Introduction to Homogenization Theory - Periodic Second Order Elliptic Equations

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This notes was given by Prof. Zhongwei Shen’s lecture in the National Taiwan University, 2015.03. For more details we refer readers to his draft lecture (with a similar title as this lecture). For simplicity, the following arguments hold for the linear elliptic system, however, we only discuss the linear scalar elliptic equation case in this lecture.

1 Introduction

We consider the following elliptic equations

\[
\begin{align*}
-\Delta u &= F \quad \text{in } \Omega \subset \mathbb{R}^d, \\
-\nabla \cdot (A(x)\nabla u) &= F \quad \text{in } \Omega \subset \mathbb{R}^d, \text{ with suitable boundary conditions,} \\
F(D^2u, Du, u, x) &= F \quad \text{in } \Omega \subset \mathbb{R}^d,
\end{align*}
\]

Medium with a fine self-similar microscopic structure. Let \( \epsilon > 0 \) be a small parameter, and consider the following homogenization problem

\[
F(D^2u_\epsilon, Du_\epsilon, u_\epsilon, x) = 0 \quad \text{in } \Omega, \text{ with suitable boundary conditions.}
\]

In the following, we only consider the simplest linear elliptic homogenization problem. Let \( A(x) = (a_{ij}(x))_{d \times d}, \ A'(x) = A(\frac{x}{\epsilon}) \) and consider

\[
\begin{align*}
-\nabla \cdot (A(x)\nabla u_\epsilon) &= F \quad \text{in } \Omega, \\
\text{Suitable boundary condition on } \partial \Omega.
\end{align*}
\]

What can we say about \( u_\epsilon \) as \( \epsilon \to 0 \). In fact, we have

\[
u_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega) \text{ and } u_\epsilon \to u_0 \text{ strongly in } L^2(\Omega).
\]

As a result, for \( F \in H^{-1}(\Omega) \), we have

\[
\begin{align*}
-\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) &= F \quad \text{in } \Omega, \\
\text{Same B.Cs}
\end{align*}
\]
as $\epsilon \to 0$ and $\mathcal{A} = (\alpha_{ij})_{d \times d}$ is a constant matrix. Here is the problem setting. Assume the second order elliptic operator

$$L_{\epsilon} := -\frac{\partial}{\partial x_i} \left( a_{ij}(\frac{x}{\epsilon}) \frac{\partial}{\partial x_j} \right)$$

with the coefficients satisfying

\[
\begin{cases}
\text{Ellipticity: } \mu |\xi|^2 \leq a_{ij}(y)\xi_i \xi_j \leq \frac{1}{\mu} |\xi|^2, \text{ for all } \xi \in \mathbb{R}^d, \\
\text{1-Periodicity: } A(y + z) = A(y) \text{ for all } y \in \mathbb{R}^d, \ z \in \mathbb{Z}^d.
\end{cases}
\]

Consider the Dirichlet problem to be

\[
\begin{aligned}
L_{\epsilon} u_{\epsilon} &= F \quad \text{in } \Omega, \\
\nu \cdot \nabla u_{\epsilon} &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $F \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial \Omega)$, and $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. By the Lax-Milgram lemma, we have

\[
\|u_{\epsilon}\|_{{H^1(\Omega)}} \leq C \{ \|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial \Omega)} \},
\]

where the constant $C$ independent of $\epsilon$. Note that $u_{\epsilon}$ is a weak solution of (2) if for all $\phi \in H^1_0(\Omega)$, we have

\[
\int_\Omega (A(x)\nabla u_{\epsilon}) \cdot \nabla \phi \, dx = \langle F, \phi \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)}.
\]

Consider the Neumann problem to be

\[
\begin{aligned}
L_{\epsilon} u_{\epsilon} &= F \quad \text{in } \Omega, \\
\frac{\partial u_{\epsilon}}{\partial \nu_{\epsilon}} &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $F \in H^{-1}_{0,\Omega} = (H^1(\Omega))^\prime$, $g \in H^{-1/2}(\partial \Omega) = (H^{1/2}(\partial \Omega))^\prime$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$ is a unit outer normal on $\partial \Omega$. Similarly, we call $u_{\epsilon}$ to be a weak solution if for all $\phi \in H^1(\Omega)$, we have

\[
\int_\Omega (A(x)\nabla u_{\epsilon}) \cdot \nabla \phi \, dx = \langle F, \phi \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} - \langle g, \phi \rangle_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)}.
\]

Moreover, by the Lax-Milgram lemma again, we have

\[
\|\nabla u_{\epsilon}\|_{L^2(\Omega)} \leq C \{ \|F\|_{H^{-1}_{0,\Omega}} + \|g\|_{H^{-1/2}(\partial \Omega)} \},
\]

where $C$ is independent of $\epsilon$. For the second order elliptic system, we consider the homogenization problem to be

\[
\begin{aligned}
L_{\epsilon} u_{\epsilon} &= -\frac{\partial}{\partial x_i} \left[ \alpha_{ij}(\frac{x}{\epsilon}) \frac{\partial u_{\epsilon}^\alpha}{\partial x_j} \right] = F_{\alpha} \quad \text{in } \Omega \\
u u_{\epsilon} &= g \text{ satisfies suitable boundary conditions}
\end{aligned}
\]

with $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. In order to demonstrate ideas, we consider $m = 1$ in most situations.
2 Derivation of the homogenized equation

The ideas of the derivation of the homogenized equation is quite clever. We consider \( u_\epsilon \) to be the perturbation of \( u_0 \) with respect to \( \epsilon \)-parameter. Moreover, by observing the elliptic operator \( L_\epsilon \), we introduce the two-scaled method, which means we consider \( x = x \), and \( y = \frac{x}{\epsilon} \) to be two independent parameters. Let

\[
  u_\epsilon := u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots
\]

be the asymptotic expansion of \( u_\epsilon \), where

\[
  u_j := u_j(x, y) = u_j(x, \frac{x}{\epsilon}).
\]

In addition,

\[
  \nabla u_j = \nabla_x u_j(x, y) + \frac{1}{\epsilon} \nabla_y u_j(x, y), \text{ as } y = \frac{x}{\epsilon},
\]

which means under our two-scaled method, the operator \( \nabla = \nabla_x + \frac{1}{\epsilon} \nabla_y \). Therefore, (1) will become

\[
  - (\nabla + \frac{1}{\epsilon} \nabla_y) \cdot \{ A(y)(\nabla_x + \frac{1}{\epsilon} \nabla_y)(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)\} = F(x) \text{ in } \Omega,
\]

where we are not concerned about the boundary condition for the equation. Expand (6) and compare it with the same orders, so we get

\[
  O(\frac{1}{\epsilon^2}) : \quad - \nabla_y \cdot (A(y)\nabla_y u_0(x, y)) = 0,
\]

\[
  O(\frac{1}{\epsilon}) : \quad - \nabla_y \cdot (A(y)\nabla_y u_1(x, y)) = \nabla_y \cdot (A(y)\nabla_x u_0) + \nabla_x \cdot (A(y)\nabla_y u_0),
\]

\[
  O(1) : \quad - \nabla_y \cdot (A(y)\nabla_y u_2(x, y)) = \nabla_y \cdot (A(y)\nabla_x u_1) + \nabla_x \cdot (A(y)\nabla_y u_1) + \nabla_x \cdot (A(y)\nabla_x u_0) + F(x).
\]

Recall that for the periodic elliptic equation

\[
  - \nabla \cdot (A(y)\nabla u(y)) = h(y), \quad A(y) \text{ is } 1\text{-periodic},
\]

then we have

\[
  \int_{\mathbb{T}^d} h(y) dy = 0
\]

by using the Stokes formula. For \( O(\frac{1}{\epsilon^2}) \) term, this equation is solvable because the right hand side is zero. In further, we multiply \( u_0(x, y) \) on both sides and integrate by parts, which will imply

\[
  0 = \int_{\mathbb{T}^d} A(y)\nabla_y u_0 \cdot \nabla_y u_0 \geq \mu \int_{\mathbb{T}^d} |\nabla_y u_0(x, y)|^2 dy \geq 0,
\]

which gives us the information that

\[
  u_0(x, y) \equiv u_0(x)
\]
independent of \( y \).

Now, for the second term \( O(\frac{1}{\epsilon}) \), the second term on the right hand side should be zero since \( \nabla_y u_0(x) = 0 \). Solve the equation

\[
-\nabla_y \cdot (A(y) \nabla_y u_1(x, y)) = \nabla_y \cdot (A(y) \nabla_x u_0) = (\nabla_y \cdot A(y))(\nabla_x u_0)
\]

formally. Note that since \( A(y) \) is 1-periodic, then the equation is solvable for \( u_1 \) if

\[
\int_{\mathbb{T}^d} (\nabla_y \cdot A(y)) \nabla_y u_0 dy = \int_{\mathbb{T}^d} (A(y) \nabla_x u_0) \cdot \nu(y) dS(y) = 0.
\]

By using the separation of variables, we put the ansatz

\[
u_1(x, y) = \chi(y)(\nabla_x u_0(x))
\]

into the equation with \( \chi(y) \) being 1-periodic, we will find that

\[
-\nabla \cdot (A(y) \nabla \chi(y))(\nabla_x u_0) = (\nabla_y \cdot A(y))(\nabla_x u_0).
\]

Moreover, we call

\[
-\nabla \cdot (A(y) \nabla \chi(y)) = (\nabla_y \cdot A(y))
\]

to be the cell problem and \( \chi(y) \) to be the corrector.

Finally, we observe the last equation carefully. Put \( u_1(x, y) = \chi(y)(\nabla_x u_0) \) into the \( O(1) \) equation and examine the solvability condition for \( u_2(x, y) \), we have

\[
0 = \int_{\mathbb{T}^d} (\nabla_y \cdot (A(y) \nabla_x u_1) + \nabla_x \cdot (A(y) \nabla_y u_1) + \nabla_x \cdot (A(y) \nabla_x u_0) + F(x)] dy
\]

where the first term vanishes by the periodicity of \( A \) and \( \chi \). Thus, we can obtain

\[
-\nabla \cdot (\overline{A} \nabla u_0) = F(x) \text{ in } \Omega,
\]

where

\[
\overline{A} = \int_{\mathbb{T}^d} \{A(y) + A(y)(\nabla_y \chi(y))\} dy
\]

is the (constant) homogenized operator and we call \( \overline{A} \) to be the homogenized equation. For the rigorous derivation of the homogenized equation, we will give the serious proof later by using the famous tool: The Div-Curl lemma.

### 3 Basic properties

Now, let us recall the corrector again. \( \chi = (\chi_1, \chi_2, \cdots, \chi_d) \in H^1_{\text{per}}(\mathbb{Z}^d) \) is the corrector, where \( H^1_{\text{per}}(\mathbb{Z}^d) \) is the closure of \( C^\infty \) 1-periodic function under the standard \( H^1 \)-norm. We rewrite the cell problem in the following way:

\[
\begin{dcases}
-\frac{\partial}{\partial y_i} (a_{ij}(y) \frac{\partial \chi_k}{\partial y_j}) = \frac{\partial}{\partial y_i} (a_{ik}(y)) \quad \text{in } \mathbb{R}^d, \\
\chi_k \in H^1_{\text{per}}(\mathbb{Z}^d), \ 1 \leq k \leq d.
\end{dcases}
\]
Note that $\frac{\partial}{\partial y_i}(a_{ik}(y)) = \frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j}(y_k))$, then the cell-problem becomes

$$L_1(x_k + x_k) = 0 \text{ in } \mathbb{R}^d.$$ 

In addition, we consider the equivalence space $H^1_{\text{per}}(\mathbb{Z}^d)/\sim$ and define the bilinear form

$$a_{\text{per}}(\phi, \psi) := \int_{\mathbb{T}^d} a_{ij}(y) \frac{\partial \phi}{\partial y_j} \frac{\partial \psi}{\partial y_i} dy,$$

then we also can prove the Lax-Milgram lemma under this equivalence Sobolev space. Moreover, when $\phi \in H^1_{\text{per}}(\mathbb{Z}^d)$, then $a_{\text{per}}(u, \phi) = 0$ for all $u \in H^1(\Omega)$.

For convenience, we set $L_0 := -\nabla \cdot (\overline{A} \nabla)$ be the homogenized second order elliptic operator with respect to $\overline{A}$.

**Proposition 3.1.** The homogenized operator $\overline{A} = (\overline{a}_{ij})$ can be rewritten in the following form

$$\overline{a}_{ij} = a_{\text{per}}(y_j + \chi_j, y_i).$$

Moreover, use the above relation, we can write $\overline{A}$ as

$$\overline{a}_{ij} = a_{\text{per}}(y_j + \chi_j, y_i + \chi_i).$$

Use the above proposition, it is not hard to obtain the following theorem.

**Theorem 3.2.** $L_0$ is elliptic, which means

$$\mu |\xi|^2 \leq \overline{a}_{ij} \xi_i \xi_j \leq \mu_1 |\xi|^2,$$

where the lower bound comes from the original $A$, and $\mu_1 = \mu_1(\mu, d)$.

**Proof.** $\overline{a}_{ij} = a_{\text{per}}(y_j + \chi_j, y_i + \chi_i)$ and $a_{\text{per}}(y_j + \chi_j, \phi) = 0$ when $\phi \in H^1_{\text{per}}(\mathbb{Z}^d)$. Therefore, we have

$$\overline{a}_{ij} \xi_i \xi_j = a_{\text{per}}((y_j + \chi_j) \xi_j, (y_i + \chi_i) \xi_i)$$

$$\geq \mu \int_{\mathbb{T}^d} |\nabla(\chi_i + y_i)\xi_i|^2 dy$$

$$= \int_{\mathbb{T}^d} \left[|\nabla(\chi_i \xi_i)|^2 + 2\nabla(\chi_i \xi_i) \cdot \nabla(y_i \xi_i) + |\nabla(y_i \xi_i)|^2 \right]$$

$$\geq \mu_1 |\xi|^2,$$

since $\nabla(\chi_i \xi_i) \cdot \nabla(y_i \xi_i) = 0$. \hfill \Box

For the rigorous proof of the $L_\epsilon \to L_0$ in a suitable sense, we use the following two useful lemmas.

**Lemma 3.3.** Let $h \in L^2_{\text{loc}}(\mathbb{R}^d)$ and 1-periodic, then

$$h(\frac{x}{\epsilon}) \rightharpoonup c_0 := \int_{\mathbb{T}^d} h(y) dy$$

weakly in $L^2(\Omega)$.
Proof. We gave a hint for readers and leave details of proof to an exercise. Hint: Try to solve
\begin{align*}
\begin{cases}
\Delta u = h \text{ in } \mathbb{T}^d, \\
u \text{ is 1-periodic}.
\end{cases}
\end{align*}

Lemma 3.4. (Div-Curl lemma) Let \( \{u_k\}, \{v_k\} \) be two bounded sequences in \( L^2(\Omega, \mathbb{R}^d) \). Suppose that \( u_k \rightharpoonup u_0, v_k \rightharpoonup v_0 \) weakly in \( L^2(\Omega, \mathbb{R}^d) \) and \( \nabla \times u_k = 0 \) for all \( k \), \( \nabla \cdot v_k \to f \) strongly in \( H^{-1}(\Omega) \). Then
\[ \int_{\Omega} u_k \cdot v_k \varphi dx \to \int_{\Omega} u_0 \cdot v_0 \varphi dx \]
for all \( \varphi \in C_0^1(\Omega) \).

Proof. Leave this lemma to an exercise.

Theorem 3.5. (Homogenization Theorem) Suppose \( A(y) \) is elliptic and 1-periodic. Let \( \Omega \) be a bounded Lipschitz domain. If \( u_\epsilon \) is a weak solution of the Dirichlet problem (2). Then
\[ u_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega), \quad u_\epsilon \to u_0 \text{ strongly in } L^2(\Omega), \]
where \( u_0 \in H^1(\Omega) \) is the weak solution to the homogenized equation
\begin{align*}
\begin{cases}
L_0 u_0 = F & \text{in } \Omega, \\
u_0 = g & \text{on } \partial \Omega.
\end{cases}
\end{align*}
Moreover, we have \( A(x/\epsilon) \nabla u_\epsilon \to \overline{A} \nabla u_0 \) weakly in \( L^2(\Omega, \mathbb{R}^d) \).

Remark 3.6. In general, \( u_\epsilon \) does not converge to \( u_0 \) strongly in \( H^1(\Omega) \).

Before giving proofs for the homogenization theorem, we introduce the dual problem for the homogenization problem. Let \( L^*_\epsilon := -\nabla \cdot (A^*(x/\epsilon) \nabla) \) be the adjoint operator of \( L_\epsilon \). Then we have the following properties.

1. \( a^*_ij = a_{ji} \).
2. \( a^*_{per}(\phi, \psi) = a_{per}(\psi, \phi) \).
3. Let \( \chi^* \) be the corrector of \( L^*_\epsilon \), then we have
\[ a^*_{per}(y_k + \chi^*_k, \psi) = 0, \text{ for all } \psi \in H^1_{per}(\mathbb{Z}^d). \]

Proposition 3.7. We have \( \overline{a^*_{ij}} = a^*_{per}(y_i + \chi_i, y_j) \) and \( \overline{A} = \overline{A^*} \). If \( A = A^* \), then \( \overline{A^*} = \overline{A} \).

Proof. Exercise.

In addition, we have
\begin{align*}
\overline{a_{ij}} &= a^*_{per}(y_i + \chi_i, y_j) = a_{per}(y_j, y_i + \chi^*_i) \\
&= \int_{\mathbb{T}^d} a_{ij}(y) \partial y_i \partial (y_i + \chi^*_i) dy \\
&= \int_{\mathbb{T}^d} a_{ij}(y) \{ \delta y_k + \frac{\partial \chi^*_i}{\partial y_k} \} dy \\
&= \int_{\mathbb{T}^d} \left\{ a_{ij}(y) + a_{ij}(y) \frac{\partial \chi^*_i}{\partial y_k} \right\} dy.
\end{align*}
Now we prove the theorem.

**Proof.** By using the Lax-Milgram theorem, we have

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C\{\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\},$$

which means \( \{u_\epsilon\} \) is bounded in \( H^1(\Omega) \). Moreover, \( u_\epsilon \) satisfies 2 and

$$\int_\Omega A(\frac{x}{\epsilon}) \nabla u_\epsilon \cdot \varphi dx = \langle F, \varphi \rangle.$$

By the duality argument, we have

$$\|A(\frac{x}{\epsilon}) \nabla u_\epsilon\|_{L^2(\Omega)} \leq C < \infty.$$ WLOG, we say \( \{u_\epsilon\} \) be the sequence such that \( u_\epsilon \rightharpoonup v \) weakly in \( H^1(\Omega) \) and \( A(\frac{x}{\epsilon}) \nabla u_\epsilon \) to \( p \) weakly in \( L^2(\Omega) \). Then remaining work is to prove

\[
\begin{aligned}
\{v &= u_0 \\
 p &= \mathbf{A} \nabla u_0.
\end{aligned}
\]

Consider

$$\int_\Omega A(\frac{x}{\epsilon}) \nabla u_\epsilon \cdot \nabla (x_k + \epsilon \chi_k(\frac{x}{\epsilon})) \psi dx = \int_\Omega \nabla u_\epsilon \cdot A(\frac{x}{\epsilon}) \nabla (x_k + \epsilon \chi_k(\frac{x}{\epsilon})) \psi dx,$$

for all \( \psi \in C^1_0(\Omega) \). For the left side, we have \( A(\frac{x}{\epsilon}) \nabla u_\epsilon \rightharpoonup p \) weakly in \( L^2(\Omega; \mathbb{R}^d) \).

\[
\nabla (x_k + \epsilon \chi_k(\frac{x}{\epsilon})) = \delta_{ik} + \frac{\partial \chi_k(\frac{x}{\epsilon})}{\partial x_i} \to L^2 \delta_{ik} + \int_{\mathbb{T}_d} \frac{\partial \chi_k}{\partial y_i} dy = \delta_{ik},
\]

and

$$\nabla \times (\nabla (x_k + \epsilon \chi_k(\frac{x}{\epsilon}))) = 0 \quad \text{and} \quad \nabla \cdot (A(\frac{x}{\epsilon}) \nabla u_\epsilon) = F.$$

The left hand side satisfies conditions of the Div-Curl lemma, then the left hand side will converge to

$$\int_\Omega p_k \delta_{ik} \psi dx = \int_\Omega p_k \psi dx.$$

For the right side, \( \nabla u_\epsilon \rightharpoonup \nabla u_0 \) in \( L^2 \),

\[
A(\frac{x}{\epsilon}) \nabla (x_k + \epsilon \chi_k(\frac{x}{\epsilon})) = A(\frac{x}{\epsilon}) \{ \nabla x_k + \nabla \chi_k(\frac{x}{\epsilon}) \}
\]

\[
= \int_{\mathbb{T}_d} a_{ij}(y) \delta_{jk} + \frac{\partial}{\partial y_j}(\chi_k(\frac{x}{\epsilon})) dy
\]

\[
= \int_{\mathbb{T}_d} a_{ki}(y) + a_{ji}(y) \frac{\partial \chi_k}{\partial y_j} dy
\]

\[
= a_{ki}.
\]

Then the right side converges to \( \int_\Omega \frac{\partial v}{\partial x_i} a_{ki} \psi dx \). Thus, \( \int_\Omega p_k \psi dx = \int_\Omega \frac{\partial v}{\partial x_i} a_{ki} \psi dx \),

which implies

$$p_k = \frac{\partial v}{\partial x_i} a_{ki} \quad \text{and} \quad A(\frac{x}{\epsilon}) \nabla u_\epsilon \rightharpoonup \mathbf{A} \nabla v$$
and
\[
\begin{aligned}
-\nabla \cdot (A \nabla v) &= F \quad \text{in } \Omega, \\
v &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]
and \(v = u_0\) (uniqueness for Dirichlet problem).

**Remark 3.8.** The homogenization theorem still holds for the Neumann problem
\[
\begin{aligned}
L_{\epsilon}(u_{\epsilon}) &= F \quad \text{in } \Omega, \\
\frac{\partial u_{\epsilon}}{\partial \nu_{\epsilon}} &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(F \in H^{-1}_0(\Omega), g \in H^{-1/2}(\partial \Omega),\) and \(\int_{\Omega} u_{\epsilon} = 0\). The proof is followed by
\[
\int_{\Omega} A(\frac{x}{\epsilon}) \nabla u_{\epsilon} \cdot \nabla \psi dx = \langle F, \psi \rangle_{H^{-1}_0 \times H^1} - \langle g, \psi \rangle_{H^{-1/2} \times H^{1/2}},
\]
and use the Div-Curl lemma argument, then we can obtain the desired results.

## 4 Rates of convergence

In the homogenization theory, it is interesting that how fast for \(u_{\epsilon}\) converging to \(u_0\).

**Theorem 4.1.** (Dirichlet) Suppose \(A\) is elliptic and 1-periodic. Let \(\Omega\) be a bounded Lipschitz domain. Then
\[
\|u_{\epsilon} - u_0 - \epsilon \chi(\frac{x}{\epsilon}) \cdot \nabla u_0\|_{H^1(\Omega)} \leq C \sqrt{\epsilon} \|u_0\|_{W^{2,\infty}(\Omega)}.
\]
If \(d \geq 3\) and \(\chi\) is Hölder continuous (if \(m = 1\)),
\[
\|u_{\epsilon} - u_0 - \epsilon \chi(\frac{x}{\epsilon}) \cdot \nabla u_0\|_{H^1(\Omega)} \leq C \sqrt{\epsilon} \|u_0\|_{W^{2,2}(\Omega)}.
\]
From \(\|u_{\epsilon} - u_0 - \epsilon \chi(\frac{x}{\epsilon}) \nabla u_0\| \leq C \sqrt{\epsilon} \|u_0\|_{W^{2,2}(\Omega)}\), we can derive
\[
\|u_{\epsilon} - u_0\|_{L^2(\Omega)} \leq \epsilon \|\chi(\frac{x}{\epsilon}) \nabla u_0\|_{L^2(\Omega)} + C \sqrt{\epsilon} \|u_0\|_{W^{2,2}(\Omega)}
\]
\[
\leq C \sqrt{\epsilon} \|u_0\|_{W^{2,2}(\Omega)}.
\]
Notice that the second inequality is nontrivial, we leave it as an exercise to readers. Moreover, we have
\[
\|u_{\epsilon} - u_0\|_{L^2(\Omega)} \leq C \epsilon \|u_0\|_{W^{2,2}(\Omega)} \quad \text{if } \begin{cases} \Omega \text{ is Lipschitz, } m = 1, \\ \Omega \text{ is } C^{1,1}, \ m \geq 2, \end{cases}
\]
and
\[
\|u_{\epsilon} - u_0\|_{H^{1/2}(\Omega)} \leq C \epsilon \|u_0\|_{W^{2,2}(\Omega)}, \quad \text{when coefficients are Hölder continuous.}
\]

**Definition 4.2.** Define \(b_{ij}(y) := a_{ij}(y) + a_{ik}(y) \frac{\partial \chi_{k}}{\partial y_{k}} - \overline{a_{ij}}\).
Note that $b_{ij} \in L^2_{loc}$, $b_{ij}$ is 1-periodic, $\int_{\mathbb{T}^d} b_{ij}(y)dy = 0$, and $\frac{\partial b_{ij}}{\partial y_i} = 0$.

**Lemma 4.3.** There exists $\phi_{kij} \in H^1_{per}(\mathbb{Z}^d)$ such that $b_{ij} = \frac{\partial}{\partial y_k} \phi_{kij}$ and $\phi_{kij} = -\phi_{ikj}$.

**Proof.** We solve

$$\begin{cases}
\Delta f_{ij} = b_{ij} \text{ in } \mathbb{R}^d, \\
f_{ij} \text{ is } 1\text{-periodic, } f_{ij} \in H^2_{per}(\mathbb{Z}^d).
\end{cases}$$

Let $\phi_{kij} := \frac{\partial}{\partial y_k} f_{ij} - \frac{\partial}{\partial y_i} f_{kj} \in H^1_{per}(\mathbb{Z}^d)$, then $\phi_{kij} = -\phi_{ikj}$. Since $\Delta \frac{\partial f_{ij}}{\partial y_i} = \frac{\partial b_{ij}}{\partial y_i} = 0$, by the Liouville’s theorem, we can get $\frac{\partial f_{ij}}{\partial y_i}$ is a constant. Hence

$$\frac{\partial}{\partial y_k} \phi_{kij} = \Delta f_{ij} - \frac{\partial^2}{\partial y_k \partial y_i} f_k = \Delta f_{ij} = b_{ij}.$$

\qed

**Remark 4.4.** If $\chi \in C^\alpha$ for some $\alpha \in (0,1)$, then $\phi = (\phi_{kij})$ is bounded.

**Lemma 4.5.** Let $w_\epsilon := u - u_0 - \epsilon \chi \frac{x}{\epsilon} \frac{\partial u_0}{\partial x_k}$ in $\Omega$. Then

$$L_\epsilon(w_\epsilon) = -\epsilon \frac{\partial}{\partial x_i} \phi_{kij}(\frac{x}{\epsilon}) \frac{\partial^2 u_0}{\partial x_j \partial x_k} + \epsilon \frac{\partial}{\partial x_i} \{a_{ij}(\frac{x}{\epsilon}) \chi(\frac{x}{\epsilon}) \frac{\partial^2 u_0}{\partial x_j \partial x_k}\}.$$

**Proof.** From direct calculation, we have

$$L_\epsilon(w_\epsilon) = L_\epsilon(u_\epsilon) - L_\epsilon(u_0) - L_\epsilon \{\epsilon \chi \frac{x}{\epsilon} \frac{\partial u_0}{\partial x_k}\}$$

$$= L_0(u_0) - L_\epsilon(u_0) - L_\epsilon \{\epsilon \chi \frac{x}{\epsilon} \frac{\partial u_0}{\partial x_k}\}$$

$$= -\epsilon \frac{\partial}{\partial x_i} \{a_{ij}(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j}\} - \epsilon \frac{\partial}{\partial x_i} \{a_{ij}(\frac{x}{\epsilon}) \chi(\frac{x}{\epsilon}) \frac{\partial^2 u_0}{\partial x_j \partial x_k}\}$$

$$= \frac{\partial}{\partial x_i} \{b_{ij}(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j}\} + I$$

$$= \epsilon \frac{\partial}{\partial x_i} \{\frac{\partial}{\partial x_k} \phi_{kij}(\frac{x}{\epsilon}) \cdot \frac{\partial u_0}{\partial x_j}\} + I$$

$$= \epsilon \frac{\partial^2}{\partial x_i \partial x_k} \{\phi_{kij}(\frac{x}{\epsilon}) \cdot \frac{\partial u_0}{\partial x_j}\} = \epsilon \frac{\partial^2}{\partial x_i \partial x_k} \{\phi_{kij}(\frac{x}{\epsilon}) \partial_x u_0\} + I$$

$$= -\epsilon \frac{\partial}{\partial x_i} \{a_{ij}(\frac{x}{\epsilon}) \chi(\frac{x}{\epsilon}) \frac{\partial^2 u_0}{\partial x_j \partial x_k}\} + I,$$

where $I = \epsilon \frac{\partial}{\partial x_i} \phi_{kij}(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j}$ and $\frac{\partial^2}{\partial x_i \partial x_k} \{\phi_{kij}(\frac{x}{\epsilon}) \frac{\partial u_0}{\partial x_j}\} = 0$ by $\phi_{kij} = -\phi_{ikj}$ (after summing over $i, j, k$). \qed
For more delicate rates of convergence, we introduce the boundary corrector. Let \( v_\epsilon \) be a solution satisfying
\[
\begin{cases}
L_\epsilon(v_\epsilon) = 0 \quad & \text{in } \Omega, \\
v_\epsilon = -\chi_k(x_\epsilon) \frac{\partial u_0}{\partial x_k} \quad & \text{on } \partial \Omega,
\end{cases}
\]
then we have the following theorem.

**Theorem 4.6.** For \( d \geq 3 \), we have
\[
\| u_\epsilon - u_0 - \epsilon \chi_k(x_\epsilon) \frac{\partial u_0}{\partial x_k} - \epsilon v_\epsilon \|_{H^1_0(\Omega)} \leq C\epsilon \| \nabla^2 u_0 \|_{L^d(\Omega)}.
\]

**Proof.** (Sketch) Let \( \tilde{w}_\epsilon := u_\epsilon - u_0 - \epsilon \chi_k(x_\epsilon) \frac{\partial u_0}{\partial x_k} - \epsilon v_\epsilon \), then it is easy to see that \( \tilde{w}_\epsilon \in H^1_0(\Omega) \) and \( L_\epsilon(\tilde{w}_\epsilon) = L_\epsilon(w_\epsilon) \). Hence by the Lax-Milgram lemma, we have
\[
\| \tilde{w}_\epsilon \|_{H^1_0(\Omega)} \leq C\| L_\epsilon(\tilde{w}_\epsilon) \|_{H^{-1}(\Omega)} \leq C\epsilon \{ \| \phi \|_{L^2(\Omega)} \| \nabla^2 u_0 \|_{L^d(\Omega)} + \| \chi(x_\epsilon) \| \nabla^2 u_0 \|_{L^2(\Omega)} \}.
\]
The remaining task is to estimate
\[
\| \phi \|_{L^2(\Omega)} \| \nabla^2 u_0 \|_{L^2(\Omega)} + \| \chi(x_\epsilon) \| \nabla^2 u_0 \|_{L^2(\Omega)} \leq C\| \nabla^2 u_0 \|_{L^d(\Omega)}.
\]
For the first term, we have
\[
\int_\Omega |\phi(x_\epsilon) \nabla^2 u_0|^2 \, dx \leq \int_\Omega |\phi(x_\epsilon)|^2 |\nabla^2 u_0|^2 \, dx \leq (\int_\Omega |\phi(x_\epsilon)|^{\frac{2d}{d-2}} \, dx)^{\frac{d-2}{2}} (\int_\Omega |\nabla^2 u_0|^{d} \, dx)^{\frac{2}{d}}.
\]
Note that \( \nabla \phi \in L^2(T^d) \) and the Sobolev embedding theorem gives \( \int_{T^d} |\phi|^{\frac{2d}{d-2}} \, dx \leq C \int_{T^d} |\nabla \phi|^2 \, dx \leq C \) and
\[
\int_\Omega |\phi(x_\epsilon)|^{\frac{2d}{d-2}} \, dx \leq \int_{\Omega} |\phi(y)|^{\frac{2d}{d-2}} \, dy \cdot \epsilon^d \leq C.
\]
For the other term, we leave it to readers. \( \square \)

**Remark 4.7.** If \( d = 2 \), \( \chi \in C^\alpha \) for some \( \alpha \in (0, 1) \), then we can use the Meyers \( L^p \) estimate to derive \( \phi \in L^\infty(T^d) \).

Now, we want to prove the fact that
\[
\| u_\epsilon - u_0 - \chi_k(x_\epsilon) \nabla u_0 \|_{H^{1,2}(\Omega)} \leq C\sqrt{\epsilon} \| u_0 \|_{W^{2,4}(\Omega)}.
\]
To prove (8), only need to check the following lemma.

**Lemma 4.8.** \( \| v_\epsilon \|_{H^{1,2}(\Omega)} \leq \frac{C}{\sqrt{\epsilon}} \| u_0 \|_{W^{2,4}(\Omega)} \).
Proof. Recall that \( v_\epsilon \) satisfies
\[
\begin{cases}
L_\epsilon(v_\epsilon) = 0 & \text{in } \Omega, \\
v_\epsilon = -\chi_k \left( \frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k} & \text{on } \partial \Omega,
\end{cases}
\]
and by the standard elliptic regularity, we have
\[
\|v_\epsilon\|_{H^1(\Omega)} \leq C\|\chi_k \left( \frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k}\|_{H^{1/2}(\partial \Omega)} \leq C\|\eta_k(x)\chi_k \left( \frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k}\|_{H^1(\Omega)},
\]
where \( \eta_k \in C^1_0(\mathbb{R}^d) \) with \( \eta_k(x) = \begin{cases} 1, & \text{if } \text{dist}(x, \partial \Omega) \leq \epsilon \\
0, & \text{if } \text{dist}(x, \partial \Omega) \geq 2\epsilon \end{cases} \) and \( \|\nabla \eta_k\|_{L^\infty} \leq \frac{C}{\epsilon} \).

Therefore, we have
\[
\int_{\Omega} |\nabla \eta_k|^2 |\chi_k(x/\epsilon)|^2 |\nabla u_0|^2 \, dx \\
\leq \frac{C}{\epsilon^2} \int_{\text{dist}(x, \partial \Omega) \leq 2\epsilon} |\chi_k(x/\epsilon)|^2 |\nabla u_0|^2 \, dx \\
\leq \frac{C}{\epsilon^2} \int_{\text{dist}(x, \partial \Omega) \leq 2\epsilon} |\chi_k(x/\epsilon)|^2 |\nabla u_0|^2 \, dx \leq \frac{C}{\epsilon^2} \int_{\text{dist}(x, \partial \Omega) \leq 2\epsilon} |\nabla u_0|^2 \, dx.
\]

Claim: \( \int_{\text{dist}(x, \partial \Omega) \leq 2\epsilon} |\nabla u_0|^2 \, dx \leq C \epsilon \) and \( \int_{\text{dist}(x, \partial \Omega) \leq 2\epsilon} |\nabla u_0|^4 \, dx \leq C \epsilon \|u_0\|_{W^{2,4}(\Omega)} \)

(this is true \( \forall u_0 \in W^{2,4}(\Omega) \)).

For the first part, we choose cubes \( Q'_j \) with \( \text{diam} Q'_j \sim O(\epsilon) \) and
\[
\{\text{dist}(x, \partial \Omega) \leq 2\epsilon\} \subset \bigcup_{j=1}^N Q'_j \subset \{\text{dist}(x, \partial \Omega) \leq 5\epsilon\}.
\]

We can estimate \( \chi \) in each cubes and use the periodicity of \( \chi \), we can find the uniform estimate for the first term.

For the second part, let \( \Omega_t := \{x \in \Omega; \text{dist}(x, \partial \Omega) > t\} \) and \( n \) be the unit outer normal on \( \partial \Omega_t \). Let \( C_0 > 0 \) be a positive constant and \( \alpha \) be the vector such that \( \langle n, \alpha \rangle \geq C_0 > 0 \). Then
\[
C_0 \int_{\Omega_t} |\nabla u_0|^q dS \leq \int_{\Omega_t} |\nabla u_0|^q \langle n, \alpha \rangle \, dS \\
\leq \int_{\Omega_t} \text{div } \alpha |\nabla u_0|^q dS + \alpha_k \frac{\partial}{\partial x_k} (|\nabla u_0|^q) \, dx \\
\leq C \int_{\Omega_t} |\nabla u_0|^q \, dx + C \int_{\Omega_t} |\nabla u_0|^{q-1} |\nabla^2 u_0| \, dx \\
\leq C \|u_0\|_{W^{2,q}}.
\]

Finally, integrate the above inequalities by \( \int_0^{2t} dt \), then we can obtain the desired estimate and the constant \( C \) is independent of \( \epsilon \). Combine the above estimates together, we will find
\[
\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C \sqrt{\epsilon} \|u_0\|_{W^{2,4}(\Omega)}.
\]

\[ \square \]

Remark 4.9. In fact, we can prove
\[
\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C \epsilon \|u_0\|_{W^{2,4}(\Omega)} \] if \( \Omega \) is Lipschitz, \( m = 1 \), \( \Omega \) is \( C^{\alpha,1} \), \( m \geq 2 \).
5 Construction of Dirichlet corrector

Let \( \Phi_\epsilon = (\Phi_{\epsilon,j}) \in H^1(\Omega) \) be the Dirichlet corrector satisfying

\[
\begin{align*}
L_\epsilon(\Phi_{\epsilon,j}) &= 0 \quad \text{in } \Omega, \\
\Phi_{\epsilon,j} &= x_j \quad \text{on } \partial \Omega.
\end{align*}
\] (9)

Note that \( x_j \) itself also satisfies (9).

**Proposition 5.1.** Assume that \( m = 1 \), then

\[ \| \Phi_{\epsilon,j} - x_j \|_{L^\infty(\Omega)} \leq C \epsilon. \]

**Proof.** Let \( u_\epsilon = x_j + \epsilon \chi_j(\frac{x}{\epsilon}) - \Phi_{\epsilon,j}(x) \), then

\[
\begin{cases}
L_\epsilon(u_\epsilon) = L_\epsilon(x_j + \epsilon \chi_j(\frac{x}{\epsilon})) - L_\epsilon(\Phi_{\epsilon,j}) = 0 \quad \text{in } \Omega, \\
u_\epsilon = \epsilon \chi_j(\frac{x}{\epsilon}) \quad \text{on } \partial \Omega.
\end{cases}
\]

By the maximum principle, we have

\[ \| u_\epsilon \|_{L^\infty(\Omega)} \leq \| u_\epsilon \|_{L^\infty(\partial \Omega)} = \epsilon \| \chi_j(\frac{x}{\epsilon}) \|_{L^\infty(\partial \Omega)} \leq \epsilon \| \chi_j(\frac{x}{\epsilon}) \|_{L^\infty(\partial \Omega)} \leq C \epsilon, \]

which implies

\[ \| \Phi_{\epsilon,j} - x_j \|_\infty \leq \| u_\epsilon \|_\infty + \| \epsilon \chi_j(\frac{x}{\epsilon}) \|_\infty \leq C \epsilon. \]

\[ \square \]

**Remark 5.2.** When \( m \geq 2 \), we consider

\[ w_\epsilon = u_\epsilon - u_0 - \epsilon \chi_k(\frac{x}{\epsilon}) S_\epsilon(\frac{\partial \tilde{u}_0}{\partial x_k}), \]

where \( \tilde{u}_0 \) is an extension of \( u_0 \) to \( \mathbb{R}^d \) by 0 outside \( \Omega \), and \( S_\epsilon(\tilde{u}_0)(x) = \int_{B_\epsilon(x)} \tilde{u}_0 \) is called the Steklov smoothing operator (the ideas were discovered by A. Suslina).

6 Uniform interior estimates

The uniform estimates are given by the compensated compactness argument.

**Theorem 6.1.** Suppose \( A = A(y) \) is elliptic and periodic. Let \( u_\epsilon \in H^1(B(x_0, R)) \) be a weak solution of \( L_\epsilon(u_\epsilon) = F \) in \( B = B(x_0, R) \). Suppose that \( F \in L^p(B) \), \( p > d \) and \( 0 < \epsilon < R \), then

\[ \left( \int_B |\nabla u_\epsilon|^2 \right)^{1/2} \leq C \left\{ \left( \int_B |\nabla u_\epsilon|^2 \right)^{1/2} + R \left( \int_B |F|^p \right)^{1/2} \right\}, \]

where \( C_p = C(d, p, \mu) \).
Theorem 6.2. (M. Avellenda, F. H. Lin, 1987) Suppose $A = A(y)$ is elliptic and periodic. $A(y)$ satisfies

$$|A(y) - A(z)| \leq \lambda |y - z|^\tau, \forall y, z \in \mathbb{R}^d,$$

for some $\tau > 0$.

Let $u_\epsilon \in H^1(B)$ be a weak solution of $L_\epsilon(u_\epsilon) = F$ in $B = B(x_0, R)$, where $F \in L^p(B)$, $p > d$. Then

$$|\nabla u_\epsilon(x_0)| \leq C_p\{(\int_B |\nabla u_\epsilon|^2)^{1/2} + R(\int_B |F|^p)^{1/p}\},$$

or

$$\|\nabla u_\epsilon\|_{L^\infty(BR/2)} \leq C_p\{(\int_B |\nabla u_\epsilon|^2)^{1/2} + R(\int_B |F|^p)^{1/p}\},$$

where $C_p = C(p, d, \mu\lambda, \tau)$.

Observations:

1. Translation.
   If $-\nabla \cdot (A(\frac{x}{\epsilon}) \nabla u_\epsilon) = F$ and $v_\epsilon = u_\epsilon(x - x_0)$, then

   $$-\nabla \cdot (\tilde{A}(\frac{x}{\epsilon}) \nabla v_\epsilon) = \tilde{F},$$

   where $\tilde{A}(y) = A(y + \frac{x_0}{\epsilon})$ and $\tilde{F} = F(x - x_0)$.

2. Dilation.
   If $-\nabla \cdot (A(\frac{x}{\epsilon}) \nabla u_\epsilon) = F$ and $v_\epsilon(x) = u_\epsilon(rx)$, $r > 0$, then

   $$L_\epsilon(v_\epsilon) = G,$$

   where $G(x) = r^2F(rx)$.

Note that Theorem 1 implies Theorem 2 by using the blow-up arguments. By translation and dilation, we may assume $x_0 = 0$ and $R = 1$. Also, we may assume $\epsilon < \frac{1}{2}$ (if $\epsilon \geq \frac{1}{2}$, then $A(\frac{x}{\epsilon})$ is good, we only need to use the standard argument for elliptic regularity).

Let $w_\epsilon(x) = \epsilon^{-1}u_\epsilon(\epsilon x)$, then $L_1(w) = \epsilon F(\epsilon x)$ in $B(0, 1)$ (note that $\frac{1}{\epsilon} > 1$).

By the interior Lipschitz estimates for $L_1$, we have

$$|\nabla w(0)| \leq C\{(\int_{B(0, 1)} |\nabla w|^2)^{1/2} + (\int_{B(0, 1)} |\epsilon F(\epsilon x)|^p)^{1/p}\}.$$

This implies

$$|\nabla u_\epsilon(0)| \leq C\{(\int_{B(0, \epsilon)} |\nabla u_\epsilon|^2)^{1/2} + \epsilon(\int_{B(0, \epsilon)} |F|^p)^{1/p}\} \leq C\{(\int_{B(0, \epsilon)} |\nabla u_\epsilon|^2)^{1/2} + \epsilon^{1-\frac{p}{2}}(\int_{B(0, 1)} |F|^p)^{1/p}\} \leq C\{(\int_{B(0, 1)} |\nabla u_\epsilon|^2)^{1/2} + (\int_{B(0, 1)} |F|^p)^{1/p}\}.$$
Lemma 7.1. (One-step improvement) Let \( A^\ell \) be one-step improvement, the second is iteration, the last is blow-up argument.

Remark 6.3. Local property needs the regularity, but the global one does not in the homogenization theory.

Theorem 6.4. (Compactness) Let \( \{ A^\ell \} \) be a sequence of 1-periodic matrices satisfying the elliptic condition with the same \( \mu \). Suppose \(-\nabla \cdot (A^\ell \nabla u^\ell) = F^\ell \) in \( \Omega \), where \( \epsilon^\ell \to 0 \). Also assume that \( u^\ell \to u \) in \( H^1(\Omega) \), \( A^\ell \to A^0 \), and \( F^\ell \to F \) in \( H^{-1}(\Omega) \). Then \( A^\ell (\frac{x}{\epsilon^\ell})\nabla u^\ell \to A^0 \nabla u \) weakly in \( L^2(\Omega) \) and \(-\nabla \cdot (A^0 \nabla u) = F \) in \( \Omega \).

Proof. The proof is similar to the proof of the homogenization theorem.

7 Compensated compactness

Here we give the compactness argument, which contains three steps. The first is one-step improvement, the second is iteration, the last is blow-up argument.

Lemma 7.1. (One-step improvement) Let \( 0 < \sigma < \rho < 1 \) and \( \rho = 1 - \frac{d}{p} \). There exist \( \epsilon_0 \in (0, \frac{1}{2}) \) and \( \theta \in (0, \frac{1}{4}) \) depending on \( \mu, \sigma, \rho, d \) such that

\[
\left( \int_{B(0, \theta)} |u_\epsilon - \int_{B(0, \theta)} u_\epsilon - x_j + \epsilon \chi_j(\frac{x}{\epsilon^\ell}) \int_{B(0, \theta)} \frac{\partial u}{\partial x_j} |^2 \right)^{1/2} \leq \theta^{1+\sigma} \max\{ (\int_{B(0, 1)} |u_\epsilon|^2)^{1/2}, (\int_{B(0, 1)} |F|^p)^{1/p} \},
\]

where \( 0 < \epsilon < \epsilon_0, u_\epsilon \in H^1(B(0, 1)) \) is a solution of \( L_\epsilon(u_\epsilon) = F \) in \( B(0, 1) \).

Proof. Prove it by contradiction. Use the fact that if \(-\nabla \cdot (A^0 \nabla u) = F \) in \( B(0, \frac{1}{2}) \), then

\[
\sup_{|x| < \theta} |u - \int_{B(0, \theta)} u - x_j \int_{B(0, \theta)} \frac{\partial u}{\partial x_j}| \leq C\theta^{1+\rho} \| \nabla u \|_{C^{0, \rho}(B(0, \theta))} \leq C\theta^{1+\rho} \| \nabla u \|_{C^{0, \rho}(B(0, \frac{1}{4}))} \leq C_0\theta^{1+\rho} \max\{ (\int_{B(0, \frac{1}{4})} |u|^2)^{1/2}, (\int_{B(0, \frac{1}{4})} |F|^p)^{1/p} \}.
\]

Since \( \sigma < \rho \), choose \( \theta \in (0, \frac{1}{4}) \) such that \( 2^{d+1} C_0\theta^{1+\rho} < \theta^{1+\sigma} \).

Claim: Suppose not, \( \exists \) sequences \( \{ \epsilon^\ell \} \subset (0, \frac{1}{2}) \) and \( \{ A^\ell \} \) satisfying periodicity and ellipticity with \( \mu \). \( \{ F^\ell \} \subset L^p(B(0, 1)) \), \( \{ u^\ell \} \subset H^1(B(0, 1)) \) such that for \( \epsilon^\ell \to 0 \), we have \(-\nabla \cdot (A^\ell \nabla u^\ell) = F^\ell \), with \( (\int_{B(0, 1)} |u^\ell|^2)^{1/2} \leq 1 \), \( (\int_{B(0, 1)} |F^\ell|^p)^{1/p} \leq 1 \) and

\[
(\int_{B(0, \theta)} |u^\ell - \int_{B(0, \theta)} u^\ell - (x_j + \epsilon \chi_j(\frac{x}{\epsilon^\ell}) \int_{B(0, \theta)} \frac{\partial u^\ell}{\partial x_j} |^2)^{1/2} > \theta^{1+\sigma},
\]
where \( \chi^\ell_x \) are correctors for \( L^\ell_x = -\nabla \cdot (A^\ell(\frac{x}{\epsilon^\ell})\nabla) \). By Cacciopoli’s inequality, \( \{u_\ell\} \) is bounded in \( H^1(B(0, \frac{1}{2})) \). By passing to subsequences, we may assume that

\[
u_\ell \to u \text{ weakly in } L^2(B(0, 1)) , \quad u_\ell \to u \text{ weakly in } H^1(B(0, 1)) , \quad F_\ell \to F \text{ weakly in } L^p(B(0, 1)) , \quad \text{and } \overline{A^\ell} \to A^0 ,
\]

which will imply \( F_\ell \to F \) strongly in \( H^{-1}(B(0, 1)) \) (since \( L^p \) compactly embedded to \( H^{-1} \) when \( p > d \)). By the compactness theorem, \( -\nabla \cdot (A^0 \nabla u) = F \) in \( B(0, \frac{1}{2}) \). Also, \( (\int_{B(0,1)} |u|^2)^{1/2} \leq 1, (\int_{B(0,1)} |F|^p)^{1/p} \leq 1 \), and

\[
\left( \int_{B(0,\theta)} |u - \int_{B(0,\theta)} u - x_j \int_{B(0,\theta)} \frac{\partial u}{\partial x_j} |^2 \right)^{1/2} \geq \theta^{1+\sigma} ,
\]

which will lead to a weird conclusion.

\( \Box \)

**Lemma 7.2.** (Iteration) Let \( 0 < \sigma < \rho < 1 \) and \( \rho = 1 - \frac{d}{p} \). Let \( (\epsilon_0, \theta) \) be given by the previous lemma. Suppose \( 0 < \epsilon < \theta^{k-1} \epsilon_0 \) for some \( k \geq 1 \). Let \( L_\ell(u_\epsilon) = F \) in \( B(0, 1) \). Then \( \exists \) constants \( E(\epsilon, \ell) = (E_j(\epsilon, \ell)) \) \( (1 \leq \ell \leq k) \) such that if \( v_\ell = u_\epsilon - (x_j + \epsilon \chi_j(\frac{x}{\epsilon})) E_j(\epsilon, \ell) \), then

\[
(\int_{B(0,\theta)} |v_\ell - \int_{B(0,\theta)} v_\ell - x_j \int_{B(0,\theta)} \frac{\partial v_\ell}{\partial x_j} |^2 )^{1/2} \leq C \theta^{(1+\sigma)} \max \{ (\int_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\int_{B(0,1)} |F|^p)^{1/p} \} .
\]

Moreover,

\[
|E(\epsilon, \ell)| \leq C \max \{ (\int_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\int_{B(0,1)} |F|^p)^{1/p} \}
\]

and

\[
|E(\epsilon, \ell + 1) - E(\epsilon, \ell)| \leq C \theta^\sigma \max \{ (\int_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\int_{B(0,1)} |F|^p)^{1/p} \} .
\]

**Proof.** Prove by induction on \( \ell \), for \( 1 \leq \ell \leq k \). When \( \ell = 1 \), by the previous lemma, \( E_j(\epsilon, 1) = \int_{B(0,\theta)} \frac{\partial u_\epsilon}{\partial x_j} \). Suppose that the constants \( E_j(\epsilon, i) \) exists, for all \( 1 \leq i \leq \ell, \ell \leq k - 1 \). To construct \( E(\epsilon, \ell + 1) \), consider

\[
w(x) = u_\epsilon(\theta^\ell x) - \{ \theta^\ell x_j + \epsilon \chi_j(\frac{\theta^\ell x}{\epsilon}) E_j(\epsilon, \ell) \}
- \int_{B(0,\theta^\ell \epsilon)} \{ u_\epsilon - (x_j + \epsilon \chi_j(\frac{x}{\epsilon})) E_j(\epsilon, \ell) \}.
\]

Then \( L^\ell_{\theta^\ell \epsilon}(w) = \theta^{2\ell} F(\theta^{2\ell} x) \) in \( B(0, 1) \). Since \( \frac{\epsilon}{\theta^\ell} \leq \frac{\epsilon}{\theta^{k-1}} < \epsilon_0 \), then we may apply the previous lemma to \( w \) in \( B(0, 1) \) and repeat all the arguments again, which will finish Lemma 7.2.

\( \Box \)
Proof of the compactness theorem: By translation and dilation, we may assume $x_0 = 0$ and $R = 1$, $0 < \epsilon < 1$. Also we assume that $0 < \epsilon < \epsilon_0$ and choose $k \geq 2$ such that $\epsilon_0\theta^k \leq \epsilon < \epsilon_0\theta^{k-1}$. By the iteration lemma, we can get

$$
(\int_{B(0,\theta^{k-1})} |u_\epsilon - \int_{B(0,\theta^{k-1})} u_\epsilon|^2)^{1/2} \leq C\theta^k \left( (\int_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\int_{B(0,1)} |F|^p)^{1/p} \right),
$$

after rescaling, the above can be rewritten as

$$
(\int_{B(0,\epsilon)} |u_\epsilon - \int_{B(0,\epsilon)} u_\epsilon|^2)^{1/2} \leq C\epsilon \left( (\int_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\int_{B(0,1)} |F|^p)^{1/p} \right),
$$

for some constant $C$ independent of $\epsilon$. Therefore,

$$
(\int_{B(0,\epsilon)} |\nabla u_\epsilon|^2)^{1/2} \leq C \epsilon \left( (\int_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\int_{B(0,1)} |F|^p)^{1/p} \right).
$$

Theorem 7.3. (Dirichlet problem, Avellenda-Lin, 87) Assume that $A$ is elliptic, periodic and $|A(y) - A(z)| \leq \lambda |y - z|$. Let $\Omega \subseteq \mathbb{R}^d$ be $C^{1,\alpha}$ Suppose

$$
\begin{cases}
L_\epsilon(u_\epsilon) = F & \text{in } \Omega, \\
u_\epsilon = g & \text{on } \partial \Omega.
\end{cases}
$$

Then

$$
\| \nabla u_\epsilon \|_{L^\infty(\Omega)} \leq C_p,\sigma \left( \| F \|_{L^p(\Omega)} + \| g \|_{C^{1,\sigma}(\partial \Omega)} \right),
$$

with $\sigma > 0$.

Proof. Compactness and Lipschitz estimates for Dirichlet correctors

$$
\begin{cases}
L_\epsilon(\Phi_{\epsilon,j}) = 0 & \text{in } \Omega, \\
\Phi_{\epsilon,j} = x_j & \text{on } \partial \Omega.
\end{cases}
$$

Show that $\| \nabla \Phi_{\epsilon,j} \|_{L^2(\Omega)} \leq C$.

\[\Box\]

Theorem 7.4. (Neumann problem, Kenig-Lin-Shen, 2013) All assumptions are the same in Theorem 7.4. Let

$$
\begin{cases}
L_\epsilon(u_\epsilon) = F & \text{in } \Omega, \\
\frac{\partial u_\epsilon}{\partial \nu_v} = g & \text{on } \partial \Omega,
\end{cases}
$$

then

$$
\| \nabla u_\epsilon \|_{L^\infty(\Omega)} \leq C \left( \| F \|_{L^p(\Omega)} + \| g \|_{C^{\sigma}(\partial \Omega)} \right),
$$

where $p > d$, $\sigma > 0$.

Proof. Compactness and Lipschitz estimates for the Neumann correctors. Let

$$
\begin{cases}
L_\epsilon(\Psi_{\epsilon,j}) = 0 & \text{in } \Omega, \\
\frac{\partial \Psi_{\epsilon,j}}{\partial \nu_v} = \frac{\partial}{\partial \nu_0}(x_j) & \text{on } \partial \Omega,
\end{cases}
$$

where $\frac{\partial u}{\partial \nu_0} = n_i\frac{\partial u}{\partial x_j}$. Show that $\| \nabla \Psi_{\epsilon,j} \| \leq C$. \[\Box\]
8 Real variable method and \( W^{1,p} \) estimates

For the simplest case,
\[
\begin{aligned}
\begin{cases}
-\Delta u = \text{div} f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

We have \( \| \nabla u \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \) and \( \| u \|_{H^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \).

**Question:** Does \( \| \nabla u \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} \) hold for \( 1 < p < \infty \)?

Similarly, for the homogenization problem, our main question is if
\[
\begin{aligned}
\begin{cases}
L_\epsilon(u_\epsilon) = \text{div} f & \text{in } \Omega, \\
u_\epsilon = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
we have \( \| \nabla u_\epsilon \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \), does \( \| \nabla u_\epsilon \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} \) hold for \( 1 < p < \infty \)?

**Theorem 8.1.** (Interior \( W^{1,p} \) estimate) Suppose that \( A \) is elliptic, periodic and continuous (VMO will be fine). Let \( L_\epsilon(u_\epsilon) = \text{div} f \) in \( B(x_0, 2R) \). Then for \( 2 < p < \infty \), we have
\[
\left( \int_{B(x_0, R)} |\nabla u_\epsilon|^p \right)^{1/p} \leq C_p \left( \int_{B(x_0, 2R)} |\nabla u_\epsilon|^2 \right)^{1/2} + \left( \int_{B(x_0, 2R)} |f|^p \right)^{1/p},
\]
where \( C_p \) depends only on \( d, \mu \) and \( \omega(t) = \sup \{|A(y) - A(z)|; |y - z| \leq t\} \).

**Lemma 8.2.** Let \( L_1(u) = 0 \) in \( B(x_0, 2r) \) for \( 0 < r < 1 \). Then
\[
\left( \int_{B(x_0, r)} |\nabla u|^p \right)^{1/p} \leq C_p \left( \int_{B(x_0, 2r)} |\nabla u|^2 \right)^{1/2}.
\]

**Proof.** This is a local \( W^{1,p} \) estimate. \( \square \)

**Lemma 8.3.** Suppose \( A \) is elliptic, periodic and continuous. Let \( L_\epsilon(u_\epsilon) = 0 \) in \( B(x_0, 2R) \). Then for \( 2 < p < \infty \), we have
\[
\left( \int_{B(x_0, R)} |\nabla u_\epsilon|^p \right)^{1/p} \leq C_p \left( \int_{B(x_0, 2R)} |\nabla u_\epsilon|^2 \right)^{1/2}.
\]

**Proof.** By translation and dilation, we may assume \( x_0 = 1, R = 1 \). If \( \epsilon \geq \frac{1}{4} \), this follows from Lemma 8.2, since \( |A(x_\epsilon) - A(y_\epsilon)| \leq \omega(|x_\epsilon - y_\epsilon|) \leq \omega(4|x - y|) \leq \omega(4r) \).

Let \( 0 < \epsilon < \frac{1}{4} \), consider \( w(x) = u_\epsilon(\epsilon x) \), then \( L_1(w) = 0 \). By Lemma 8.2,
\[
\left( \int_{B(0, 1)} |\nabla w|^p \right)^{1/p} \leq C_p \left( \int_{B(0, 2)} |w|^2 \right)^{1/2},
\]
which implies
\[
\left( \int_{B(0, \epsilon)} |\nabla u_\epsilon|^p \right)^{1/p} \leq C_p \left( \int_{B(0, 2\epsilon)} |\nabla u_\epsilon|^2 \right)^{1/2} \leq C_p \left( \int_{B(0, 1)} |\nabla u_\epsilon|^2 \right)^{1/2}.
\]

By translation,
\[
\left( \int_{B(x_0, \epsilon)} |\nabla u_\epsilon|^p \right)^{1/p} \leq C_p \left( \int_{B(x_0, 2\epsilon)} |\nabla u_\epsilon|^2 \right)^{1/2} \leq C_p \left( \int_{B(0, 2)} |\nabla u_\epsilon|^2 \right)^{1/2}.
\]
for all \( x_0 \in B(0, 1) \). Therefore, we have
\[
\int_{B(x_0, \epsilon)} |\nabla u_\epsilon|^p \leq C \epsilon^d \left( \int_{B(0, 2)} |\nabla u_\epsilon|^2 \right)^{p/2},
\]
and
\[
\int_{B(0, 1)} |\nabla u_\epsilon|^p \leq C \left( \int_{B(0, 2)} |\nabla u_\epsilon|^2 \right)^{p/2}.
\]

The ideas of real variable method come from the Calderon-Zygmund decomposition.

**Theorem 8.4.** (A real variable theorem) Let \( B_0 \) be ball in \( \mathbb{R}^d \) and \( F \in L^2(4B_0) \). Let \( q > 2 \) and \( f \in L^p(4B) \) for \( 2 < p < q \). Suppose that for each ball \( B \subset 2B_0 \) with \( |B| \leq C_1|B_0| \), there exist two functions \( F_B \) and \( R_B \) on \( 2B \) such that
1. \( |F| \leq |F_B| + |R_B| \) on \( 2B \),
2. \( \left( \int_{2B} |R_B|^q \right)^{1/q} \leq N_1 \left( \int_{4B} |F|^2 \right)^{1/2} + \sup_{B \subset B' \subset 4B_0} \left( \int_B |f|^2 \right)^{1/2} \),
3. \( \left( \int_{2B} |F_B|^q \right)^{1/q} \leq N_2 \sup_{B \subset B' \subset 4B_0} \left( \int_{B'} |f|^2 \right)^{1/2} \).

Then
\[
\left( \int_{B_0} |F|^p \right)^{1/p} \leq C \left( \int_{4B_0} |F|^2 \right)^{1/2} + \left( \int_{4B_0} |f|^p \right)^{1/p},
\]
where \( C \) depends on \( N_1, N_2, C_1, p, q, d \).

**Proof.** (Proof of interior \( W^{1,p} \)) Let \( F = |\nabla u_\epsilon| \), \( f = f \). Show
\[
\left( \int_{B(x_0, R)} |F|^p \right)^{1/p} \leq C \left( \int_{B(x_0, 2R)} |F|^2 \right)^{1/2} + \left( \int_{B(x_0, 2R)} |f|^p \right)^{1/p},
\]
For each \( B \) with \( 4B \subset B(x_0, 2R) \), we write \( u_\epsilon = v_\epsilon + w_\epsilon \), where
\[
\begin{align*}
L_\epsilon(v_\epsilon) &= \text{div} v_\epsilon & \quad & \text{in } \Omega = 4B, \\
v_\epsilon &= 0 & \quad & \text{on } \partial \Omega = \partial(4B).
\end{align*}
\]
Let \( F_B = |\nabla v_\epsilon| \), \( R_B = |\nabla w_\epsilon| \). Clearly, \( |F| \leq F_B + R_B \) in \( 2B \) and by the energy estimate, we have
\[
\left( \int_{4B} |F_B|^2 \right)^{1/2} = \left( \int_{4B} |\nabla v_\epsilon|^2 \right)^{1/2} \leq C \left( \int_{4B} |f|^2 \right)^{1/2}.
\]
Note that \( L_\epsilon(w_\epsilon) = 0 \) in \( 4B \) and take \( q = p + 1 \). By Lemma 8.3, we have
\[
\left( \int_{4B} |\nabla w_\epsilon|^q \right)^{1/q} \leq C \left( \int_{4B} |\nabla w_\epsilon|^2 \right)^{1/2} \leq C \left( \int_{4B} |\nabla v_\epsilon|^2 \right)^{1/2} + C \left( \int_{4B} |\nabla u_\epsilon|^2 \right)^{1/2}
\]
\[
\leq C \left( \int_{4B} |f|^2 \right)^{1/2} + C \left( \int_{4B} |F|^2 \right)^{1/2}.
\]
Take \( B' = B \) and apply the real variable theorem, we can obtain the \( W^{1,p} \) estimate. \( \square \)
9 Singular integrals

Finally, we discuss the singular integrals, which will be useful to the homogenization theorem.

Let $Tf(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x,y)f(y)dy$, where $K(x,y)$ is the Calderon-Zygmund kernel satisfying

$$
\begin{aligned}
|K(x,y)| &\leq \frac{C}{|x-y|^d}, \\
|K(x,y) - K(x,z)| &\leq \frac{C|x-z|^\sigma}{|x-y|^{d+\sigma}}, \quad |x-z| < \frac{1}{2}|x-y|, \\
|K(y,x) - K(z,x)| &\leq \frac{C|x-z|^\sigma}{|x-y|^{d+\sigma}}, \quad |x-z| < \frac{1}{2}|x-y|.
\end{aligned}
$$

$T$ is called a Calderon-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^d)$ and $T$ is associated with CZ kernel ($K(x,y) = C_d \frac{x_j - y_j}{|x-y|^{d+1}}$).

**Lemma 9.1.** (Calderon-Zygmund) All CZ operators are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and is of weak type $(1,1)$.

Use the Marcinkiewitz interpolation, we have $1 < p < \infty$ and duality argument, we have $2 < p < \infty$.

**Theorem 9.2.** Let $T$ be a bounded sublinear operator on $L^2(\mathbb{R}^d)$ and $q > 2$. Suppose that

$$
(\int_B |Tg(y)|^q)^{1/q} \leq N\{\int_{2B} |Tg(y)|^{q/2} + \sup_{B \subset B'} (\int_B |g(y)|^2)^{1/2}\}
$$

for all $B \subset \mathbb{R}^d$ and for all $g \in C^\infty_0(\mathbb{R}^d)$ with supp($g$) $\subset \mathbb{R}^d \setminus AB$. Then

$$
\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall 2 < p < q.
$$

**Remark 9.3.** In the previous theorem, we did not use any conditions on CZ kernel.

10 Exercises

Consider the second order elliptic operator

$$
L_\epsilon = -\nabla \cdot (A(\frac{x}{\epsilon})\nabla),
$$

where $\epsilon > 0$. In the following problems we always assume that the coefficients matrix

$$
A = A(y) = (a_{ij}(y))_{d \times d}
$$

is real, bounded measurable, and satisfies the ellipticity condition

$$
\frac{1}{\mu} |\xi|^2 \leq a_{ij}(y)\xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for} \quad y \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,
$$

where $\mu > 0$, and periodicity condition

$$
A(y + z) = A(y) \quad \text{for} \quad y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d.
$$

The summation convention that the repeated indices are summed is used.
1. Define the weak solution of the Dirichlet problem

\[ L_\epsilon(u_\epsilon) = F \text{ in } \Omega \text{ and } u_\epsilon = f \text{ on } \partial\Omega, \]

where \( \Omega \) is a bounded Lipschitz domain, \( F \in H^{-1}(\Omega) \) and \( f \in H^{1/2}(\partial\Omega) \). Prove that the existence and uniqueness of the weak solutions.

2. Define the weak solutions of the Neumann problem

\[ L_\epsilon(u_\epsilon) = F \text{ in } \Omega \text{ and } \frac{\partial u_\epsilon}{\partial \nu_\epsilon} = g \text{ on } \partial\Omega, \]

where \( \Omega \) is a bounded Lipschitz domain, \( F \in H^{-1}_0(\Omega) \) and \( g \in H^{-1/2}(\partial\Omega) \). Prove the existence and uniqueness of the weak solutions.

3. Show that if \( L_\epsilon(u_\epsilon) = \text{div} f \) in \( B(x_0, 2r) \), then

\[ \hat{B}(x_0, r) \left| \nabla u_\epsilon \right|^2 \, dx \leq C \left\{ \frac{1}{r^d} \int_{B(x_0, 2r)} |u_\epsilon|^2 \, dx + \int_{B(x_0, 2r)} |f|^2 \, dx \right\}. \]

4. Show that there exists a 1-periodic function \( u \) such that \( u \in H^1_{\text{loc}}(\mathbb{R}^d) \) and

\[ L_1(u) = -L_1(x_j) \text{ in } \mathbb{R}^d. \]

Such \( u \), unique up to constants, are called the correctors and denote by \( \chi_j \). The homogenized matrix \( \bar{A} = (\bar{a}_{ij}) \) is defined by

\[ \bar{a}_{ij} = \int_{\mathbb{T}^d} \left\{ a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial x_k} \right\} dy. \]

5. Find the correctors \( \bar{A} \), defined in Problem 4, in the case \( d = 1 \) (Signolo example).

6. Finish the proofs of Lemma 3.3 and Lemma 3.4 (the Div-Curl lemma).

7. Let

\[ S_\epsilon(u) = \int_{B(x, \epsilon)} u(y) \, dy. \]

Prove that

\[ \| g S_\epsilon(u) \|_{L^p(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \left( \int_{B(x, \epsilon)} |g|^p \right)^{1/p} \| u \|_{L^p(\mathbb{R}^d)} \]

for any \( 1 \leq p < \infty \).

8. Let \( u \in H^1(\mathbb{R}^d) \). Show that

\[ \| S_\epsilon(u) - u \|_{L^2(\mathbb{R}^d)} \leq \epsilon \| \nabla u \|_{L^2(\mathbb{R}^d)}. \]

9. Suppose that \( u_\epsilon \in H^1_0(\Omega) \) and \(-\nabla \cdot (A(x) \nabla u_\epsilon) = F \) in \( \Omega \). Show that \( \epsilon \to 0 \),

\[ \int_\Omega A(\frac{x}{\epsilon}) \nabla u_\epsilon \cdot (\nabla u_\epsilon) \varphi \, dx \to \int_\Omega \bar{A} \nabla u_0 \cdot (\nabla u_0) \varphi \, dx \]

for any \( \varphi \in C_0^\infty(\Omega) \), where \( u_0 \) is the solution of the homogenized problem.
10. Let $u_\epsilon$ be the weak solution of
\[
\begin{cases}
(\partial_t + L_\epsilon)u_\epsilon = F & \text{in } \Omega \times (0, T), \\
u_\epsilon = 0 & \text{on } \partial\Omega \times (0, T), \\
u_\epsilon = f & \text{on } \Omega \times \{t = 0\},
\end{cases}
\]
where $F \in L^2(\Omega \times (0, T))$, $f \in L^2(\Omega)$ and $\Omega$ is a bounded Lipschitz domain. Show that $u_\epsilon \to u_0$ strongly in $L^2(\Omega \times (0, T))$ and $A(\frac{\partial}{\partial \epsilon})\nabla u_\epsilon \rightharpoonup A\nabla u_0$ weakly in $L^2(\Omega \times (0, T))$, where $u_0$ is the solution of
\[
\begin{cases}
(\partial_t + L_0)u_0 = F & \text{in } \Omega \times (0, T), \\
u_0 = 0 & \text{on } \partial\Omega \times (0, T), \\
u_0 = f & \text{on } \Omega \times \{t = 0\},
\end{cases}
\]
and $L_0 = -\nabla \cdot (A\nabla)$. 
