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Spectral radius of bipartite graphs

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Let k, p, q be positive integers with $k < p < q + 1$. We prove that the maximum spectral radius of a simple bipartite graph obtained from the complete bipartite graph $K_{p,q}$ of bipartition orders p and q by deleting k edges is attained when the deleted edges are all incident on a common vertex which is located in the partite set of order q. Our method is based on new sharp upper bounds on the spectral radius of bipartite graphs in terms of their degree sequences.

1. Introduction

Let G be a simple graph of order n. The *adjacency matrix* $A = (a_{ij})$ of G is a 0-1 square matrix of order n with rows and columns indexed by the vertex set VG of G such that for any $i, j \in VG$, $a_{ij} = 1$ iff i, j are adjacent in G. The *spectral radius* $\rho(G)$ of G is the largest eigenvalue of the adjacency matrix A of G.

Brualdi and Hoffman proposed the problem of finding the maximum spectral radius of a graph with precisely e edges in 1976 [3, p. 438], and ten years later they gave a

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conjecture in [6] that the maximum spectral radius of a graph with e edges is attained by taking a complete graph and adding a new vertex which is adjacent to a corresponding number of vertices in the complete graph. This conjecture was proved by Peter Rowlinson in [16]. See [18,10] also for the proof of partial cases of this conjecture.

The next problem is then to determine graphs with maximum spectral radius in the class of connected graphs with n vertices and e edges. The cases $e \leq n + 5$ when n is sufficiently large are settled by Brualdi and Solheid [7], and the cases $e - n = \binom{r}{2} - 1$ by F.K. Bell [1].

The bipartite graphs analogue of the Brualdi–Hoffman conjecture was settled by A. Bhattacharya, S. Friedland, and U.N. Peled [4] with the following statement: For a connected bipartite graph G, $\rho(G) \leq \sqrt{e}$ with equality iff G is a complete bipartite graph. Moreover, they proposed the problem to determine graphs with maximum spectral radius in the class of bipartite graphs with bipartition orders p and q, and e edges. They then gave Conjecture 1.1 below.

From now on the graphs considered are simple bipartite. Let $\mathcal{K}(p, q, e)$ denote the family of e-edge subgraphs of the complete bipartite graph $K_{p,q}$ with bipartition orders p and q.

Conjecture 1.1. *Let $1 < e < pq$ be integers. An extremal graph that solves*

$$\max_{G \in \mathcal{K}(p,q,e)} \rho(G)$$

is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

Moreover, in [4, Theorem 8.1] Conjecture 1.1 was proved in the case that $e = st - 1$ for some positive integers s, t satisfying $2 \leq s \leq p < t \leq q \leq t + \frac{t-1}{s-1}$. They also indicated that the only extremal graph is obtained from $K_{s,t}$ by deleting an edge.

Conjecture 1.1 does not indicate that the adding vertex goes into which partite set of a complete bipartite graph. For $e \geq pq - \max(p, q)$ (resp. $e \geq pq - \min(p, q)$), let ${}^{e}K_{p,q}$ (resp. $K^{e}_{p,q}$) denote the graph which is obtained from $K_{p,q}$ by deleting $pq - e$ edges incident on a common vertex in the partite set of order no larger than (resp. no less than) that of the other partite set. Then the extremal graph in Conjecture 1.1 is either ${}^{e}K_{s,t}$ or $K^{e}_{s,t}$ for some positive integers $s \leq p$ and $t \leq q$ which meet the constraints of the number of edges. Fig. 1 gives two such graphs.

In 2010 [8], Yi-Fan Chen, Hung-Lin Fu, In-Jae Kim, Eryn Stehr and Brendon Watts determined $\rho(K^{e}_{p,q})$ and gave an affirmative answer to Conjecture 1.1 when $e = pq - 2$. Furthermore, they refined Conjecture 1.1 for the case when the number of edges is at least $pq - \min(p, q) + 1$ to the following conjecture.

Conjecture 1.2. *Suppose $0 < pq - e < \min(p, q)$. Then for $G \in \mathcal{K}(p, q, e)$,*

$$\rho(G) \leq \rho(K^{e}_{p,q}).$$

$K^5_{2,3} = {}^5K_{2,3}$ ${}^5K_{2,4}$

Fig. 1. The graphs $K^5_{2,3}$, ${}^5K_{2,3}$ and ${}^5K_{2,4}$.

The paper is organized as follows. Preliminary contents are in Section 2. Theorem 3.3 in Section 3 presents a series of sharp upper bounds of $\rho(G)$ in terms of the degree sequence of G. Some special cases of Theorem 3.3 are further investigated in Section 4 in which Corollary 4.2 is the most useful in this paper. We prove Conjecture 1.2 as an application of Corollary 4.2 in Section 5. Finally we propose another conjecture which is a general refinement of Conjecture 1.1 in Section 6.

2. Preliminary

Basic results are provided in this section for later use.

Lemma 2.1. *(See [4, Proposition 2.1].) Let G be a simple bipartite graph with e edges. Then*

$$\rho(G) \leq \sqrt{e}$$

with equality iff G is a disjoint union of a complete bipartite graph and isolated vertices.

Let G be a simple bipartite graph with bipartition orders p and q, and degree sequences $d_1 \geq d_2 \geq \cdots \geq d_p$ and $d'_1 \geq d'_2 \geq \cdots \geq d'_q$ respectively. We say that G is *biregular* if $d_1 = d_p$ and $d'_1 = d'_q$.

Lemma 2.2. *(See [2, Lemma 2.1].) Let G be a simple connected bipartite graph. Then*

$$\rho(G) \leq \sqrt{d_1 d'_1}$$

with equality iff G is biregular.

Let M be a real matrix of the following block form

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ \vdots & & \vdots \\ M_{m,1} & \cdots & M_{m,m} \end{pmatrix},$$

where the diagonal blocks $M_{i,i}$ are square. Let b_{ij} denote the average row-sums of $M_{i,j}$, i.e. b_{ij} is the sum of entries in $M_{i,j}$ divided by the number of rows. Then $B = (b_{ij})$ is called a *quotient matrix* of M. If in addition for each pair i, j, $M_{i,j}$ has constant row-sum, then B is called an *equitable quotient matrix* of M. The following lemma is direct from the definition of matrix multiplication [5, Chapter 2].

Lemma 2.3. *Let B be an equitable quotient matrix of M with an eigenvalue θ. Then M also has the eigenvalue θ.*

The following lemma is a part of the Perron–Frobenius Theorem [15, Chapter 2].

Lemma 2.4. *If M is a nonnegative $n \times n$ matrix with largest eigenvalue $\rho(M)$ and row-sums $r_1, r_2, \ldots, r_n$, then*

$$\rho(M) \leq \max_{1 \leq i \leq n} r_i.$$

Moreover, if M is irreducible then the above equality holds if and only if the row-sums of M are all equal.

3. A series of sharp upper bounds of $\rho(G)$

We give a series of sharp upper bounds of $\rho(G)$ in terms of the degree sequence of a bipartite graph G in this section. The following set-up is for the description of extremal graphs of our upper bounds.

Definition 3.1. Let H, H' be two bipartite graphs with given ordered bipartitions $VH = X \cup Y$ and $VH' = X' \cup Y'$, where $VH \cap VH' = \phi$. The *bipartite sum* $H + H'$ of H and H' (with respect to the given ordered bipartitions) is the graph obtained from H and H' by adding an edge between x and y for each pair $(x, y) \in X \times Y' \cup X' \times Y$.

Example 3.2. Let $N_{s,t}$ denote the bipartite graph with bipartition orders s, t and without any edges. Then for $p \leq q$ and e meeting the desired constraints, ${}^eK_{p,q} = K_{p-1,q-pq+e} + N_{1,pq-e}$ and $K^e_{p,q} = K_{p-pq+e,q-1} + N_{pq-e,1}$.

Theorem 3.3. *Let G be a simple bipartite graph with bipartition orders p and q, and corresponding degree sequences $d_1 \geq d_2 \geq \cdots \geq d_p$ and $d'_1 \geq d'_2 \geq \cdots \geq d'_q$. For $1 \leq s \leq p$ and $1 \leq t \leq q$, let $X_{s,t} = d_s d'_t + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t)$ and $Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{j=1}^{t-1}(d'_j - d'_t)$. Then the spectral radius*

$$\rho(G) \leq \phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}}.$$

Furthermore, if G is connected then the above equality holds if and only if there exist nonnegative integers $s' < s$ and $t' < t$, and a biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively such that $G = K_{s',t'} + H$.

Before proving Theorem 3.3, we mention some simple properties of $\phi_{s,t}$.

Lemma 3.4.

(i) $\phi_{1,1} = \sqrt{d_1 d'_1}$.

(ii) *If $d_{s'} = d_s$ then $\phi_{s',t} = \phi_{s,t}$. If $d'_{t'} = d'_t$ then $\phi_{s,t'} = \phi_{s,t}$.*

(iii)

$$\phi_{s,t}^2 \geq \max\left(\sum_{i=1}^{s-1}(d_i - d_s), \sum_{j=1}^{t-1}(d'_j - d'_t)\right)$$

with equality iff $\phi_{s,t}^2 = e$.

(iv) $\phi_{s,t}^4 - X_{s,t}\phi_{s,t}^2 + Y_{s,t} = 0$.

Proof. (i), (ii), (iv) are immediate from the definition of $\phi_{s,t}$. Clearly $d_s d'_t = 0$ if and only if

$$\max\left(\sum_{i=1}^{s-1}(d_i - d_s), \sum_{j=1}^{t-1}(d'_j - d'_t)\right) = e.$$

Hence (iii) follows by using $X_{s,t} \geq \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t)$ with equality iff $d_s d'_t = 0$ to simplify $\phi_{s,t}$. □

We set up notations for the use in the proof of Theorem 3.3. For $1 \leq k \leq s-1$, let

$$x_k = \begin{cases} 1 + \dfrac{d'_t(d_k - d_s)}{\phi_{s,t}^2 - \sum_{i=1}^{s-1}(d_i - d_s)}, & \text{if } \phi_{s,t}^2 > \sum_{i=1}^{s-1}(d_i - d_s); \\ 1, & \text{if } \phi_{s,t}^2 = \sum_{i=1}^{s-1}(d_i - d_s), \end{cases} \tag{3.1}$$

and for $1 \leq \ell \leq t-1$ let

$$x'_\ell = \begin{cases} 1 + \dfrac{d_s(d'_\ell - d'_t)}{\phi_{s,t}^2 - \sum_{j=1}^{t-1}(d'_j - d'_t)}, & \text{if } \phi_{s,t}^2 > \sum_{j=1}^{t-1}(d'_j - d'_t); \\ 1, & \text{if } \phi_{s,t}^2 = \sum_{j=1}^{t-1}(d'_j - d'_t). \end{cases} \tag{3.2}$$

Note that $x_k, x'_\ell \geq 1$ because of Lemma 3.4(iii). The relations between the above parameters are given in the following.

Lemma 3.5.

(i) *Suppose* $\phi_{s,t}^2 > \sum_{a=1}^{s-1}(d_a - d_s)$. *Then*

$$\frac{1}{x_i}\left(d_i d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t) + \sum_{k=1}^{s-1}(x_k - 1)d_i\right) = \phi_{s,t}^2$$

for $1 \le i \le s-1$, *and*

$$d_s d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t) + \sum_{k=1}^{s-1}(x_k - 1)d_s = \phi_{s,t}^2.$$

(ii) *Suppose* $\phi_{s,t}^2 > \sum_{b=1}^{t-1}(d'_b - d'_t)$. *Then*

$$\frac{1}{x'_j}\left(d_s d'_j + \sum_{h=1}^{s-1}(d_h - d_s) + \sum_{\ell=1}^{t-1}(x'_\ell - 1)d'_j\right) = \phi_{s,t}^2$$

for $1 \le j \le t-1$, *and*

$$d_s d'_t + \sum_{h=1}^{s-1}(d_h - d_s) + \sum_{\ell=1}^{t-1}(x'_\ell - 1)d'_t = \phi_{s,t}^2.$$

Proof. Referring to (3.1) and Lemma 3.4(iv),

$$\begin{aligned}
&\frac{1}{x_i}\left(d_i d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t) + \sum_{k=1}^{s-1}(x_k - 1)d_i\right)\\
&= \frac{1}{\phi_{s,t}^2 - \sum_{k=1}^{s-1}(d_k - d_s) + d'_t(d_i - d_s)}\\
&\quad \times \left(\phi_{s,t}^2\left(d_i d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t)\right) - \sum_{h=1}^{t-1}(d'_h - d'_t)\sum_{k=1}^{s-1}(d_k - d_s)\right)\\
&= \phi_{s,t}^2
\end{aligned}$$

for $1 \le i \le s-1$, and

$$\begin{aligned}
&d_s d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t) + \sum_{k=1}^{s-1}(x_k - 1)d_s\\
&= \frac{1}{\phi_{s,t}^2 - \sum_{k=1}^{s-1}(d_k - d_s)}\left(\phi_{s,t}^2\left(d_s d'_t + \sum_{h=1}^{t-1}(d'_h - d'_t)\right) - \sum_{h=1}^{t-1}(d'_h - d'_t)\sum_{k=1}^{s-1}(d_k - d_s)\right)\\
&= \phi_{s,t}^2.
\end{aligned}$$

Hence (i) follows. Similarly, referring to (3.2) and Lemma 3.4(iv) we have (ii). □

Let $U = \{u_i \mid 1 \le i \le p\}$ and $V = \{v_j \mid 1 \le j \le q\}$ be the bipartition of G such that the degree sequences $d_1 \ge d_2 \cdots \ge d_p$ and $d'_1 \ge d'_2 \cdots \ge d'_q$, respectively are according to the list. For $1 \le i, j \le p$, let n_{ij} denote the numbers of common neighbors of u_i and u_j, i.e., $n_{ij} = |G(u_i) \cap G(u_j)|$ where $G(u)$ is the set of neighbors of the vertex u in G. Similarly, for $1 \le i, j \le q$ let $n'_{ij} = |G(v_i) \cap G(v_j)|$. Since G is bipartite, the adjacency matrix A and its square A^2 look like the following in block form:

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} BB^T & 0 \\ 0 & B^T B \end{pmatrix} = \begin{pmatrix} (n_{ij})_{1 \le i,j \le p} & 0 \\ 0 & (n'_{ij})_{1 \le i,j \le q} \end{pmatrix}. \quad (3.3)$$

We have the following properties of n_{ij} and n'_{ij}.

Lemma 3.6.

(i) *For $1 \le i \le p$ and $1 \le j \le q$, $n_{ii} = d_i$ and $n'_{jj} = d'_j$.*
(ii) *For $1 \le i, j \le p$, $n_{ij} \le d_i$ with equality if and only if $G(u_j) \supseteq G(u_i)$.*
(iii) *For $1 \le i, j \le q$, $n'_{ij} \le d'_i$ with equality if and only if $G(v_j) \supseteq G(v_i)$.*
(iv) *For $1 \le i \le p$,*

$$\sum_{k=1}^{p} n_{ik} = \sum_{j:\ u_i v_j \in EG} d'_j \le (d_i - t + 1)d'_t + \sum_{h=1}^{t-1} d'_h.$$

(v) *For $1 \le j \le q$,*

$$\sum_{k=1}^{q} n'_{jk} = \sum_{i:\ u_i v_j \in EG} d_i \le (d'_j - s + 1)d_s + \sum_{h=1}^{s-1} d_h.$$

Proof. (i)–(iii) are immediate from the definition of n_{ij}. Counting the pairs (u_k, v_j) such that $v_j \in G(u_i) \cap G(u_k)$ in two orders (j, k) and (k, j), we have the first equality in (iv). The second inequality of (iv) is clear since d'_j is non-increasing. (v) is similar to (iv). □

Proof of Theorem 3.3. It is well known that $\rho(A)^2 = \rho(A^2)$. In the following we will show that $\rho(A^2) \le \phi_{s,t}^2$. Let

$$U = diag(\underbrace{x_1, x_2, \cdots, x_{s-1}, 1, \cdots, 1}_{p}, \underbrace{x'_1, x'_2, \cdots, x'_{t-1}, 1, \cdots, 1}_{q})$$

be a diagonal matrix of order $p + q$. Let $C = U^{-1} A^2 U$. Then A^2 and C are similar and with the same spectrum. Let $r_1, \cdots, r_p, r'_1, \cdots, r'_q$ be the row-sums of C. Referring to (3.3), we have

$$r_i = \sum_{k=1}^{s-1} \frac{x_k}{x_i} n_{ik} + \sum_{k=s}^{p} \frac{1}{x_i} n_{ik} = \frac{1}{x_i} \sum_{k=1}^{p} n_{ik} + \frac{1}{x_i} \sum_{k=1}^{s-1} (x_k - 1) n_{ik}$$
$$\text{for} \quad 1 \le i \le s-1; \tag{3.4}$$

$$r_i = \sum_{k=1}^{s-1} x_k n_{ik} + \sum_{k=s}^{p} n_{ik} = \sum_{k=1}^{p} n_{ik} + \sum_{k=1}^{s-1} (x_k - 1) n_{ik} \quad \text{for} \quad s \le i \le p; \tag{3.5}$$

$$r'_j = \sum_{\ell=1}^{t-1} \frac{x'_\ell}{x'_j} n'_{j\ell} + \sum_{\ell=t}^{q} \frac{1}{x'_j} n'_{j\ell} = \frac{1}{x'_j} \sum_{\ell=1}^{q} n'_{j\ell} + \frac{1}{x'_j} \sum_{\ell=1}^{t-1} (x'_\ell - 1) n'_{j\ell}$$
$$\text{for} \quad 1 \le j \le t-1; \tag{3.6}$$

$$r'_j = \sum_{\ell=1}^{t-1} x'_\ell n'_{j\ell} + \sum_{\ell=t}^{q} n'_{j\ell} = \sum_{\ell=1}^{q} n'_{j\ell} + \sum_{\ell=1}^{t-1} (x'_\ell - 1) n'_{j\ell} \quad \text{for} \quad t \le j \le q. \tag{3.7}$$

If $\phi_{s,t}^2 = \sum_{a=1}^{s-1}(d_a - d_s)$ then $x_k = 1$ for $1 \le k \le s-1$ by (3.1) and $\phi_{s,t}^2 = e$ by Lemma 3.4(iii). Hence (3.4) and (3.5) become

$$r_i = \sum_{k=1}^{p} n_{ik} = \sum_{j:\ u_i v_j \in EG} d'_j \le e = \phi_{s,t}^2 \tag{3.8}$$

for $1 \le i \le p$. Suppose $\phi_{s,t}^2 > \sum_{a=1}^{s-1}(d_a - d_s)$. Referring to (3.4) and (3.5), for $1 \le i \le s-1$

$$r_i \le \frac{1}{x_i} \left((d_i - t + 1) d'_t + \sum_{h=1}^{t-1} d'_h \right) + \frac{1}{x_i} \sum_{k=1}^{s-1} (x_k - 1) d_i = \phi_{s,t}^2 \tag{3.9}$$

and for $s \le i \le p$

$$r_i \le (d_i - t + 1) d'_t + \sum_{h=1}^{t-1} d'_h + \sum_{k=1}^{s-1} (x_k - 1) d_i \tag{3.10}$$

$$\le (d_s - t + 1) d'_t + \sum_{h=1}^{t-1} d'_h + \sum_{k=1}^{s-1} (x_k - 1) d_s = \phi_{s,t}^2, \tag{3.11}$$

where the inequalities are from Lemma 3.6(ii)–(iv) and the non-increasing of degree sequence, and the equalities are from Lemma 3.5(i). Thus, $r_i \le \phi_{s,t}^2$ for $1 \le i \le p$. Similarly, referring to (3.6), (3.7), Lemma 3.6(iii)–(v), the non-increasing of degree sequence, and Lemma 3.5(ii) we have $r'_j \le \phi_{s,t}^2$ for $1 \le j \le q$. Hence $\rho(A^2) = \rho(C) \le \phi_{s,t}^2$ by Lemma 2.4.

To verify the second part of Theorem 3.3, assume that G is connected. We prove the sufficient conditions of $\rho(G) = \phi_{s,t}$. If $s' = 0$ or $t' = 0$ then G is biregular. By Lemmas 2.2 and 3.4(i)–(ii), $\rho(G) = \sqrt{d_1 d'_1} = \phi_{s,t}$. Suppose $s' = 0$ and $t' \ge 1$. Then $d_1 = d_p$ and

$p = d'_1 = d'_{t'} \geq d'_{t'+1} = d'_q$. We take the equatable quotient matrix E of A with respect to the partition $\{\{1, \dots, p\}, \{p+1, \dots, p+t'\}, \{p+t'+1, \dots, p+q\}\}$. Hence

$$E = \begin{pmatrix} 0 & t' & d_s - t' \\ p & 0 & 0 \\ d'_t & 0 & 0 \end{pmatrix}.$$

The eigenvalues of E are 0 and $\pm\sqrt{d_s d'_t + t'(p - d'_t)} = \pm\phi_{s,t}$. By Lemma 2.3, $\phi_{s,t}$ is also an eigenvalue of A. Since $\rho(G) \leq \phi_{s,t}$ has been shown in the first part, we have $\rho(G) = \phi_{s,t}$. Similarly for the case $s' \geq 1$ and $t' = 0$. Suppose $s' \geq 1$ and $t' \geq 1$. Then $q = d_1 = d_{s'} \geq d_{s'+1} = d_p$ and $p = d'_1 = d'_{t'} \geq d'_{t'+1} = d'_q$. We take the equatable quotient matrix F of A with respect to the partition $\{\{1, \dots, s'\}, \{s'+1, \dots, p\}, \{p+1, \dots, p+t'\}, \{p+t'+1, \dots, p+q\}\}$. Hence

$$F = \begin{pmatrix} 0 & 0 & t' & q - t' \\ 0 & 0 & t' & d_s - t' \\ s' & p - s' & 0 & 0 \\ s' & d'_t - s' & 0 & 0 \end{pmatrix}.$$

Then the eigenvalues of F are

$$\pm\sqrt{\frac{X_{s,t} \pm \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}}.$$

We see $\phi_{s,t}$ is an eigenvalue of F, and by Lemma 2.3 $\phi_{s,t}$ is also an eigenvalue of A. Hence $\rho(G) = \phi_{s,t}$. Here we complete the proof of the sufficient conditions of $\phi_{s,t} = \rho(G)$.

To prove the necessary conditions of $\rho(G) = \phi_{s,t}$, suppose $\rho(G) = \phi_{s,t}$. Then by Lemma 2.4 $r_i = r'_j = \phi_{s,t}^2$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Let $s' < s$ and $t' < t$ be the smallest nonnegative integers such that $d_{s'+1} = d_s$ and $d'_{t'+1} = d'_t$, respectively. We prove either $d_1 = d_p$ or $q = d_1 = d_{s'} > d_{s'+1} = d_p$ in the following. The connectedness of G implies $d_s d'_t > 0$ so that

$$\phi_{s,t}^2 > \max\left(\sum_{i=1}^{s-1}(d_i - d_s), \sum_{j=1}^{t-1}(d'_j - d'_t)\right)$$

by Lemma 3.4(iii). Hence the equalities in (3.9) to (3.11) all hold. The choose of s' and the equalities in (3.11) imply that $d_{s'+1} = d_s = d_p$. If $s' = 0$ then $d_1 = d_p$. Suppose $s' \geq 1$. For $1 \leq i \leq s'$, since $d_i > d_s$ and $\phi_{s,t}^2 > \sum_{a=1}^{s-1}(d_a - d_s)$, we have $x_i > 1$ by (3.1). The equalities in (3.9) imply $n_{ik} = d_i$ and then $G(u_k) \supseteq G(u_i)$ by Lemma 3.6(ii) for $1 \leq k \leq s'$ and $1 \leq i \leq s - 1$. Similarly the equalities in (3.10) imply $G(u_k) \supseteq G(u_i)$ for $1 \leq k \leq s'$ and $s \leq i \leq p$ by Lemma 3.6(ii). That is,

$$G(u_1) = G(u_2) = \cdots = G(u_{s'}) \supseteq G(u_i) \quad \text{for} \quad s'+1 \le i \le p.$$

Due to the connectedness of G, $d_1 = d_{s'} = q$. The result follows. Similarly, either $d'_1 = d'_q$ or $p = d'_1 = d'_{t'} > d'_{t'+1} = d'_q$. Clearly that the graphs with those degree sequences are $K_{s',t'} + H$ for some biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively. Here we complete the proof for the necessary conditions of $\phi_{s,t} = \rho(G)$, and also for Theorem 3.3. □

Remark 3.7. Other previous results shown by the style of the above proof can be found in [17,14,9,13]. Similar earlier results are referred to [6,7,18,11,12].

4. A few special cases of Theorem 3.3

In this section we study some special cases of $\phi_{s,t}$ in Theorem 3.3. We follow the notations in Theorem 3.3. As $\phi_{1,1} = \sqrt{d_1 d'_1}$ in Lemma 3.4(i), Theorem 3.3 provides another proof of $\rho(G) \le \sqrt{d_1 d'_1}$ in Lemma 2.2. Applying Theorem 3.3 and simplifying the formula $\phi_{s,t}$ in cases $(s,t) = (1,q)$ and $(s,t) = (p,1)$, we have the following corollary.

Corollary 4.1.

(i) $\rho(G) \le \phi_{1,q} = \sqrt{e - (q - d_1)d'_q}$.
(ii) $\rho(G) \le \phi_{p,1} = \sqrt{e - (p - d'_1)d_p}$. □

We can quickly observe that

$$X_{p,q} = d_p d'_q + (e - p d_p) + (e - q d'_q) = 2e - (p d_p + q d'_q - d_p d'_q) \tag{4.1}$$

and

$$Y_{p,q} = (e - p d_p)(e - q d'_q). \tag{4.2}$$

Hence we have the following corollary.

Corollary 4.2.

$$\rho(G) \le \sqrt{\frac{2e - (p d_p + q d'_q - d_p d'_q) + \sqrt{(p d_p + q d'_q - d_p d'_q)^2 - 4 d_p d'_q (pq - e)}}{2}}. \quad \square$$

By adding an isolated vertex if necessary, we might assume $d_p = 0$ and find $\phi_{p,q} = \sqrt{e}$ from Corollary 4.2. Hence Theorem 3.3 provides another proof of $\rho(G) \le \sqrt{e}$ in Lemma 2.1.

5. Proof of Conjecture 1.2

When e, p, q are fixed, the formula

$$\phi_{p,q}(d_p, d'_q) = \sqrt{\frac{2e - (pd_p + qd'_q - d_pd'_q) + \sqrt{(pd_p + qd'_q - d_pd'_q)^2 - 4d_pd'_q(pq - e)}}{2}} \tag{5.1}$$

obtained in Corollary 4.2 is a 2-variable function. The following lemma will provide shape of the function $\phi_{p,q}(d_p, d'_q)$.

Lemma 5.1. *If $1 \le d'_q \le p - 1$ and $qd'_q \le e$ then*

$$\frac{\partial \phi_{p,q}(d_p, d'_q)}{\partial d_p} < 0.$$

Proof. Referring to (5.1), it suffices to show that

$$\begin{aligned} &\frac{\partial}{\partial d_p}\left(2e - (pd_p + qd'_q - d_pd'_q) + \sqrt{(pd_p + qd'_q - d_pd'_q)^2 - 4d_pd'_q(pq - e)}\right) \\ &\quad = -p + d'_q + \frac{(pd_p + qd'_q - d_pd'_q)(p - d'_q) - 2d'_q(pq - e)}{\sqrt{(pd_p + qd'_q - d_pd'_q)^2 - 4d_pd'_q(pq - e)}} \end{aligned} \tag{5.2}$$

is negative. If $qd'_q = e$ then (5.2) has negative value $2(d'_q - p)$. Indeed if the numerator of the fraction in (5.2) is not positive then (5.2) has negative value. Thus assume that it is positive and $qd'_q < e$. From simple computation to have the fact that

$$\begin{aligned} &\left((pd_p + qd'_q - d_pd'_q) - 2d'_q \cdot \frac{pq - e}{p - d'_q}\right)^2 - \left((pd_p + qd'_q - d_pd'_q)^2 - 4d_pd'_q(pq - e)\right) \\ &= \frac{4d'^2_q(pq - e)}{(p - d'_q)^2} \cdot (qd'_q - e) < 0, \end{aligned}$$

we find that the fraction in (5.2) is strictly less than $p - d'_q$, so the value in (5.2) is negative. □

Remark 5.2. From Example 3.2, if $p \le q$ then the graphs ${}^eK_{p,q} = K_{p-1,q-pq+e} + N_{1,pq-e}$ and $K^e_{p,q} = K_{p-pq+e,q-1} + N_{pq-e,1}$ satisfy the equalities in Theorem 3.3. Hence $\rho({}^eK_{p,q}) = \phi_{p,q}(q - pq + e, p - 1)$ and $\rho(K^e_{p,q}) = \phi_{p,q}(q - 1, p - pq + e)$; the latter is expanded as

$$\rho(K^e_{p,q}) = \sqrt{\frac{e + \sqrt{e^2 - 4(q-1)(p-pq+e)(pq-e)}}{2}} \tag{5.3}$$

by (5.1).

Lemma 5.3. *Suppose* $0 < pq - e < \min(p, q)$, $1 \le d_p \le q - 1$, $1 \le d'_q \le p - 1$ *and*

$$d_p + d'_q = e - (p-1)(q-1). \tag{5.4}$$

Then

$$\phi_{p,q}(d_p, d'_q) \le \rho(K^e_{p,q}).$$

Proof. From symmetry, we can assume $p \le q$. Referring to (5.1) and (5.3), we only need to show that

$$e - (pd_p + qd'_q - d_pd'_q) + \sqrt{(pd_p + qd'_q - d_pd'_q)^2 - 4d_pd'_q(pq-e)} \tag{5.5}$$

$$\le \sqrt{e^2 - 4(q-1)(p-pq+e)(pq-e)}. \tag{5.6}$$

From (5.4), we have

$$e - (pd_p + qd'_q - d_pd'_q) = (p - d'_q - 1)(q - d_p - 1) \ge 0 \tag{5.7}$$

and

$$\begin{aligned} d_pd'_q &= \frac{(d_p + d'_q)^2 - [2d_p - (d_p + d'_q)]^2}{4} \\ &\ge \frac{(e - (p-1)(q-1))^2 - [2(q-1) - (e - (p-1)(q-1))]^2}{4} \\ &= (q-1)(p-pq+e). \end{aligned} \tag{5.8}$$

Hence Eq. (5.5) is at most

$$e - (pd_p + qd'_q - d_pd'_q) + \sqrt{(pd_p + qd'_q - d_pd'_q)^2 - 4(q-1)(p-pq+e)(pq-e)}. \tag{5.9}$$

Set $a = e - (pd_p + qd'_q - d_pd'_q)$ and $b = 4(q-1)(p-pq+e)(pq-e)$. Note that $a \ge 0$ by (5.7) and $b \ge 0$ by the relations between p, q, e. Using the fact that

$$\sqrt{e^2 - b} - \sqrt{(e-a)^2 - b} \ge \sqrt{e^2} - \sqrt{(e-a)^2} = a \tag{5.10}$$

from the concave property of the function $y = \sqrt{x}$, we find the value in (5.9) is at most that in (5.6) and the result follows. □

Proof of Conjecture 1.2. By Theorem 3.3, $\rho(G) \leq \phi_{p,q}(d_p, d'_q)$. Note that the assumption $0 < pq - e < \min(p, q)$ implies $1 \leq d_p \leq q-1$ and $1 \leq d'_q \leq p-1$. Let $e_p = e-(p-1)(q-1) - d'_q$. Clearly that $1 \leq e_p \leq d_p$ and $qd'_q \leq e$. By Lemma 5.1, $\phi_{p,q}(d_p, d'_q) \leq \phi_{p,q}(e_p, d'_q)$. With e_p in the role of d_p in Lemma 5.3, we have $\phi_{p,q}(e_p, d'_q) \leq \rho(K^e_{p,q})$. This completes the proof. □

6. Concluding remark

We give a series of sharp upper bounds for the spectral radius of bipartite graphs in Theorem 3.3. One of these upper bounds can be presented only by five variables: the number e of edges, bipartition orders p and q, and the minimal degrees d_p and d'_q in the corresponding partite sets as shown in Corollary 4.2. We apply this bound when three variables e, p, q are fixed to prove Conjecture 1.2, a refinement of Conjecture 1.1 in the assumption that $0 < pq - e < \min(p, q)$. To conclude this paper we propose the following general refinement of Conjecture 1.1.

Conjecture 6.1. *Let $G \in \mathcal{K}(p, q, e)$. Then*

$$\rho(G) \leq \rho(K^e_{s,t})$$

for some positive integers $s \leq p$ and $t \leq q$ such that $0 \leq st - e \leq \min(s, t)$.

We believe that the function $\phi_{p,q}(d_p, d'_q)$ in (5.1) will still play an important role in solving Conjecture 6.1. Two of the key points might be to investigate the shape of the 4-variable function $\phi_{p,q}(d_p, d'_q)$ with variables p, q, d_p, d'_q, and to check that for which sequence s, t, d_s, d'_t such that $s \leq p$ and $t \leq q$ and $0 \leq st - e \leq \min(s, t)$, there exists a bipartite graph H with e edges whose spectral radius satisfying $\rho(H) = \phi_{s,t}(d_s, d'_t)$, where s, t are the bipartition orders of H and d_s and d'_t are corresponding minimum degrees.

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