1_{\{X_s' > X_s\}} (\sigma(s, X_s') - \sigma(s, X_s)) dB_s + \int_0^t 1_{\{X_s' > X_s\}} (b(s, X_s') - b(s, X_s)) ds \]

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for all \( t \geq 0 \) (a.s.). Since
\[
\sigma(s, (X'_t \lor X_s)) = 1_{\{X'_t > X_s\}} \sigma(s, X'_s) + 1_{\{X_s \geq X'_t\}} \sigma(s, X_s)
\]
and
\[
b(s, (X'_t \lor X_s)) = 1_{\{X'_t > X_s\}} b(s, X'_s) + 1_{\{X_s \geq X'_t\}} b(s, X_s),
\]
we get
\[
(X'_t \lor X_t) = X_t + (X'_t - X_t)^+
\]
\[
= X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds
+ (X_0 - X_0)^+ + \int_0^t 1_{\{X'_t > X_s\}} (\sigma(s, X'_s) - \sigma(s, X_s)) dB_s + \int_0^t 1_{\{X'_t > X_s\}} (b(s, X'_s) - b(s, X_s)) ds
\]
\[
= (X'_0 \lor X_0) + \int_0^t \sigma(s, (X'_s \lor X_s)) dB_s + \int_0^t b(s, (X'_s \lor X_s)) ds
\]
for all \( t \geq 0 \) (a.s.) and so \( X \lor X' \) is a solution of \( E(\sigma, b) \). Note that
\[
(X_t \land X'_t) = X_t - (X_t - X'_t)^+.
\]
By similar argument, we see that \( X \land X' \) is a solution of \( E(\sigma, b) \).

2. Suppose \( \sigma(t, x) = 1 \) for all \( t, x \). Then
\[
X_t - X'_t = X_0 - X'_0 + \int_0^t (b(s, X_s) - b(s, X'_s)) ds
\]
for all \( t \geq 0 \) (a.s.) and so \( L^0_0(X - X') = 0 \) for all \( t \geq 0 \) (a.s.). Suppose in addition that weak uniqueness holds for \( E(\sigma, b) \) and \( X_0 = X'_0 = x \in \mathbb{R} \). By question 1, \( X \lor X' \) and \( X \land X' \) are solutions of \( E(\sigma, b) \) and so \( X \lor X' \overset{d}{=} X \land X' \). It’s clear that
\[
X_t \lor X'_t = X_t \land X'_t \quad \text{(a.s.)}
\]
for all \( t \geq 0 \). Indeed, if \( P(X_t \lor X'_t > X_t \land X'_t) > 0 \), then \( E[X_t \land X'_t] < E[X_t \lor X'_t] \) which contradict to \( X_t \lor X'_t \overset{d}{=} X_t \land X'_t \). Thus, we have \( X_p = X'_p \) for all \( p \in \mathbb{Q}_+ \) (a.s.) and so
\[
X_t = \lim_{p \in \mathbb{Q}_+ \to t} X_p = \lim_{p \in \mathbb{Q}_+ \to t} X'_p = X'_t
\]
for all \( t \geq 0 \) (a.s.). Therefore \( X \) and \( X' \) are indistinguishable.

\[\square\]

### 9.10 Exercise 9.25 (Another look at the Yamada–Watanabe criterion)

Let \( \rho \) be a nondecreasing function from \([0, \infty)\) into \([0, \infty)\) such that, for every \( \epsilon > 0 \),
\[
\int_0^\epsilon \frac{du}{\rho(u)} = \infty.
\]
Consider then the one-dimensional stochastic differential equation
\[
E(\sigma, b) : \quad dX_t = \sigma(X_t) dB_t + b(X_t) dt
\]
where one assumes that the functions \( \sigma \) and \( b \) satisfy the conditions
\[
(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|), \quad |b(x) - b(y)| \leq K|x - y|,
\]
for every \( x, y \in \mathbb{R} \), with a constant \( K < \infty \). Our goal is use local times to give a short proof of pathwise uniqueness for \( E(\sigma, b) \) (this is slightly stronger than the result of Exercise 8.14).
1. Let $Y$ be a continuous semimartingale such that, for every $t > 0$,
\[ \int_0^t \frac{d(Y,Y)_s}{\rho(|Y_s|)} < \infty \quad (a.s.). \]

Prove that $L_t^0(Y) = 0$ for every $t \geq 0$ (a.s.).

2. Let $X$ and $X_0$ be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion $B$. By applying question 1. to $Y = X - X'$, prove that $L_t^0(X - X')$ for every $t \geq 0$ (a.s.) and therefore,
\[ |X_t - X'_t| = |X_0 - X'_0| + \int_0^t (\sigma(X_s) - \sigma(X'_s))dB_s + \int_0^t (b(X_s) - b(X'_s))ds. \]

3. Using Gromwall’s lemma, prove that if $X_0 = X'_0$, then $X_t = X'_t$ for every $t \geq 0$ (a.s.).

**Proof.**

1. Since $L_t^a(Y) \overset{a.s.}{\to} L_t^0(Y)$ $\forall t \geq 0$ (a.s.), there exists $C = C(w) > 0$ and $\epsilon = \epsilon(w) > 0$ such that
\[ L_t^a(Y) \geq CL_t^0(Y) \quad \forall 0 < a < \epsilon \quad \forall t \geq 0 \quad (a.s.). \]

By Density of occupation time formula (Corollary 9.7), we have
\[ \infty > \int_0^t \frac{d(Y,Y)_s}{\rho(|Y_s|)} = \int_0^t \frac{1}{\rho(|a|)} L_t^a(Y)da \geq CL_t^0(Y) \int_0^t \frac{1}{\rho(a)} da \quad \forall t \geq 0 \quad (a.s.). \]

Since $\int_0^t \frac{da}{\rho(a)} = \infty$ for all $\epsilon > 0$, we get $L_t^0(Y) = 0$ for all $t \geq 0$ (a.s.).

2. Set $Y = X - X'$. Then
\[ Y_t = X_0 - X'_0 + \int_0^t (\sigma(X_s) - \sigma(X'_s))dB_s + \int_0^t (b(X_s) - b(X'_s))ds \]
and so
\[ d(Y,Y)_t = (\sigma(X_t) - \sigma(X'_t))^2dt. \]

Thus,
\[ \int_0^t \frac{d(Y,Y)_s}{\rho(|Y_s|)} = \int_0^t \frac{(\sigma(X_s) - \sigma(X'_s))^2}{\rho(|Y_s|)}ds \leq \int_0^t \frac{\rho(|X_s - X'_s|)}{\rho(|X_s - X'_s|)}ds = t < \infty \quad \forall t \geq 0 \quad (a.s.). \]

By question 1., we get $L_t^0(X - X') = 0$ for every $t \geq 0$ (a.s.). By Tanaka’s formula, we have
\[ |X_t - X'_t| = |X_0 - X'_0| + \int_0^t (\sigma(X_s) - \sigma(X'_s))sgn(X_s - X'_s)dB_s + \int_0^t (b(X_s) - b(X'_s))sgn(X_s - X'_s)ds \]
for every $t \geq 0$ (a.s.).

3. By continuity, it suffices to show that $X_t = X'_t$ (a.s.) for every $t \geq 0$. Fix $t_0 > 0$ and choose $L > t_0$. Define
\[ T_M = \inf\{s \geq 0 \mid |X_s| \geq M \text{ or } |X'_s| \geq M\} \quad \forall M > 0. \]

Fix $M > 0$. Since
\[ E[\int_0^t (\sigma(X_s) - \sigma(X'_s))sgn(X_s - X'_s)1_{[0,T_M]}dB_s, \int_0^t (\sigma(X_s) - \sigma(X'_s))sgn(X_s - X'_s)1_{[0,T_M]}dB_s] \]
\[ = E[\int_0^t (\sigma(X_s) - \sigma(X'_s))^21_{[0,T_M]}ds] \leq E[\int_0^t \rho(|X_s - X'_s|)1_{[0,T_M]}ds] \leq \rho(2M)t < \infty \quad \forall t > 0, \]

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we see that \((\int_0^t (\sigma(X_s) - \sigma(X'_s)) sgn(X_s - X'_s) 1_{[0,T_M]} dB_s)_{t \geq 0}\) is a martingale. Thus
\[
0 \leq g(t) \equiv E[|X_t - X'_t|1_{[0,T_M]}(t)] \leq 2M
\]
and
\[
g(t) = E[|X_t - X'_t|1_{[0,T_M]}(t)] = E[\int_0^t (b(X_s) - b(X'_s)) sgn(X_s - X'_s) 1_{[0,T_M]} ds] \leq 2K \int_0^t g(s) ds
\]
for every \(t \in [0,L]\). By Gromwall’s lemma, we get \(g(t) = 0\) in \([0,L]\) and so \(E[|X_{t_0 \wedge T_M} - X'_{t_0 \wedge T_M}|] = 0\). By letting \(M \uparrow \infty\), we have \(E[|X_{t_0} - X'_{t_0}|] = 0\) and so \(X_{t_0} = X'_{t_0}\).
Chapter 10

Appendices

10.1 Skorokhod’s Lemma

Let \( y \) be a real-valued continuous function on \([0, \infty)\) such that \( y(0) \geq 0 \). There exists a unique pair \((z, a)\) of functions on \([0, \infty)\) such that

1. \( z(t) = y(t) + a(t) \),
2. \( z(t) \) is nonnegative,
3. \( a(t) \) is increasing, continuous, vanishing at zero and \( supp(da_s) \subseteq \{ s \geq 0 : z(s) = 0 \} \).

Moreover, the function \( a(t) \) is given by

\[
a(t) = \sup_{s \leq t} (-y(s) \vee 0).
\]

Proof.

It’s clear that \((y - a, a)\) satisfies all properties above, where \( a(t) = \sup_{s \leq t} (-y(s) \vee 0) \), and so, it suffices to prove the uniqueness of the pair \((z, a)\). Suppose that \((z, a)\) and \((\overline{z}, \overline{a})\) satisfy all properties above. Then

\[
z(t) - \overline{z}(t) = a(t) - \overline{a}(t) \quad \forall t \geq 0
\]

and so

\[
0 \leq (a(t) - \overline{a}(t))^2 = 2 \int_0^t (z(s) - \overline{z}(s))d(a - \overline{a})(s) \quad \forall t \geq 0.
\]

Since

\[
\int_0^t z_s da(s) = \int_0^t \overline{z}(s)d\overline{a}(s) = 0 \quad \forall t \geq 0,
\]

we see that

\[
2 \int_0^t z(s) - \overline{z}(s)d(a - \overline{a})(s) = -2(\int_0^t z(s)d\overline{a}(s) + \int_0^t \overline{z}da(s)) \leq 0 \quad \forall t \geq 0
\]

and so \( z(t) = \overline{z}(t) \) for every \( t \geq 0 \). 

\[\square\]
References